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Week 4: Algorithms for Decentralized Finance

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In an exchange, the total amount of each asset will remain constant before and after the finalization of the exchange period. To understand an exchange system, it is, therefore, beneficial to look at the case of centralized allocation first.

1 Centralized Allocation

For simplicity, we will focus on the case when there are multiple goods, each of which is **"divisible"**. This means that we can allocate one unit of an item to more than one people in the system.[1]

Let assume that there exists J products, each of which has its total quantity normalized to be 1, and I agents. Our task as a centralized allocator is to determine the (deterministic) distribution of products among each agent. Mathematically, we have to decide x_{ij} that specifies the amount of good j allocated to the agent i. Therefore, the constraints for our decision are

$$\sum_{i=1}^{I} x_{ij} \leq 1 \qquad \forall j \in \{1, 2, \ldots, J\}, \text{ and}$$
$$x_{ij} \geq 0 \qquad \forall i \in \{1, 2, \ldots, I\}; \forall j \in \{1, 2, \ldots, J\}.$$

These two constraints are from that we cannot allocate more amount of a product than what we have in the beginning, and we cannot remove any amount of any product from any agent. The constraints are usually referred to as feasibility constraint.

[1] In some cases, this assumption of divisibility can be limited. For example, in a house allocation mechanism, it does not make sense for a person to get 0.4 house A and 0.1 house B. In such case of indivisible products with agents having linear utility, we often replace the allocation of the **actual** product, which requires integer optimization, with the allocation of the **marginal probability** of the rights to the product instead. A number of works on centralized allocation have followed this approach, which can easily incorporate ex ante fairness and can be easily computed.

1.1 Objective

In this section, we assume that utility functions of all agents are known or given exogenously. By doing so, we avoid implication of strategic reporting, where each agent can report its own preference to the centralized allocator, and the allocator does not have an ability to verify the truthfulness of such report.[2]

In general settings, a utility function of an agent i will be a function of the allocation X, which specifies x_{ij} for all $i \in \{1, 2, \dots, I\}$ and for all $j \in \{1, 2, \dots, J\}$. For example, an agent may get a lower utility if it gets to know that its rival get a good allocation. However, from now on, we will narrow the scope to when the utility of an agent i is only dependent on its own allocation.

For further simplification, we will enforce a linear utility, so we get that the utility of an agent i is

$$U_i(\mathbf{x}_i) = \sum_{j=1}^{J} w_{ij} x_{ij}.$$

The assumption of no strategic behavior suggests that every w_{ij} is globally known.

Note that most analysis is applicable in the case when the utility is an increasing concave utility function.

The next natural question is how the individual level utility is useful for our allocation task, when

1) The stakeholders, which are the agents in the system, does not have agency in the allocation decision.

2) The centralized allocator is not a direct stakeholder of the allocation.

To solve this objective confusion implication, we then have to suggest the allocator to "internalize" the utilities of the others as its own utility. This is done by setting the utility to be maximize as

$$U = f(U_1, U_2, \dots, U_I),$$

when f is some aggregation function, which is usually chosen to be non-decreasing in every argument. By doing so, we can hope that the utility maximization will likely lead to an overall higher utilities among agents.

The next question is then what function f is the most suitable.

1.1.1 Utilitarianism

In utilitarianism approach, we want to maximize the sum of utilities. Thus,

$$U = \sum_{i=1}^{I} U_i.$$

[2]For some problem specification, the centralized allocator may be able to pay some additional cost to observe the true preference of an agent. This model is usually called "costly state verification".

This is perhaps one of the most natural specification of utilities aggregator. However, this approach can sometimes result in having an agent having all allocations, which may not be ideal. This approach is then questionable on whether it actually leads to **"the greater good"** as its name suggests.[3]

For example, imagine a case when there are 100 agents and 100 products with $w_{1j} = 1.0001$ for all $j \in \{1, 2, \ldots, J\}$, and $w_{ij} = 1$ for all $i \neq 1$. It is obvious that the agent 1 will get everything, and everyone else will not get anything.

1.1.2 Egalitarianism

In egalitarianism approach, we will care about **"the good of the least favored"**. The excerpt below is from John Rawls' remark in "The Theory of Justice"[1], suggesting why this system may be more fair for the allocation system.

> These principles rule out justifying institutions on the grounds that the hardships of some are offset by a greater good in the aggregate. It may be expedient but it is not just that some should have less in order that others may prosper. But there is no injustice in the greater benefits earned by a few provided that the situation of persons not so fortunate is thereby improved. [1]

Mathematically, this is equivalence to having

$$U = \min_{i \in \{1,2,\ldots,I\}} U_i.$$

In both cases, U is a concave function of $U_1, U_2, \ldots, U_I$, so the allocation problem can be easily solved via convex optimization.

1.2 Desiderata

1.2.1 Pareto Optimality

This occurs when we cannot possibly strictly improve a utility of an agent without hurting (adversely affecting) any other agent's utility. Mathematically, there exists no feasible X' such that

$$U_i(\mathbf{x}'_i) \geq U_i(\mathbf{x}_i)$$

for all $i \in \{1, 2, \ldots, I\}$ and the inequality is strict for some $i \in \{1, 2, \ldots, I\}$.

[3] In some school of thought, utilitarianism will not lead to the problem of drastic inequality if the allocation other people get also affect an agent utility. In that case, we will see that the "internalization" process has also been carried on in an individual level, so its importance in the aggregation level is not as critical. However, according to our simplification assumption, the individual level internalization is not allowed.

1.2.2 Envy Freeness

This occurs when an agent will not have a strict preference over any allocation for any other agent over its own allocation. Mathematically, for all $i, k \in \{1, 2, \ldots, I\}$,

$$U_i(\mathbf{x}_i) \geq U_i(\mathbf{x}_j).$$

1.2.3 Proportionality

This occurs when an agent will not have a strict preference over equal allocation. Mathematically, for all $i \in \{1, 2, \ldots, I\}$,

$$U_i(\mathbf{x}_i) \geq U_i\left(\left(\frac{1}{I}, \frac{1}{I}, \ldots, \frac{1}{I}\right)\right).$$

It is easy to show that utilitarianism approach will result in allocations that are Pareto optimal, and egalitarian approach will have at least 1 member of the optimal allocations set that is Pareto-optimal.[4] However, both allocations are not necessarily envy free and proportional.

For example, let consider a case when there are 2 agents and 2 products, with $w_{11} = w_{12} = 70$ and $w_{21} = w_{22} = 630$. Utilitarianism approach will give the agent 1 zero allocation, so it will envy the agent 2, and it will strictly prefer the equal distribution over what it gets, which is zero. Egalitarianism approach will give the agent 1 0.9 of each item and give the agent 2 0.1 of each item. The agent 2 will be envy of the agent 1 and prefer the equal allocation over the current allocation.

In order to incorporate the two desiderata into an allocation problem, one can easily explicitly enforce them as a set of linear constraints.

1.3 Nash Welfare

Nash welfare is $\sum_{i=1}^{I} \log U_i$. The Nash welfare optimizing centralized allocator can be considered as a compromise between the utilitarianism approach and the egalitarianism approach. In this approach, every increase in the utility will strictly increase the aggregated utility (unlike that in the egalitarian framework). However, the increase in the individual utility for an agent with lower utility will yield a strictly greater increase in the aggregated utility than the increase in the individual utility for an agent with higher utility (while the utilitarianism approach will be indifferent between the two increments).

1.4 Properties

It is obvious that every Nash welfare optimizing allocation will be Pareto optimal.

[4]Normally, there can be some elements of the optimal allocations set that is not Pareto-optimal, due to the fact that the marginal change of an agent with higher than the least utility will not affect the allocator utility.

In the next section, we will also see that every Nash welfare optimizing allocation will be envy-free and proportional.

From the concavity of log function, the Nash welfare maximization is a convex optimization known as Gale-Eisenberg convex program and can be solved easily.

2 Fisher Market

In Fisher market, we, as the centralized allocator, will not actively perform the allocation decision. However, we will assign monopoly money that does not have any external value to each agent, and we will set price p_j for each product. The agents then make decisions to buy products with the objective of maximizing their own utilities. This, therefore, utilizes the market system to facilitate our allocation decision. In other words, our task, as the centralized allocator, is shifted from allocating to pricing.

The price to be set by the centralized allocator has to be a market clearing price $\mathbf{p}$, which is the price such that, for each agent i with an initial budget B_i, there exists some $\mathbf{x}_i$ that can solve the

$$\begin{array}{rl} \max_{\mathbf{x}} & U_i(\mathbf{x}) \\ \text{subject to} & \sum_{j=1}^{J} p_j x_j \leq B_i \\ & \mathbf{x} \geq 0 \quad \text{elementwise} \end{array}$$

, and

$$\sum_{i=1}^{I} x_{ij} = 1 \; \forall j \in \{1, 2, \ldots, J\}.$$

This means that it is a price that, if each agent optimize its own utility constrained to its own budget (which is normalized to be 1), then the supply **can** match the demand.[5]

2.1 Competitive Equilibrium with Equal Endowments

In this section, we will evaluate a special case of Fisher market, when every agent gets the same amount of monopoly money. This is called CEEI or Competitive Equilibrium with Equal Endowments.

2.1.1 Equivalence

In this subsection, we will show that CEEI will maximize the Nash welfare of the system.

We first consider the Nash welfare maximization allocation. The problem we face is the following:

[5] In some cases, there may exist an aggregation of personal decisions that does lead to market clearing, because $\mathbf{x}_i$ may not be unique for each agent i.

$$\begin{array}{rll} \max\limits_{X} & \sum_{i=1}^{I} \log U_i(\mathbf{x}_i) & \\ \text{subject to} & \sum_{i=1}^{I} x_{ij} \leq 1 & \forall j \in \{1, 2, \ldots, J\} \\ & X \geq 0 & \text{elementwise} \end{array}$$

We introduce a dual variable λ_j for each product j for the first constraint. Therefore, we can get that the Lagrangian is

$$\mathcal{L}\left(X, \lambda\right) = \sum_{i=1}^{I} \log U_i(\mathbf{x}_i) - \sum_{j=1}^{J} \left(\lambda_i \left(\sum_{i=1}^{I} x_{ij} - 1 \right) \right).$$

Note that this is a concave function, and for any realized λ, we get that the first order condition, if exists within the constraint set, will lead to a (conditional) global maximum. Consider

$$\frac{\partial}{\partial x_{ij}} \mathcal{L} = \frac{w_{ij}}{\sum_{k=1}^{J} w_{ik} x_{ik}} - \lambda_i.$$

If the second constraint on non-negativity is non-binding (i.e. if the result X are all strictly positive), then

$$\begin{aligned} \lambda_i &= \frac{w_{ij}}{\sum_{k=1}^{J} w_{ik} x_{ik}} \\ \sum_{j=1}^{J} \lambda_i x_{ij} &= \frac{\sum_{j=1}^{J} w_{ij} x_{ij}}{\sum_{k=1}^{J} w_{ik} x_{ik}} \\ &= 1. \end{aligned}$$

Thus, we can think of λ_j as a price for the product j. The equation

$$\frac{\partial}{\partial x_{ij}} \mathcal{L} = \frac{w_{ij}}{\sum_{k=1}^{J} w_{ik} x_{ik}} - \lambda_j$$

then suggests that the agent i will choose to buy more x_j only when $\frac{w_{ij}}{\lambda_j} > \sum_{k=1}^{J} w_{ik} x_{ik}$. That is when the "bang for buck" is greater than a reserve level.

Next, we can consider the individual optimization for the agent i in Fisher market subjected to the central price $\mathbf{p}$ and a personal budget 1. From the monotonicity of log function, we get that the problem is equivalence (in terms of maximizer) to

$$\begin{array}{rll} \max\limits_{\mathbf{x}} & \log\left(U_i(\mathbf{x})\right) & \\ \text{subject to} & \sum_{j=1}^{J} p_j x_j \leq 1 & \\ & \mathbf{x} \geq 0 & \text{elementwise} \end{array}$$

Next, we can introduce a dual variable, and get that the individual Lagrangian is

$$\mathcal{L}_i(\mathbf{x}_i, \lambda_i) = \log\left(U_i(\mathbf{x}_i)\right) - \lambda_i \left(\sum_{j=1}^{J} p_j x_{ij} - 1 \right).$$

Therefore,

$$\frac{\partial}{\partial x_{ij}}\mathcal{L}_i = \frac{w_{ij}}{\sum_{k=1}^{J} w_{ik}x_{ik}} - \lambda_i p_j.$$

Thus, we then see that the Nash welfare maximization can be reformulated as a CEEI, and vice versa.

This equivalence suggests that Nash welfare maximization will inherit the properties of CEEI.

2.1.2 Envy-Freeness

Under CEEI is envy free, if an agent i envies an agent j, the agent i can use the same amount of budgets to copy the purchase decision of the agent j. Therefore, Nash welfare maximization will also lead to an envy free allocation. However, it is obvious that a general Fisher market with different initial budgets will not inherit this property; a richer agent can always copy a poorer agent, but a poorer agent may not be able to copy a richer agent.

2.1.3 Proportionality

Under CEEI, the sum of all prices will be I. Thus, every agent can choose to buy $\frac{1}{I}$ of each product if that choice increases its utility. Therefore, Nash welfare maximization will also lead to a proportional allocation.

2.2 Unequal Endowment

Note that we can extend the concept of a Fisher market from CEEI to the case when not every agent is equally important by allocate the monopolistic budget unequally.

Can we think of a case where it is better to allocate the initial budget unequally?

1) We do not take into account the initial wealth of each person. In order to do so, we may want to increase a priority for low-income people, who is more in need, by allocating more monopolistic budget to them.

2) If we always give people equal budget, there is an incentive for any agent to register in the system with 2 accounts, thereby being 2 agents in the allocation system.

3 Arrow-Debreu Market

In Fisher market, each person will have some initial wealth as determined by the monopolistic budget allocation. Although we can change the budget allocation to create unequal initial wealth, the wealth for each agent will still be exogenously decided.

In Arrow-Debreu market, the initial wealth is not exogenous, but is highly dependent on the demand-supply system. Specifically, in Arrow-Debreu "exchange" market, each agent will be exogenously endowed with some initial endowment of products. Afterwards, each agent can "trade" with one another at some trade ratios. It turns out that even when we do not have any actual money in the exchange market at all, the trade ratios between each pair of product can be thought of as a price ratio between them. These price ratios are the main tools to inform the values of each product, and thereby the initial wealth of each agent. This means that an agent can get richer, if what it has in the beginning has high aggregated demand.

The idea that the initial wealth is dependent on the price makes the equilibrium price much more difficult to be found. In a general specification of utility functions, there can also be issues on existence and uniqueness of the equilibrium. A more detailed discussion can be seen in Prof. Levin's lecture note on General Equilibrium[2].

4 SPEEDEX: Scalable, Parallelizable, and Economically Efficient Distributed EXchange

SPEEDEX is a design for distributed exchange or DEX. For a full information on SPEEDEX, one can read "SPEEDEX: A Scalable, Parallelizable, and Economically Efficient Digital EXchange" , Ramseyer, et. al., 2021[3]. Before diving into the mechanism of SPEEDEX, it is beneficial for us to review the current exchanges systems.

4.1 Pairwise Exchange

The existing exchanges normally rely on pairwise trading. For example, a limit order book is a continuous "double" auction. An agent can place bid and ask orders. The exchange will aggregate those orders **in order** to determine what transactions will occur and at which price or prices. The ordered nature of the mechanism of a limit order book prevents the algorithm from being parallelized.

Although some exchanges may wait for an aggregation of orders before applying a static double auction, which does not consider the orders of any orders as long as they are in the batch, the focus is mainly on pairwise trading.

4.2 SPEEDEX

SPEEDEX also utilizes the idea of batch trading, meaning that the internal order can be neglected. However, it also addresses the problem of trading multi-assets at the same time. In order to do so, it runs an Arrow-Debreu market algorithm for each batch to determine the central price for the transaction to occur and the final allocation of assets.

The order sent from an agent will be processed by the mechanism in a parallel manner to get the inferred utility function and endowment. From the utility functions and endowments

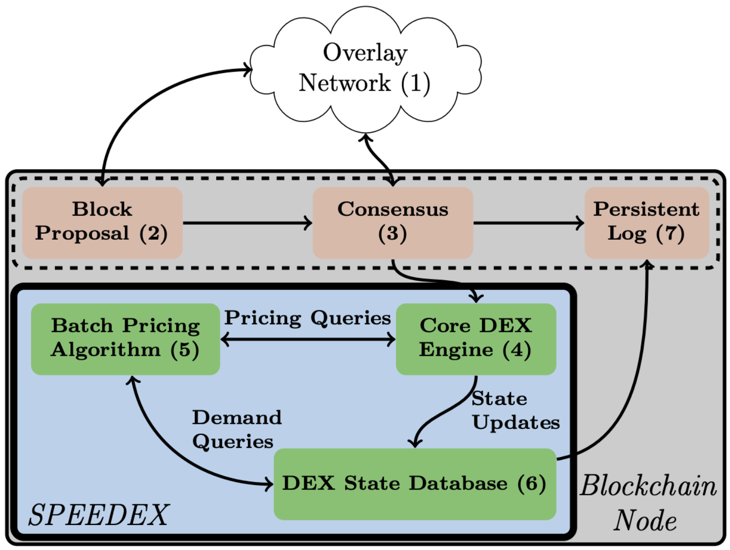

Figure 1: Diagram explaining the mechanism of SPEEDEX from "SPEEDEX: A Scalable, Parallelizable, and Economically Efficient Digital EXchange" , Ramseyer, et. al., 2021[3].

of the population, the central price, as well as the final allocation, can be computed.[6]

Since all transactions in the same batch will be done with the same equilibrium price instead of a limit price as in a vanilla limit order book, there will not be an arbitrage from "front running", which utilizes the discrepancies in the limit prices. Moreover, the bundling of multiple assets can facilitate the trading in the sense that each agent does not have to hold any intermediate currency, as in a normal pairwise exchange scheme.

References

[1] John Rawls. *A Theory of Justice*. Universal Law Publishing Co Ltd, 2013.

[2] Jonathan Levin. General equilibrium, 01 2021.

[3] Geoffrey Ramseyer, Ashish Goel, and David Mazières. Speedex: A scalable, parallelizable, and economically efficient digital exchange, 11 2021.

[6]There is a computational challenge in the equilibrium computation for Arrow-Debreu market, and SPEEDEX also inherits the challenge.

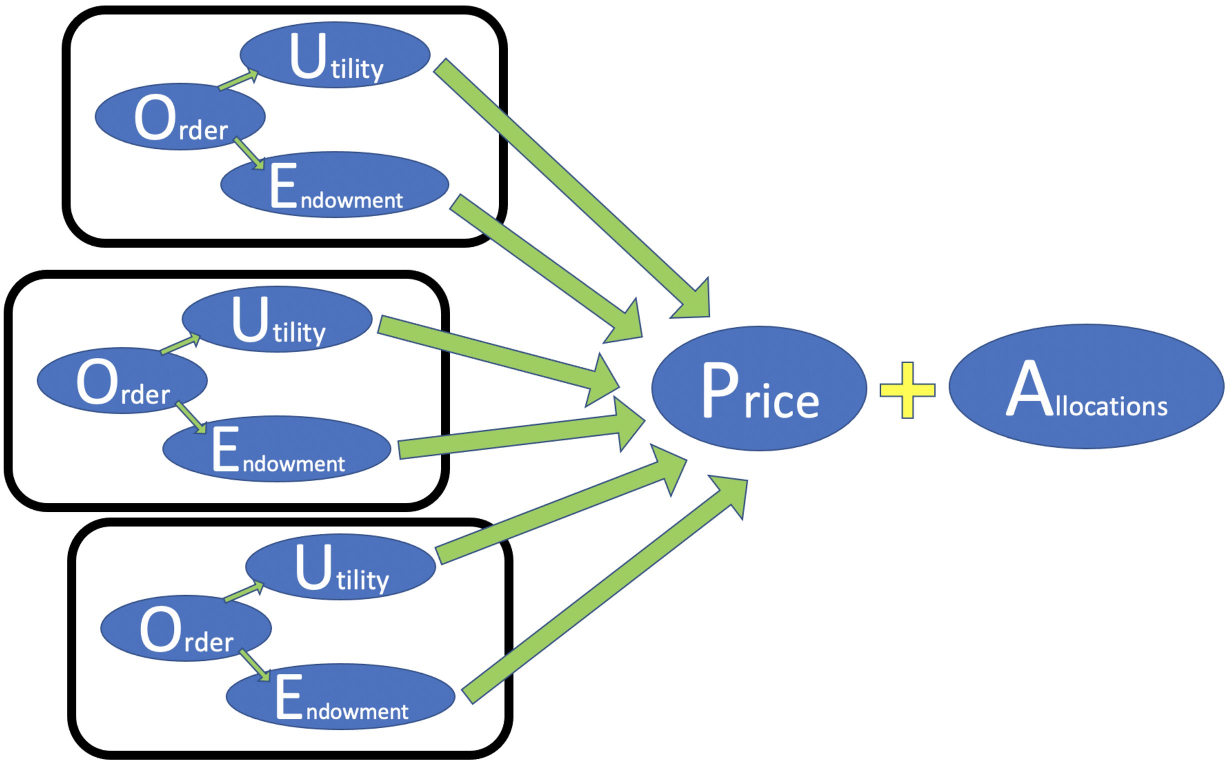

Figure 2: Diagram explaining the algorithmic model for the allocation used in SPEEDEX. In this diagram, each box corresponds to a different agent within the system. It can be seen that, unlike a more conventional limit order book, which is a special case of double auction, the allocation for SPEEDEX is not directly related to the placed orders itself. Since the price and the final allocation will be determined via the inferred utility function and endowment.