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Week 5: Algorithms for Decentralized Finance

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1 Automated Market Maker

The main task of a market maker is to set a price for B in terms of A and that for A in terms of B perhaps via bid and ask. We will assume that the two prices will be such that the product is 1, suggesting there is no bid-ask spread. The notion of "automated" suggests that the price is generated using a computer code. Moreover, the code is publicly known. However, the information used for the algorithm can also include history and information on reserves. This suggests that the complexity of the function being used to set a price can be very complex and time-dependent.

Therefore, we first consider a specific type of automated market maker, "constant function market maker"

2 Constant Function Market Maker

The main components of CFMM (Constant Function Market Maker) are

1. Liquidity pool will specify (x, y) when there exists x units of A and y units of B based on the current reserves. This can be observed at any given time.

2. Some known function $f(\cdot, \cdot)$, which is weakly increasing in each variate, has to be declared and the CFMM has to commit to such function. Normally, f will be increasing in each variate, continuous, and differentiable.

These two suggests that all subsequent behaviors of CFMM will only use current reserves states and not be time-dependent.

CFMM permits any trade that would incur the changes Δx and Δy only when

$$f(x + \Delta x, y + \Delta y) \geq f(x, y).$$

This suggests the trading will then normally occur along the contour curve to preserve the value of $f(x, y)$ before and after each transaction.

To avoid the occurrence of x or y dropping to 0, we normally consider a function f such that its contour line $\{(x, y)|f(x, y) = k\}$ has **non-exhaustion** property. That is, as $x \to 0$, $y \to \infty$, and, as $y \to 0$, $x \to \infty$. However, this is not a strict requirement.

2.1 Price

The **spot price** of A in terms of B can then be found via the negative of the slope of such contour line at the point corresponding to the current reserves status (x, y). Thus, price A in terms of B is

$$-\frac{dy}{dx},$$

when x, y is constrained to the contour line.

Since we want the price to be higher when A is scarce and the price to be decreasing as there is more A in the reserves, we, therefore, usually consider a function f whose corresponding contour $\{(x, y)|f(x, y) = k\}$ can be written as $\{(x, g(x))|x > 0\}$ for some $g : \mathbb{R}^+ \to \mathbb{R}^+$ being a strictly decreasing strictly convex function.

Note that the **spot price** is defined along the contour curve. For a non-zero order, the average price will then be different from the initial price, since price is updated continuously. Specifically, we can think of an order as an collection of many smaller orders, each of which will be subjected to different price, since after the previous order has been carried on, the status of the reserves changes. This phenomenon of the change in price due to a transaction is called a **slippage**.

2.1.1 Example: $f(x, y) = xy$

The trading occurs along a curve $xy = k$ for some constant $k \in \mathbb{R}^+$. The spot price of A in terms of B is

$$-\frac{dy}{dx} = \frac{k}{x^2} = \frac{y}{x}.$$

Thus, we can see that price will be higher when the ratio of y comparing to x is higher.

2.2 Liquidity

There can be multiple notions for liquidity. One possible desiderata for the liquidity measure is such that the liquidity is high whenever the rate of the change of price is slow, and is low when the rate of the change of price is high. The next question is what measure of change for price should we use, linear or fractional, and what other changes we should be comparing it to, x, y, or theirs fractional. In this case, we let liquidity be defined to be

$$\frac{dy}{d \log p} \text{ or } \frac{p dy}{dp}. \tag{1}$$

Let denote the liquidity at a spot price p by

$$L(p) = \frac{pdy}{dp}.$$

If the trade size in units of B is k, then we will define the failure probability as

$$\frac{k}{L(p)}.$$

This is also called CFMM inefficiency, which we want to minimize.

2.3 Impermanent Loss

Let the initial state be x_0, y_0 with the initial spot price being p_0.

After the trade occurs, we get that the state changes into x, y with a new spot price being p.

We can say that the value in terms of B of the current pool is

$$y + px,$$

and that the value in terms of B of the original pool is

$$y_0 + px_0.$$

We can see that at a given price p, the status (x, y) whose spot price is p will be a minimizer of the following problem:

$$\begin{aligned} \min_{x',y'} \quad & px' + y' \\ \text{subject to} \quad & f(x', y') = f(x_0, y_0) \end{aligned}$$

This follows from the convexity of the contour curve of f. Therefore,

$$y + px \leq y_0 + px_0.$$

This suggests that a liquidity pool will always lose value.

3 Multi-Asset CFMM

In a multi-asset case, the structure is similar to the case of 2 assets CFMM, we have discussed above. Let there be a total of $n \geq 2$ assets.

The steps are the following:

0) The current state of the reserves $R \in \mathbb{R}_0^{+n}$ is globally known.

1) An agent will propose the trade, specified by $\Delta, \Lambda \in \mathbb{R}_0^{+n}$.

2) The market maker will decide whether to accept or reject the proposal. The decision is based on a pre-specified and globally known **trading function** $\psi : \mathbb{R}_0^{+n} \to \mathbb{R}$ along with $\gamma in (0, 1]$. The trade proposal will be accepted if and only if

$$\psi\left(R + \gamma\Delta - \Lambda\right) \geq \psi\left(R\right).$$

(γ will be specified by a linear transaction fee, and $\gamma = 1$ when there is no transaction fee.)

3) If the proposal is accepted, a transaction will occur. The agent will have to pay Δ and will receive Λ in return.

The trading function ψ is generally up to the market maker to decide. In practice, ψ is a concave function. One can consider a "balancer" trading function, which is

$$\psi(R) = \prod_{i=1}^{n} R_i^{w_i},$$

where $w_i \geq 0$ for all i and $\sum_{i=1}^{n} w_i = 1$.

The price of a product i with respect to a product j is then

$$p = \frac{\frac{\partial}{\partial R_i}\psi(R)}{\frac{\partial}{\partial R_j}\psi(R)}.$$

4 Optimal Routing

Let there be m CFMMs in the market and n assets. In order to trade more than 1 asset, it is probably better to trade through a series of transactions with different CFMMs. This is to avoid the change in the price due to slippage (which prevents us from trading myopically). Let the trading function for the i^{th} CFMM be $\psi_i : \mathbb{R}_0^{+n} \to \mathbb{R}$ and the linear transaction fee of each CFMM be specified by γ_i. The current reserves for all CFMMs are observable and specified by $R_1, R_2, \ldots, R_m$.

By assuming that the utility function $U : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is concave in the total transactions (with sign) and that all ψ_i's are concave, we can formulate optimal routing as a convex optimization problem:

$$\begin{array}{rl} \underset{\Delta_i, \Lambda_i \in \mathbb{R}_0^{+n}; \forall i \in \{1,2,\ldots,m\}}{\max} & U\left(\sum_{i=1}^{m}\left(\Lambda_i - \Delta_i\right)\right) \\ \text{subject to} & \psi_i\left(R_i +_i \Delta_i - \Lambda_i\right) \geq \psi_i\left(R_i\right) \quad ; \forall i \in \{1, 2, \ldots, m\} \end{array}$$

For a detailed discussion and example code implemented in CVXPY, one can read "Optimal Routing for Constant Function Market Makers", Angeris, et. al. [1].

References

[1] Guillermo Angeris, Alex Evans, Tarun Chitra, and Stephen Boyd. Optimal routing for constant function market makers. In *Proceedings of the 23rd ACM Conference on Economics and Computation*, EC '22, page 115–128, New York, NY, USA, 2022. Association for Computing Machinery.