

MATH 205A HOMEWORK 2 (FALL 2018)

0. [Not to turn in]. Let (X, \mathcal{A}, μ) be a measure space. For $A \subset X$, define

$$\mu^*(A) = \inf\{\mu(B) : A \subset B \text{ and } B \in \mathcal{A}\}.$$

(a) Prove that the infimum is attained. (b) Prove that μ^* is an outer measure. (c) Prove that if $A \in \mathcal{A}$, then A is μ^* -measurable and $\mu^*(A) = \mu(A)$.

(Thus every measure can be gotten from an outer measure.)

1. Let X be any set. Let \mathcal{C} be a collection of subsets of X such that $\emptyset \in \mathcal{C}$. Let $\phi : \mathcal{C} \rightarrow [0, \infty]$ be any function such that $\phi(\emptyset) = 0$. Define

$$\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$$

by

$$\mu^*(S) = \inf \left\{ \sum_{i=1}^{\infty} \phi(T_i) : S \subset \cup_{i=1}^{\infty} T_i, \quad T_i \in \mathcal{C} \right\}$$

Prove that μ^* is an outer measure.

2. Suppose that X is an uncountable set. Define outer measures μ^* and ν^* on X by:

$$\mu^*(S) = \begin{cases} 0 & \text{if } S \text{ is countable} \\ 1 & \text{if } S \text{ is uncountable} \end{cases}, \quad \nu^*(S) = \begin{cases} 0 & \text{if } S \text{ is countable} \\ +\infty & \text{if } S \text{ is uncountable.} \end{cases}$$

(a) Find all sets that are μ^* -measurable. (b) Find all sets that are ν^* -measurable. (Prove that your answers are correct.)

3. Let $A \subset \mathbf{R}$ be a Lebesgue measurable set with positive Lebesgue measure. Prove that for every $\epsilon > 0$, there is a nonempty open interval I such that

$$\lambda(A \cap I) > (1 - \epsilon)\lambda(I).$$

4. Let μ^* be an outer measure on a set X . Suppose $A \subset X$ is not μ^* -measurable and that $\mu^*(A) < \infty$. Prove that A has a subset S such that

$$\mu^*(S) > 0$$

and such that S has no μ^* -measurable subsets with positive measure.

5. Let $A \subset \mathbf{R}$ be a Lebesgue measurable set. Define $1_A : \mathbf{R} \rightarrow \mathbf{R}$ by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \text{ and} \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that for every $\epsilon > 0$, there is a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\lambda\{x : f(x) \neq 1_A(x)\} < \epsilon.$$

Here λ is Lebesgue measure.

6. On the first day of class, we proved (using the axiom of choice) that there is a set $A \subset [0, 1]$ with the following property: for every $x \in \mathbf{R}$, there is exactly one $y \in A$ such that $x - y$ is rational. Prove that $\lambda^*(A) > 0$, but that if $S \subset A$ is Lebesgue measurable, then $\lambda(S) = 0$.