Optimal Routing for Constant Function Market Makers

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Presented at EC '22

Outline

- Overview of constant function market makers
- Routing problem
- Problem and solution properties
- Conclusion

A quick overview of DEXs

- *Decentralized exchanges* (DEXs) are venues for exchanging assets requiring no intermediary
- The main trading venue on blockchains (Ethereum, Solana, *etc.*)

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- *Decentralized exchanges* (DEXs) are venues for exchanging assets requiring no intermediary
- The main trading venue on blockchains (Ethereum, Solana, *etc.*)
- Billions of dollars in volume per day!

Constant function market makers

- Usually DEXs are organized as *constant function market makers* (CFMMs)
- Anyone can propose a trade to a CFMM
- The CFMM accepts or rejects this trade based on a simple rule
- If a trade is accepted then the CFMM pays out the trade from its reserves

Constant function market makers (math version)

- CFMMs can trade baskets of n tokens
- Define a *trading function* $\varphi : \mathbf{R}_+^n \to \mathbf{R}$ and *reserves* $R \in \mathbf{R}_+^n$
- Anybody can propose a *trade*, written $\Delta, \Lambda \in \mathbf{R}_+^n$
- CFMM accepts if

$$\varphi(R + \gamma\Delta - \Lambda) \geq \varphi(R)$$

  where $0 < \gamma \leq 1$ is the *trading fee*
- Pays out Λ to user, gets Δ from user
- New reserves: $R + \Delta - \Lambda$

Examples

- Many example trading functions
- The most common (Uniswap v1/v2, *etc.*)

$$\varphi(R_1, R_2) = \sqrt{R_1 R_2}$$

- More general (Balancer)

$$\varphi(R) = \prod_{i=1}^{n} R_i^{w_i}$$

with $w \geq 0$ and $\mathbf{1}^T w = 1$

- Many others...

Concavity and trading functions

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Concavity and trading functions

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Concavity and trading functions

- Will mostly use one property:
- Trading function φ is concave
- This is true in all(!) practical examples
- (There are other natural generalizations)

Outline

Why routing?

- Usually there is not just one CFMM, even on a single chain
- Often there are many (with overlapping tokens!)
- For example, user wants to trade $A \to B$

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- Usually there is not just one CFMM, even on a single chain
- Often there are many (with overlapping tokens!)
- For example, user wants to trade $A \to B$
- What about $A \to C \to B$? $A \to D \to B$? Splitting orders?
- Potentially very complicated...

The routing problem (set up)

- Let's write it as an optimization problem!

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- Let's write it as an optimization problem!
- Say we have a *network* of $i = 1, \ldots, m$ CFMMs
- The network also has n tokens, labeled $1, \ldots, n$
- CFMM i trades subset of $n_i \le n$ tokens
- Has trading function $\varphi_i : \mathbf{R}_+^{n_i} \to \mathbf{R}$, fee γ_i, reserves $R_i \in \mathbf{R}_+^{n_i}$

The routing problem (set up, cont.)

- We write $\Delta_i, \Lambda_i \in \mathbf{R}_+^{n_i}$ for trade with CFMM i
- In terms of the global list of tokens, trader receives net amount from i:

$$A_i(\Lambda_i - \Delta_i)$$

- Here $A_i \in \mathbf{R}^{n \times n_i}$ maps local token indices to global ones
- Net amount received over CFMMs is the *network trade vector*:

$$\Psi = \sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i)$$

The routing problem

- This gives the *optimal routing problem*:

$$
\begin{array}{ll}
\text{maximize} & U(\Psi) \\
\text{subject to} & \Psi = \sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i) \\
& \varphi_i(R_i + \gamma\Delta_i - \Lambda_i) \geq \varphi(R_i), \quad i = 1, \ldots, m \\
& \Delta_i \geq 0, \quad \Lambda_i \geq 0, \quad i = 1, \ldots, m
\end{array}
$$

  with variables $\Psi \in \mathbf{R}^n$, $\Delta_i, \Lambda_i \in \mathbf{R}^{n_i}$ for $i = 1, \ldots, m$

- $U : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ is the user's *utility* over resulting tokens

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\text{subject to} & \Psi = \displaystyle\sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i) \\
& \varphi_i(R_i + \gamma\Delta_i - \Lambda_i) \geq \varphi(R_i), \quad i = 1, \ldots, m \\
& \Delta_i \geq 0, \;\; \Lambda_i \geq 0, \quad i = 1, \ldots, m
\end{array}
$$

- If the utility U is concave, then *problem is convex*

Example utility functions

- Private valuations $c \in \mathbf{R}^n_+$

$$U(\Psi) = c^T \Psi$$

- Liquidating basket $\Delta^{\text{in}} \in \mathbf{R}^n_+$ to token j

$$U(\Psi) = \Psi_j - I(\Psi + \Delta^{\text{in}} \geq 0)$$

- Optimal arbitrage with valuation c (remember this!)

$$U(\Psi) = c^T \Psi - I(\Psi \geq 0)$$

- Markowitz mean $\mu \in \mathbf{R}^n$, covariance $\Sigma \in \mathbf{S}^n_+$, risk $\lambda \geq 0$

$$U(\Psi) = \mu^T \Psi - \frac{\lambda}{2} \Psi^T \Sigma \Psi$$

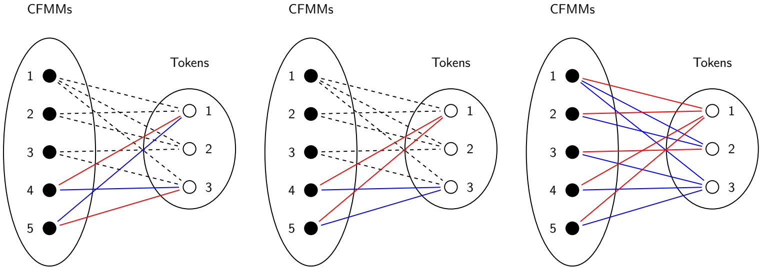
Outline

Convexity and tractability

- Problem is convex ≈ easy to solve in practice
- Even for moderately large n (tokens) and m (CFMMs)
- Problem lends itself nicely to decomposition methods
- See our open source Julia package: `CFMMRouter.jl`
  (Additional work done with Theo Diamandis)

Example solution

- Solutions are rarely intuitive
- Objective: liquidate t of token 1 for token 3



(a) $t = 0$ (b) $t = 20$ (c) $t = 50$

General optimality conditions

- When are the zero trades $\Delta_i = \Lambda_i = 0$ optimal?
- When there exist multipliers $\lambda_i \geq 0$ with

$$\gamma \nabla \varphi_i(R_i) \leq \lambda_i A_i^T g \leq \nabla \varphi_i(R_i),$$

where $g \in -\partial(-U)(0)$, for $i = 1, \ldots, m$

- We can interpret $P_i = \nabla \varphi_i(R_i)$ as the marginal prices of CFMM i

No-arbitrage conditions

- When is there no arbitrage? $(U(\Psi) = c^T\Psi + I(\Psi \geq 0))$
- Apply condition from before: there exists $\lambda_i \geq 0$ such that

$$\gamma P_i \leq \lambda_i A_i^T g \leq P_i,$$

where $g \in \mathbf{R}^n_+$ and $P_i = \nabla \varphi_i(R_i)$, for $i = 1, \ldots, m$

- This means there exists a *market clearing price*, g, consistent with all CFMMs' prices

More general conditions

- Can derive general first-order solution conditions
- (Not just for $\Delta_i = \Lambda_i = 0$)
- See paper for more details and examples!

Outline

Overview of constant function market makers

Routing problem

Problem and solution properties

Conclusion

Conclusion and future directions

- Problem of optimally routing through DEXs is (usually) easy!
- Is being used to fairly great effect in practice
- Convexity implies good computational properties
- And also some interesting mathematical ones!

Acknowledgements

- Theo Diamandis (MIT)