205A LECTURE 4, FALL 2018

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1. Other Measures on R

Let $F : \mathbf{R} \to \mathbf{R}$ be any increasing function. Define $\mu^* : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ by letting $\mu^*(S)$ be the infimum of

$$\sum_{n=1}^{\infty}(F(y_i) - F(x_i))$$

over all countable coverings

$$S \subset \cup_{n=1}^{\infty}(x_i, y_i)$$

of S by open intervals. Almost exactly as we did for Lebesgue measure, one proves that that μ^* is an outer measure, that all Borel sets are μ^*-measurable, and that

$$\mu([a, b]) = \mu^*([a, b]) = F(b^+) - F(a^-),$$

where

$$\begin{aligned} F(b^+) &= \lim_{t \downarrow b} F(t) = \inf_{t>b} F(t), \\ F(a^-) &= \lim_{t \uparrow a} F(t) = \sup_{t<a} F(t). \end{aligned}$$

Of course if F is continuous, then $\mu([a, b]) = F(b) - F(a)$.

2. Some Approximation Theorems for Lebesgue Measure

Theorem 1. *Given $S \subset \mathbf{R}^N$ and $\epsilon > 0$, there is an open set $W \subset \mathbf{R}^N$ such that*

$$\lambda(W) \le \lambda^*(S) + \epsilon. \tag{*}$$

(Note: if $\mu^*(S) = \infty$, then of course $\lambda^*(W)$ and $\lambda^*(S) + \epsilon$ are also ∞. For that reason, we have $\le$ and not $<$ in (*).)

Proof. By definition of λ^*, there exist open intervals $I_1, I_2, \dots$ such that

$$S \subset \cup_n I_n$$

and such that

$$\sum_n |I_n| \le \lambda^*(S) + \epsilon.$$

Let $W = \cup_n I_n$. Then W is open. Since W is covered by the intervals I_n,

$$\begin{aligned} \lambda(W) &= \lambda^*(W) \\ &\le \sum_n |I_n| \\ &\le \lambda^*(S) + \epsilon. \end{aligned}$$

□

If S is Lebesgue measurable, one can say much more:

Theorem 2. *Suppose that* $S \subset \mathbf{R}^N$ *is* λ^**-measurable and that* $\epsilon > 0$*.*

(1) *There exists an open set* W *such that* $S \subset W$ *and* $\lambda(W \setminus S) \le \epsilon$*.*
(2) *There exists a closed set* K *such that* $K \subset S$ *and* $\lambda(S \setminus K) \le \epsilon$*.*

Remark. If $\lambda(S) < \infty$, then $\lambda(W \setminus S) \le \epsilon$ is equivalent to $\mu(W) \le \mu(S) + \epsilon$. But if $\lambda(S) = \infty$, the former statement is much stronger.

Proof. Let

$$S_n = S \cap \{x : |x| < n\}.$$

The set S_n is Lebesgue measurable since it is the intersection of the measurable sets S and $\{x : |x| < n\}$. Since S_n is contained in the cube $(-n, n)^N$, $\mu(S) = \mu^*(S_n)$ is finite (it is $\le (2n)^N$). By Theorem 1, there is an open set W_n containing S_n such that

$$\mu(W_n) \le \mu(S_n) + \frac{\epsilon}{2^n}.$$

Since $\mu(W_n) = \mu(S_n) + \mu(W_n \setminus S_n)$,

$$\mu(S_n) + \mu(W_n \setminus S_n) \le \mu(S_n) + \frac{\epsilon}{2^n}.$$

Since $\mu(S_n) < \infty$, this implies that

$$\mu(W_n \setminus S_n) \le \frac{\epsilon}{2^n}$$

and thus

$$\mu(W_n \setminus S) \le \frac{\epsilon}{2^n}.$$

Now let $W = \cup_n W_n$. Then W is an open set containing S, and

$$W \setminus S = (\cup W_n) \setminus S = \cup (W_n \setminus S),$$

so

$$\lambda(W) \le \sum_n \lambda(W_n \setminus S) \le \sum_n \frac{\epsilon}{2^n} = \epsilon.$$

This proves assertion (1).

By assertion (1) applied to S^c, there is an open set U containing S such that

$$\mu(U \setminus S^c) \le \epsilon.$$

Consequently, $K := U^c$ is a closed set contained in S and $\mu(S \setminus K) \le \epsilon$. (Note that $S \setminus K = U \setminus S^c$.) ☐

Theorem 3. *If* $S \subset \mathbf{R}^N$ *is a Lebesgue measurable set, then there is a Borel set* T *such that* $\lambda(T \setminus S) = 0$*.*

Proof. By Theorem 2, there is an open set U_n containing S such that $\lambda(U_n \setminus S) < \frac{1}{n}$. Let $T = \cup_n U_n$. Then T is a Borel set containing S, and $T \setminus S \subset U_n \setminus S$ for every n, so

$$\lambda(T \setminus S) \le \lambda(U_n \setminus S) \le \frac{1}{n}.$$

Since this holds for every n, $\lambda(T \setminus S) = 0$. ☐

Recall that a set T in a topological space is called a G_δ set if it can be expressed as a countable intersection of open sets. Note that the proof of Theorem ? shows that we can choose T to be a G_δ set.

The same method of proof shows that S contains a F_σ set R such that $\lambda(S \setminus R) = 0$. (Recall that an F_σ set is a set that can be expressed as a countable union of closed sets.)

3. FINITE BOREL MEASURES ON $\mathbf{R}^N$

A measure μ on a measurable space $(X, \mathcal{A})$ is said to be **finite** if $\mu(X) < \infty$. If X is a topological space, a **Borel measure** is a measure on $(X, \mathcal{B}(X))$.

Theorem 4. *Suppose that μ is a finite Borel measure on $\mathbf{R}^N$. If S is a Borel set and $\epsilon > 0$, then there exists a closed set K and an open set W such that $K \subset S \subset W$ and such that*

$$(*) \qquad \mu(W \setminus S) < \epsilon \text{ and } \mu(S \setminus K) < \epsilon.$$

Proof. Let $\mathcal{F}$ be the family of all Borel sets S that satisfy the conclusion of the theorem for every $\epsilon > 0$.

Claim 1: $\mathcal{F}$ contains all the open sets.

Proof of claim 1: If S is open, we let $W = S$. Let

$$S_n = \left\{ x \in S : \operatorname{dist}(x, S^c) \geq \frac{1}{n} \right\}$$

Then S_n is closed, $S_1 \subset S_2 \subset \dots$, and $\cup_n S_n = S$. Hence

$$\mu(S) = \lim_n \mu(S_n).$$

Thus there exist n for which $\mu(S_n) > \mu(S) - \epsilon$. It follows that $\mu(S \setminus S_n) < \epsilon$. Let $K = S_n$. This proves Claim 1.

Claim 2: $\mathcal{F}$ is closed under complements.

For suppose $K \subset S \subset W$ such that K is closed, W is open, and such that (*) holds. Then W^c is closed, K^c is open, $W^c \subset S^c \subset K^c$, and

$$\mu(K^c \setminus S^c) = \mu(S \setminus K) < \epsilon,$$
$$\mu(S^c \setminus W^c) = \mu(W \setminus S) < \epsilon.$$

This proves Claim 2.

Claim 3: $\mathcal{F}$ is closed under countable unions.

To prove Claim 3, let $S_1, S_2, \dots$ be in $\mathcal{F}$, let $\epsilon > 0$, and let $S = \cup_n S_n$. Then for each n, there exist an open set W_n and a closed set K_n such that $K_n \subset S_n \subset W_n$ and

$$\mu(W_n \setminus S_n) < \frac{\epsilon}{2^n},$$
$$\mu(S_n \setminus K_n) < \frac{\epsilon}{2^n}.$$

Now let $W = \cup_n W_n$. Then W is open and

$$W \setminus S = (\cup_n W_n) \setminus (\cup_n S_n) \subset \cup_n (W_n \setminus S_n),$$

so

$$\mu(W \setminus S) \leq \sum_n \mu(W_n \setminus S_n) \leq \sum_n \frac{\epsilon}{2^n} = \epsilon.$$

Likewise, let $C = \cup_n K_n$. Then

$$S \setminus C = (\cup_n S_n) \setminus (\cup_n K_n) \subset \cup_n (S_n \setminus K_n),$$

so

$$\mu(S \setminus C) \leq \sum_n (S_n \setminus K_n) = \sum_n \frac{\epsilon}{2^n} = \epsilon,$$

or, equivalently,

$$\mu(C) > \mu(S) - \epsilon. \tag{1}$$

Now we are not quite done, because C is an F^σ set, but is not necessarily a closed set.

Let $C_n = K_1 \cup K_2 \cup \dots \cup K_n$. Then each C_n is closed, $C_1 \subset C_2 \subset \dots$, and $\cup_n C_n = C$. Thus $\mu(C) = \lim_n \mu(C_n)$. By (1), we can choose n so that

$$\mu(C_n) > \mu(S) - \epsilon,$$

or, equivalently, so that $\mu(S \setminus C_n) < \epsilon$. This completes the proof of Claim 3.

We have showed that $\mathcal{F}$ is a σ-algebra that contains all the open sets. Hence it contains all the Borel sets. □

We can choose the K in Theorem 4 to be compact:

Corollary 5. *Suppose that $S \subset \mathbf{R}^N$ is a Borel set and that $\epsilon > 0$. Then there is a compact set $K \subset S$ such that $\mu(S \setminus K) < \epsilon$, or, equivalently, such that*

$$\mu(K) > \mu(S) - \epsilon.$$

Proof. By the theorem, there is a closed set $C \subset S$ such that

$$\mu(C) > \mu(S) - \epsilon.$$

Now let $C_n = C \cap \{x : |x| \le n\}$. Then $C_1 \subset C_2 \subset \dots$ and $\cup_n C_n = C$, so $\lim_n \mu(C_n) = \mu(C)$. Hence

$$\mu(C_n) > \mu(S) - \epsilon$$

for all sufficiently large n. Since C_n is compact, we are done. □

4. Measurable Functions

In this section, $(X, \mathcal{A})$ be a measurable space (i.e., X is a space and $\mathcal{A}$ is a σ-algebra of subsets of X), and Y is a topological space.

Definition 6. We say that a function $F : X \to Y$ is **$\mathcal{A}$-measurable** provided the inverse image of every Borel set is in $\mathcal{A}$:

$$F^{-1}(S) \in \mathcal{A} \quad \text{for every } S \in \mathcal{B}(Y).$$

Theorem 7. *Suppose that $\mathcal{F}$ is a family of subsets of Y such that $\sigma(\mathcal{F}) = \mathcal{B}(Y)$. The following are equivalent:*

1. *$F^{-1}(S) \in \mathcal{A}$ for every $S \in \mathcal{F}$.*
2. *F is $\mathcal{A}$-measurable.*

Thus to check that $F : X \to Y$ is $\mathcal{A}$ measurable, it is not necessary to check that $F^{-1}(S) \in \mathcal{A}$ for every Borel set $S \subset Y$; it suffices to check that this holds for every S in some family of sets that generates all the Borel sets.

Some important examples of such families $\mathcal{F}$ are

1. The family of open subset of Y.
2. The family of closed subsets of Y.
3. In case $Y = \mathbf{R}$, the family of intervals of the form $[a, \infty)$, $a \in \mathbf{R}$ (or the family of all intervals (a, ∞)).

Remark. The text defines an $\mathcal{A}$-measurable function $F : X \to \mathbf{R}$ to be a function such that $F^{-1}([a, \infty))$ is in $\mathcal{A}$ for every $a \in \mathbf{R}$. By Theorem 7, that definition is equivalent to Definition 6 above. Definition 6 has the advantage that it makes sense for any topological target space.

Proof of Theorem 7. Let $\mathcal{C}$ be the collection of all sets $S \subset Y$ such that $F^{-1}(S) \in \mathcal{A}$.

Since $F^{-1}(\emptyset) = \emptyset$, $F^{-1}(Y \setminus S) = X \setminus F^{-1}(S)$, $F^{-1}(\cup_\alpha S_\alpha) = \cup_\alpha F^{-1}(S_\alpha)$ and $F^{-1}(\cap_\alpha S_\alpha) = \cap_\alpha F^{-1}(S_\alpha)$, we see that $\mathcal{C}$ is a σ-algebra. Thus since $\mathcal{C}$ contains $\mathcal{F}$, it contains the σ-algebra generated by $\mathcal{F}$. $\square$