MATH 205A HOMEWORK 2 SOLUTIONS (FALL 2018)

0. Let $A \subset X$. By definition of μ^*, there exist $B_n \in \mathcal{A}$ with $A \subset B_n$ such that $\mu(B_n) \to \mu^*(A)$. Let $B = \cap_n B_n$. Then $A \subset B$ and $B \in \mathcal{A}$. Thus

$$\mu^*(A) \le \mu(B). \tag{*}$$

Also, for every n we have $B \subset B_n$, so $\mu(B) \le \mu(B_n)$. Letting $n \to \infty$, we have $\mu(B) \le \mu^*(A)$. Hence by (*), $\mu(B) = \mu^*(A)$. This proves (a).

Proof of (b): Clearly $\mu^*(\emptyset) = 0$. Now suppose $A \subset \cup_n A_n$. By part (a), for each n there exists $B_n \in \mathcal{A}$ with $A_n \subset B_n$ and $\mu^*(A_n) = \mu(B_n)$. Thus

$$\begin{aligned}
\mu^*(A) &\le \mu(\cup B_n) \qquad \text{(def of } \mu^*) \\
&\le \sum_n \mu(B_n) \\
&= \sum_n \mu^*(A_n) \qquad \text{(by choice of } B_n)
\end{aligned}$$

This proves that μ^* is an outer measure.

Proof of (c) Suppose $A \in \mathcal{A}$. Then for $B \in \mathcal{A}$ with $A \subset B$, $\mu(A) \le \mu(B)$. Thus the infimum in the definition of $\mu^*(A)$ is attained by A itself, so $\mu^*(A) = \mu(A)$.

Let S be any set in X. Let T be a set such that $S \subset T \in \mathcal{A}$ and $\mu^*(S) = \mu(T)$. (This T exists by part (a).) Then

$$\begin{aligned}
\mu^*(S) &= \mu(T) \\
&= \mu(T \cap A) + \mu(T \cap A^c) \\
&\ge \mu^*(S \cap A) + \mu^*(S \cap A^c)
\end{aligned}$$

(The last inequality holds because μ of any measurable set is $\ge \mu^*$ of any subset.) Since this is true for all S, we see that A is μ^*-measurable. □

1. This is exactly like the proof that Lebesgue outer measure is an outer measure. That $\mu^*(\emptyset) = 0$ is trivial:

$$\emptyset \subset \cup_{n=1}^{\infty} \emptyset$$

so

$$\mu^*(\emptyset) \le \sum_{n=1}^{\infty} \phi(\emptyset) = 0.$$

Suppose $A \subset \cup_n A_n$. We must show that

$$\mu^*(A) \le \sum_n \mu^*(A_n). \tag{*}$$

We may assume each $\mu^*(A_n) < \infty$ since otherwise (*) is trivially true. Let $\epsilon > 0$. By definition of inf, we can cover A_n by sets $B_{ni} \in \mathcal{C}$:

$$A_n \subset \cup_i B_{ni}$$

such that

$$\sum_i \phi(B_{ni}) < \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then $A \subset \cup_{n,i} B_{ni}$, so

$$\begin{aligned}\mu^*(A) &\le \sum_{n,i} \phi(B_{ni}) \\ &= \sum_n \sum_i \phi(B_{ni}) \\ &\le \sum_n (\mu^*(A_n) + \tfrac{\epsilon}{2^n}) \\ &= \left(\sum_n \mu^*(A_n)\right) + \epsilon.\end{aligned}$$

This is true for each $\epsilon > 0$, so we have (*). $\square$

2. In general, any set of outer measure 0 is measurable (as proved in class). Thus the complement of any set of outer measure 0 is also measurable.

In part (a), this means that if S or $X \setminus S$ is countable, then S is μ^*-measurable. Suppose neither S nor $X \setminus S$ is countable. Then $\mu^*(S) = \mu^*(X \setminus S) = 1$. Thus

$$\mu^*(X) = 1 < 2 = \mu^*(S) + \mu^*(X \setminus S)$$

so S is not μ^*-measurable.

Thus we have shown: S is μ^*-measurable if and only if either S or $X \setminus S$ is countable.

For part (b), let S and A be any sets in X. If A is countable, then so are $A \cap S$ and $A \cap S^c$, so $\nu^*(A) = \nu^*(A \cap S) = \nu^*(A \cap S^c) = 0$, so

$$(*) \qquad \nu^*(A) = \nu^*(A \cap S) + \nu^*(A \cap S^c).$$

If on the other hand A is uncountable, then either $A \cap S$ or $A \cap S^c$ (or both) must be uncountable. Thus in this case $\nu^*(A) = \infty$ and at least one of $\nu^*(A \cap S)$ and $\nu^*(A \cap S^c)$ is ∞, so again we have (*). Thus S is ν^*-measurable (for every $S \subset X$). $\square$

3. Suppose first that $\lambda(A) < \infty$. Let s be the supremum of

$$\frac{\lambda(A \cap I)}{\lambda(I)}$$

over all nonempty open intervals I. We must show that $s = 1$.

Consider any covering $A \subset \cup_n I_n$ of A by nonempty open intervals. Then

$$\lambda(A) \le \sum_n \lambda(A \cap I_n) = \sum_n \frac{\lambda(A \cap I_n)}{\lambda(I_n)} \lambda(I_n) \le \sum s\, \lambda(I_n),$$

so

$$\lambda(A) \le s \sum_n |I_n|.$$

Taking the infimum over all such coverings gives

$$\lambda(A) \le s\lambda(A).$$

Since $0 < \lambda(A) < \infty$, this means $s \ge 1$.

If $\lambda(A) = \infty$, then let $A_n = A \cap [-n, n]$. Then $A_1 \subset A_2 \subset \dots$ and $A = \cup A_n$, so $\lambda(A) = \lim \lambda(A_n)$, so $\lambda(A_n) > 0$ for sufficiently large n. Also, $\lambda(A_n) < 2n < \infty$. Thus (as we have already proved) the assertion of the problem is true for A_n. But then it is clearly true for any set (e.g. A) that contains A_n. □

4.

Lemma 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\mathcal{F}$ be a nonempty collection of sets in $\mathcal{A}$ such that $\mathcal{F}$ is closed under countable unions. Then there is set $B \in \mathcal{F}$ such that $\mu(B) = \max_{C \in \mathcal{F}} \mu(C)$.*

Proof. Let α be the supremum of $\mu(C)$ among all $C \in \mathcal{F}$. By definition of sup, there exist sets $B_n \in \mathcal{F}$ such that

$$\lim_{n \to \infty} \mu(B_n) = \alpha.$$

Let $B = \cup_n B_n$. Then $B \in \mathcal{F}$ since $\mathcal{F}$ is closed under countable unions. Since $B_n \subset B$, $\mu(B_n) \le \mu(B)$. Letting $n \to \infty$ gives $\alpha \le \mu(B)$. In other words, $\mu(T) \le \mu(S)$ for all $T \in \mathcal{F}$. □

To prove the assertion of problem 4, we apply the lemma to the measure space $(X, \mathcal{M}_{\mu^*}, \mu)$. We let $\mathcal{F}$ be the colletion of $B \in \mathcal{M}_{\mu^*}$ such that $B \subset A$. Then $\mathcal{F}$ is nonempty (since $\emptyset \in \mathcal{F}$) and is closed under countable unions. By the lemma, there is a set $B \in \mathcal{F}$ such that

$$\mu(B) \ge \mu(Z)$$

for all $Z \in \mathcal{F}$. Since $B \subset A$,

$$\mu(B) = \mu^*(B) \le \mu^*(A) < \infty.$$

Let $S = A \setminus B$.

Since $A = B \cup S$ is not μ^*-measurable and since B is μ^*-measurable, the set S cannot be μ^*-measurable.

Suppose T is a μ^*-measurable subset of S. Then $B \cup T \in \mathcal{F}$ so (by choice of B)

$$(*) \qquad \mu(B) \ge \mu(B \cup T) = \mu(B) + \mu(T)$$

since B and T are disjoint. Since $\mu(B)$ is finite, (*) implies that $\mu(T) = 0$. □

5. Recall that we proved in class that A contains a closed set C with $\lambda(A\setminus C) < \epsilon/2$. We also proved that A is contained in an open set W with $\lambda(U \setminus A) < \epsilon/2$. Thus

$$\lambda(W \setminus C) = \lambda(W \setminus A) + \lambda(A \setminus C) < \epsilon.$$

Since $W \setminus C$ is open, it is a countable union of disjoint open intervals I_n, the endpoints of which are in $C \cup W^c$. Now define f by

$$f(x) = \begin{cases} 0 & \text{if } x \in W^c, \\ 1 & \text{if } x \in C, \end{cases}$$

and extend f by linear interpolation to each of the intervals $\overline{I_n}$. Then f is continuous and

$$\{x : f(x) \neq 1_A(x)\} \subset W \setminus C$$

so $\lambda\{x : f(x) \neq 1_A(x)\} \leq \lambda(W \setminus C) < \epsilon$. □

Another way to define f that is more elegant and that works in $\mathbf{R}^n$: Note that for any nonempty closed set K, the function

$$\text{dist}(x, K) = \min\{|x - y| : y \in K\}$$

is continuous and $\text{dist}(x, K) = 0$ if and only if $x \in K$.

We may assume $W \neq \mathbf{R}$ and $C \neq \emptyset$ (since otherwise we can just let $f \equiv 1$ or $f \equiv 0$.) Now define

$$f(x) = \frac{\text{dist}(x, W^c)}{\text{dist}(x, W^c) + \text{dist}(x, C)}.$$

Since W^c and C are disjoint, the denominator is never 0. Thus f is continuous. Also, note that $f(x) = 0$ if $x \in W^c$ and $f(x) = 1$ if $x \in C$.

6. Recall that if p and q are rational numbers with $p \neq q$, then $A + p$ and $A + q$ are disjoint. [Proof: if x belonged to both, then $x - p$ and $x - q$ would both be in A, which is impossible since they differ by a rational number, namely $p - q$.] Note that any subset of A has the same property.

Now suppose S is a Lebesgue measurable subset of A. By translational invariance of Lebesgue measure, $S + q$ is also Lebesgue measurable (for any q) and has the same measure as S. Note also that

$$[0, 2] \supset \cup_{n=1}^{\infty} (S + (1/n)).$$

Thus

$$2 = \lambda[0, 1] \geq \lambda(\cup_n (S + \tfrac{1}{n})) = \sum_n \lambda(S + \tfrac{1}{n}) = \sum_n \lambda(S)$$

so $\lambda(S)$ must be 0. □