

MATH 205A HOMEWORK 1 SOLUTIONS

1. Let us use the finite decimal expansion when there is one. (Thus we write 1 instead of .999..., etc.) Note the set of numbers with a 7 in the n th decimal place is a finite union of intervals, namely:

$$S_n = \bigcup_{k=0}^{10^{n-1}-1} \left[\frac{k}{10^{n-1}} + \frac{7}{10^n}, \frac{k}{10^{n-1}} + \frac{8}{10^n} \right).$$

In particular each S_n is a Borel set. Thus $S = \cup_n S_n$ is a countable union of Borel sets and is therefore a Borel set.

2. For $x \in [0, 1]$, let x_n denote the n decimal digit of x . Then

$$\begin{aligned} S &= \{x \in [0, 1] : \text{for every } k, \text{ there is an } n \geq k \text{ such that } x_n = 7\} \\ &= \bigcap_{k=1}^{\infty} \{x \in [0, 1] : \text{there is an } n \geq k \text{ such that } x_n = 7\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{x \in [0, 1] : x_n = 7\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n \end{aligned}$$

(where S_n is as in problem 1.) Thus S is a Borel set. (It is a countable intersection of countable unions of finite unions of intervals.)

3. Let \mathcal{F} be the family of all subsets S of \mathbf{R} such that $f^{-1}(S)$ is a Borel set. We will show that \mathcal{F} is a σ -algebra:

(i) $f^{-1}(\emptyset) = \emptyset \in \mathcal{B}(\mathbf{R})$, so $\emptyset \in \mathcal{F}$.

(ii) Note for $S \in \mathbf{R}$, $f^{-1}(S^c) = (f^{-1}(S))^c$. If $S \in \mathcal{F}$, then $f^{-1}(S) \in \mathcal{B}(\mathbf{R})$, so $f^{-1}(S^c) = (f^{-1}(S))^c \in \mathcal{B}(\mathbf{R})$, so $S^c \in \mathcal{F}$.

(iii) If $S_n \in \mathcal{F}$ for $n = 1, 2, 3, \dots$, then $f^{-1}(S_n) \in \mathcal{B}(\mathbf{R})$, so

$$f^{-1}(\cup_n S_n) = \cup_n f^{-1}(S_n) \in \mathcal{B}(\mathbf{R}),$$

and thus $\cup_n S_n \in \mathcal{F}$.

By (i), (ii), and (iii), \mathcal{F} is a σ -algebra. By hypothesis, \mathcal{F} contains all sets of the form $[a, \infty)$ (where $a \in \mathbf{R}$). Thus it contains the σ -algebra generated by such sets, i.e., the σ -algebra of Borel sets.

Remark. Be sure you understand (and can prove) that $f^{-1}(S^c) = (f^{-1}S)^c$ and that $f^{-1}(\cup_n S_n) = \cup_n f^{-1}(S_n)$. Note also that is **not** true in general that $f(S^c) = f(S)^c$. Similarly, $f^{-1}(\cap_n S_n) = \cap_n f^{-1}(S_n)$, but it is not true (in general) that $f(\cap_n S_n) = \cap_n f(S_n)$.

4. By definition, there exist closed sets $C_i \subset \mathbf{R}^n$ (where $i = 1, 2, \dots$) such that

$$S = \cup_i C_i$$

For $j = 0, 1, 2, \dots$, let $K_i^j = \{x \in C_i : |x| \leq j\}$. Then $C_i = \cup_j K_i^j$. Thus

$$S = \cup_{i,j} K_i^j$$

so

$$(*) \quad f(S) = f(\cup_{i,j} K_i^j) = \cup_{i,j} f(K_i^j).$$

Now K_i^j is closed and bounded and therefore compact. Thus $f(K_i^j)$ is the continuous image of a compact set and is therefore compact. In particular, it is closed. Thus by (*), $f(S)$ is a countable union of closed sets.

5*.

Proof. If $f^*(x) < a$ then there is a neighborhood of x such that $f(y) < a$ for every y in that neighborhood. But then $f^*(y) < a$ for every y in that neighborhood. This proves that the set

$$\{x : f^*(x) < a\}$$

is an open set. Equivalently,

$$\{x : f^*(x) \geq a\}$$

is a closed set. Similarly,

$$\{x : f_*(x) \leq a\}$$

is a closed set.

Now f is discontinuous at x if and only if $f_*(x) < f^*(x)$, which happens if and only if there are rational numbers p and q with $f_*(x) \leq p < q \leq f^*(x)$. Thus

$$\begin{aligned} S &= \{x : f_*(x) \leq p < q \leq f^*(x) \text{ for some rational } p \text{ and } q\} \\ &= \bigcup_{p,q \in \mathbf{Q}, p < q} \{x : f_*(x) \leq p \text{ and } q \leq f^*(x)\} \\ &= \bigcup_{p,q \in \mathbf{Q}, p < q} (\{x : f_*(x) \leq p\} \cap \{x : q \leq f^*(x)\}). \end{aligned}$$

Thus S is a countable union of closed sets, so it is a Borel set. (Indeed it's an F_σ set.) \square

6. Let $S = \{s_1, s_2, s_3, \dots\}$ be any countable dense set (such as the rationals). Now S is an F_σ set since it is a countable union of points.

We will prove that S is not a G_δ set. Suppose that $S \subset \cap_n U_n$ where the U_n are open sets. It suffices to prove that $S \neq \cap_n U_n$, i.e., that $\cap_n U_n$ contains a point that is not in S .

Now U_n is dense since it contains S . Note that the set $U_n \setminus \{s_n\} = U_n \cap \{s_n\}^c$ is also open and dense. Thus by the Baire category theorem,

$$\cap_n (U_n \setminus \{s_n\})$$

is dense. But

$$\bigcap_n (U_n \setminus \{s_n\}) = \bigcap_n (U_n \cap \{s_n\}^c) = (\bigcap_n U_n) \setminus S.$$

Since this set is dense in \mathbf{R} , it is certainly nonempty. \square

Remark. Suppose more generally that S is a subset of \mathbf{R} with the following property: there is some nonempty interval I such that $S \cap I$ is a countable dense subset of I . Then essentially the same argument shows that S is not a G_δ set.

7. Note that if $t > 0$, then

$$t \cdot \mathcal{H}^0\{x : f(x) \geq t\} \leq \sum_{x \in X} f(x).$$

where $\mathcal{H}^0(S)$ denotes the number of elements of S .

In particular, if the sum is finite, then $\{x : f(x) \geq t\}$ is finite. Thus

$$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) \geq 1/n\}$$

is a countable union of finite sets and is therefore countable.