THE FIVE TIMES COVERING THEOREM

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Let X be a metric space. The **diameter** of a set $S \subset X$ is the supremum of the distance between points in S:

$$\mathrm{diam}(S) = \sup_{x,y \in S} d(x,y).$$

If $x \in X$, the **distance** from x to S is defined to be $\mathrm{dist}(x,S) = \inf_{y \in S} d(x,y)$.

It will be convenient here to use the following notation. If $S \subset X$, we let

$$\widehat{S} = \{x : \mathrm{dist}(x,S) < 2\,\mathrm{diam}(S)\}$$

Note that if $\mathbf{B}$ is a ball in $\mathbf{R}^N$, then $\widehat{\mathbf{B}}$ is the open ball with the same center and with 5 times the radius.

A collection $\mathcal{F}$ of subsets of X **covers** a set S if

$$S \subset \cup_{B \in \mathcal{F}} B,$$

or, equivalently, if for every $x \in S$, there is a $B \in \mathcal{F}$ with $x \in B$. We say that $\mathcal{F}$ **covers** S **finely** if for every $\epsilon > 0$,

$$S \subset \cup\{B : B \in \mathcal{F} \text{ and } \mathrm{diam}(B) < \epsilon\},$$

or, equivalently, if for every $x \in S$ and $\epsilon > 0$, there is a $B \in \mathcal{F}$ with $x \in B$ and $\mathrm{diam}(B) < \epsilon$.

Theorem (The 5-Times Covering Theorem)**.** *Let $\mathcal{F}$ be a collection of closed sets in a metric space such that the diameters of the sets in $\mathcal{F}$ are all positive and bounded above. Suppose that $\mathcal{F}$ satisfies the following finiteness condition:*

> *There is no infinite collection of disjoint sets in $\mathcal{F}$ having diameters bounded away from 0.*

Then there is a sequence (finite or infinite) $B_1, B_2, \dots$ of disjoint sets in $\mathcal{F}$ with the following property: for every $B \in \mathcal{F}$, there is a k such that $B \cap B_k \neq \emptyset$ and $B \subset \widehat{B}_k$.

Furthermore, if $\mathcal{F}$ covers a set S finely, then for every n,

$$S \subset (\cup_{k \le n} B_k) \cup (\cup_{k > n} \widehat{B}_k).$$

Proof. For $n = 1, 2, \dots$, we define subsets $\mathcal{F}_n \subset \mathcal{F}$, numbers d_n, and sets $B_n \in \mathcal{F}$ inductively as follows.

We let

$$\mathcal{F}_n = \{B \in \mathcal{F} : B \text{ is disjoint from } B_j \text{ for every } j < n\}.$$

(Thus $\mathcal{F}_1 = \mathcal{F}$.) If $\mathcal{F}_n$ is empty, we stop. (In that case our sequence is the finite sequence $B_1, \dots, B_{n-1}$.) If $\mathcal{F}_n$ is not empty, we let

$$d_n = \sup\{\mathrm{diam}(B) : B \in \mathcal{F}_n\}.$$

Note that $d_n < \infty$ since the diameters of $B \in \mathcal{F}$ are bounded away from ∞. We choose $B_n \in \mathcal{F}_n$ with

$$\operatorname{diam}(B_n) > \frac{1}{2} d_n. \tag{1}$$

Note that $\mathcal{F}_n \subset \mathcal{F}_k$ for $k < n$, so the d_n's form a decreasing sequence:

$$d_1 \geq d_2 \geq \dots.$$

Clearly the B_k's are disjoint. Thus by the finiteness condition and by (1), either the sequence is finite or else $d_n \to 0$.

Let $B \in \mathcal{F}$. We claim that there is an n such that $B \cap B_n \neq \emptyset$. If $\mathcal{F}_m$ is empty for some m, that is trivially true. Thus we may assume that $\mathcal{F}_m$ is nonempty for each m. Thus the sequence of B_i's is an infinite sequence. Note that

$$\operatorname{diam}(B) > d_n$$

for some n since $\operatorname{diam}(B) > 0$ and since $d_n \to 0$. Thus $B \notin \mathcal{F}_n$ by definition of d_n, so B intersects B_k for some $k < n$.

Now let k be the smallest number such that

$$B \cap B_k \neq \emptyset. \tag{2}$$

Now B is disjoint from $B_1, \dots, B_{k-1}$, so $B \in \mathcal{F}_k$, and thus

$$\operatorname{diam}(B) \leq d_k.$$

Also, $d_k < 2 \operatorname{diam}(B_k)$ (by choice of B_k), so

$$\operatorname{diam}(B) < 2 \operatorname{diam}(B_k). \tag{3}$$

From (2) and (3), it follows that $B \subset \widehat{B_k}$.

This completes the proof except for the assertion about fineness. Thus suppose that $\mathcal{F}$ covers S finely. Fix an n. Let $x \in S$. We must show that

$$x \in (\cup_{j \leq n} B_j) \cup (\cup_{j > n} \widehat{B}_j).$$

We may suppose that

$$x \notin \cup_{j \leq n} B_j \tag{4}$$

since otherwise we are done. Because the B_i's are closed, $\cup_{j \leq n} B_j$ is closed. Thus by (4), there is an $r > 0$ so that the ball $B(x, r)$ is disjoint from

$$\cup_{j \leq n} B_j.$$

Since $\mathcal{F}$ covers S finely, there is a $B \in \mathcal{F}$ with $x \in B$ and $\operatorname{diam}(B) < r$. Hence $B \subset B(x, r)$, so

$$B \cap B_j = \emptyset \text{ for all } j \leq n. \tag{5}$$

By the first part of the theorem, there is a k such that

$$B \cap B_k \neq \emptyset \tag{6}$$

and

$$B \subset \widehat{B}_k. \tag{7}$$

By (5) and (6), $k > n$. Since $x \in B$,

$$x \in \widehat{B}_k$$

by (7). □