MATH 205A HOMEWORK 3 (FALL 2018)

1. Let μ^* be an outer measure on X, and let $A_1, A_2, \dots$ be a sequence of sets in X such that $\sum_n \mu^*(A_n) < \infty$. Let $Z = \{x \in X : x \in A_i \text{ for infinitely many } i\}$. Prove that $\mu^*(Z) = 0$.

2. Let X be a metric space. A function $f : X \to [-\infty, \infty]$ is said to be **lower semicontinuous** at the point $x \in X$ provided

$$f(x) = \lim_{r \to 0^+} \inf\{f(y) : d(y, x) < r\}.$$

We that f is lower semicontinuous provided it is lower semicontinuous at every point in its domain. Prove that if f is lower-semicontinuous, then it is a Borel function.

3. (a) Let $\mathcal{F}$ be any family (not necessarily countable) of lower semicontinuous functions from metric space X to $[-\infty, \infty]$. Prove that the function

$$h(x) = \sup\{f(x) : f \in \mathcal{F}\}$$

is lower semicontinuous (and therefore a borel function.) (b) Give an example of an uncountable family $\mathcal{F}$ of borel functions $f : \mathbf{R} \to \mathbf{R}$ such that $\sup\{f(x) : f \in \mathcal{F}\}$ is not a borel function.

4. Suppose μ and ν are finite measures defined on the same measurable space $(X, \mathcal{A})$. Prove that there is a set $N \in \mathcal{A}$ with the following properties:

(1) $\mu(N) = 0$, and
(2) If S is a measurable subset of $X \setminus N$ with $\mu(S) = 0$, then $\nu(S) = 0$.

5.(a) Show that for every $\epsilon > 0$ there exists an open dense subset W of $\mathbf{R}$ with Lebesgue measure $\lambda(W) < \epsilon$. (b) Show that there exists two disjoint borel sets A and B of $\mathbf{R}$ such that $A \cap I$ and $B \cap I$ both have positive Lebesgue measure for every every nonempty open interval I. (c). Show that there is an infinite sequence $A_1, A_2, \dots$ of disjoint borel sets in $\mathbf{R}$ such that $\lambda(I \cap A_n) > 0$ for every nonempty open interval I.

6. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_1, f_2, \dots : X \to [0, \infty]$ be $\mathcal{A}$-measurable functions that converge pointwise to a limit f. Show that if $f_n \le f$ for each n, then $\int f \, d\mu = \lim_n \int f_n \, d\mu_n$.

7. (a). Let $(X, \mathcal{A}, \mu)$ be a measure space and $f_n : X \to \mathbf{R}$ $(n = 1, 2, \dots)$ be a sequence of $\mathcal{A}$-measurable functions such that $\int |f_n| \, d\mu \to 0$. Prove that there is a subsequence $f_{n(i)}$ such that $f_{n(i)} \to 0$ almost everywhere, i.e., such that

$$\mu\{x \in X : f_{n(i)}(x) \text{ does not converge to } f(x)\} = 0.$$

(b). Give an example of such f_n for which the original sequence does not converge anywhere. (c). Let E be the set of Lebesgue measurable functions from $[0, 1]$ to $[0, 1]$. Prove that there is no pseudometric d on E with the following property:

$f_n \in E$ converges to $f \in E$ almost everywhere if and only if $d(f_n, f) \to 0$.

[Note: a pseudometric is just like a metric (as in the definition of metric space), except that the distance between distinct points is allowed to be 0. Thus a pseudometric d on E is a function $d : E \times E \to [0, \infty)$ such that

(1) $d(f, g) = d(g, f)$,
(2) $d(f, h) \le d(f, g) + d(g, h)$

for all f, g, and h in E.]