MATH 205A HOMEWORK 3 SOLUTIONS (FALL 2018)

1. As in the solution to problem 2 of hw1,

$$Z = \bigcap_{n \in \mathbf{N}} \bigcup_{k \geq n} A_k.$$

In particular, for every n we have

$$Z \subset \bigcup_{k \geq n} A_k$$

so

$$\mu^*(Z) \leq \mu^*(\cup_{k \geq n} A_k) \leq \sum_{k \geq n} \mu^*(A_k). \tag{*}$$

Since $\sum_n \mu^*(A_n) < \infty$, we have that $\sum_{k \geq n} \mu^*(A_k) \to 0$ as $n \to \infty$. Thus letting $n \to \infty$ in (*) shows that $\mu^*(Z) = 0$. □

2. Suppose $f(x) > t$. Then there is an $r > 0$ such that

$$\inf\{f(y) : d(y,x) < r\} > t$$

so in particular,

$$d(y,x) < r \implies f(y) > t.$$

We have just shown that if $x \in f^{-1}(t, \infty]$, then all points in some ball around x also belong to $f^{-1}(t, \infty]$. In other words, $f^{-1}(t, \infty]$ is open, and therefore borel. Since this is true for all t, f is $\mathcal{B}$-measurable. □

Remark 1. In fact lower semicontinuity of f is equivalent to openness of $f^{-1}(t, \infty]$ for all t. We just proved one direction. To prove the other, suppose $f^{-1}(t, \infty]$ is open for all t. Let $x \in X$. Let t be any number $< f(x)$. Since $f^{-1}(t, \infty]$ is open and contains x, it contains a ball of some radius r around x. But that implies that

$$t \leq \inf\{f(y) : d(y,x) < r\} \leq f(x).$$

Note this is also true for any $\rho \leq r$. Thus letting $\rho \to 0^+$:

$$t \leq \lim_{\rho \to 0^+} \inf\{f(y) : d(y,x) < \rho\} \leq f(x). \tag{*}$$

Since this is true for all $t < f(x)$, it must also be true for $t = f(x)$. Thus (*) becomes

$$f(x) \leq \lim_{\rho \to 0^+} \inf\{f(y) : d(y,x) < \rho\} \leq f(x)$$

which immediately gives lower semicontinuity.

Remark 2. We defined f to be lower semicontinuous at x if

$$f(x) = \lim_{r \to 0} \inf\{f(y) : d(y,x) < r\}.$$

Lower semicontinuity is usually defined in a slightly different way: f is lower semicontinuous at x provided

$$f(x) \leq \liminf_{y \to x} f(y).$$

In fact, the two definitions are equivalent. To see that they are equivalent, note that

$$\liminf_{y \to x} f(y) = \lim_{r \to 0} \inf\{f(y) : 0 < d(y,x) < r\},$$

so

$$\lim_{r \to 0} \inf\{f(y) : d(y,x) < r\} = \min\left\{f(x), \liminf_{y \to x} f(y)\right\}.$$

Equivalence of the two definitions follows immediately.

3(a). Suppose $h(x) > t$. Then $f(x) > t$ for some $f \in \mathcal{F}$ (by def of sup). Since f is lower semicontinuous, this means that for some $r > 0$,

$$d(y,x) < r \implies f(y) > t.$$

Since $h \geq f$,

$$d(y,x) < r \implies h(y) > t.$$

Of course this is also true if we replace r by any smaller number ρ. Thus

$$h(x) \geq \inf\{h(y) : d(y,x) < \rho\} \geq t$$

for all $\rho \leq r$, and thus

$$h(x) \geq \lim_{\rho \to 0^+} \inf\{h(y) : d(y,x) < \rho\} \geq t.$$

Since this holds for all $t < h(x)$, it must also hold for $t = h(x)$:

$$(*) \qquad h(x) \geq \lim_{\rho \to 0^+} \inf\{h(y) : d(y,x) < \rho\} \geq h(x).$$

In fact we must have equality in (*) since the first and last terms are the same. Thus h is lower-semicontinuous. □

Remark 3. We give a shorter proof by using remark 1 above. Namely,

$$\begin{aligned} h^{-1}(t,\infty] &= \{x : f(x) > t \text{ for some } f \in \mathcal{F}\} \\ &= \cup_{f\in\mathcal{F}} f^{-1}(t,\infty]. \end{aligned}$$

Each $f^{-1}(t,\infty]$ is open since f is lowersemicontinuous. Thus $h^{-1}(t,\infty]$ is open, because every union of open sets is open.

3(b). Let A be a non-borel subset of $\mathbf{R}$. (For example, any set that is not Lebesgue measurable will do.) Then

$$1_A(x) = \sup\{1_{\{a\}}(x) : a \in A\}.$$

Each of the functions $1_{\{a\}}$ is a borel function, but 1_A is not. □

4. Let $\mathcal{N}$ be the collection of sets $A \in \mathcal{A}$ for which $\mu(A) = 0$. Then $\mathcal{N}$ is nonempty (it contains the emptyset) and it is closed under countable unions. Thus by the lemma in the solution to hw2, problem 4, the collection $\mathcal{N}$ contains a set N for which $\nu(N)$ is a maximum, i.e., such that

$$(*) \qquad \nu(N) \geq \nu(A) \text{ for all } A \in \mathcal{N}.$$

Now suppose S is a measurable subset of $X \setminus N$ with $\mu(S) = 0$. Then $\mu(N \cup S) = \mu(N) + \mu(S) = 0$, so $N \cup S \in \mathcal{N}$. Thus by (*),

$$\nu(N) \geq \nu(N \cup S) = \nu(N) + \nu(S)$$

which implies that $\nu(S) = 0$. □

5(a). Let $\{q_1, q_2, \dots\}$ be a countable dense subset of $\mathbf{R}$. (We can use the rationals, for instance.) Let I_n be the open interval of length $2^{-n}\epsilon$ centered at q_n. Let $W = \cup_n I_n$. Then W is a union of open sets, so it is open. It is dense since it contains a dense set. And

$$\lambda(W) = \lambda(\cup_n I_n) \leq \sum_n \lambda(I_n) = \sum_{n=1}^{\infty} 2^{-n}\epsilon = \epsilon.$$

□

5(b). Claim 1: If I is any nonempty open interval, then I contains a closed nowhere dense set K of positive measure. (A set is called nowhere dense if its closure contains no nonempty open set. Thus a closed nowhere dense of $\mathbf{R}$ is a closed set that contains no nonempty open interval.) To prove claim 1, let J be a closed interval in I of nonzero length. By (a), there is an open dense subset W of $\mathbf{R}$ such that $\lambda(W) < \lambda(J)$. Let $K = J \setminus W$. Then K is closed and nowhere dense, and

$$\lambda(K) \geq \lambda(J) - \lambda(W) > 0.$$

This proves the claim 1.

Claim 2: If I is any nonempty open interval, and n is any finite number, then I contains n disjoint closed nowhere dense subsets each of positive lebesgue measure. Proof of claim 2: Choose n disjoint nonempty open subintervals of I, and use claim 1 to make a closed nowhere dense set of positive measure in each of those subintervals. This proves claim 2.

Now let $I_1, I_2, I_3, \dots$ be an enumeration of the open intervals (of positive length) with rational endpoints in $\mathbf{R}$. Choose disjoint closed nowhere dense sets $F_1, F_1', F_2, F_2', \dots$ inductively as follows.

At stage n, note that $\cup_{i<n}(F_i \cup F_i')$ is closed and nowhere dense. Thus I_n contains an open subinterval I_n^* disjoint from that set. Now by claim 2, I_n^* contains two disjoint closed nowhere dense subsets F_n and F_n', each of positive measure.

Now let $A = \cup_n F_n$ and $B = \cup_n F_n'$. Then A and B are disjoint. Also, if I is any open interval, then I contains an open subinterval with rational endpoints. That is, $I_n \subset I$ for some n. But then

$$\lambda(A \cap I) \geq \lambda(A \cap I_n) \geq \lambda(F_n \cap I_n) = \lambda(F_n) > 0.$$

Likewise $\lambda(B \cap I) > 0$. This proves (b). □

5(c). As in (b), let $I_1, I_2, I_3, \dots$ be an enumeration of the nonempty open subintervals of $[0,1]$ with rational endpoints. Define disjoint closed nowhere dense sets A_k^n (with $1 \leq k \leq n < \infty$) as follows.

At the nth stage, note that $\cup_{j \leq m < n} A_j^m$ is closed and nowhere dense. Thus I_n contains an open subinterval I_n^* disjoint from that set. By claim 2, we can find disjoint closed nowhere dense subsets $A_1^n, A_2^n, \dots, A_n^n$ of I_n^*, each of positive lebesgue measure.

Now let $A_k = \cup_{n \geq k} A_k^n$. Then $A_1, A_2, \dots$ are disjoint borel (indeed F_σ) sets. Let I be a nonempty open interval and k be any natural number. Then I contains a nonempty open subinterval I' that is shorter than each of $I_1, \dots, I_k$. Now I' contains an interval I_n with rational endpoints. Note that $n > k$ since I_n is shorter than I_i for $i \leq k$. Thus

$$\begin{aligned} 0 &< \mu(I_n \cap A_k^n) \\ &\leq \mu(I \cap A_k) \end{aligned}$$

□

6. Since $f_n \leq f$, we have $\int f_n \leq \int f$ for each n, and thus

$$\limsup \int f_n \leq \int f.$$

On the other hand, $\int f \leq \liminf \int f_n$ by Fatou's Lemma. These two statements together imply that $\lim \int f_n$ exists and equals $\int f$.

7.(a). Since $\int |f_n| \to 0$, for each $k = 1, 2, 3, \dots$, there exists an $n(k)$ such that $n \geq n(k)$ implies $\int |f_n| < \frac{1}{2^k}$. Note we can choose the $n(k)$ so that $n(1) < n(2) < n(3) < \dots$. By Beppo Levi's Theorem,

$$\int \sum_k |f_{n(k)}| \, du = \sum_k \int |f_{n(k)}| \, d\mu \leq \sum_k \frac{1}{2^k} < \infty.$$

Thus for almost every x, $\sum_k |f_{n(k)}(x)| < \infty$. For every such x, $|f_{n(k)}(x)| \to 0$. □

7(b). Consider the following sequence of intervals: first the interval $[0,1]$, then the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, then the intervals $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$, and $[\frac{2}{3}, 1]$, and so on. (At the nth stage, we have the intervals $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \dots, [\frac{n-1}{n}, 1]$.) Let $f_1, f_2, \dots$ be the characteristic functions (i.e., indicator functions) of those intervals. Then $\int |f_n| = \int f_n \to 0$. But

$$\limsup_n f_n(x) = 1$$

for every $x \in [0,1]$. Thus $f_n(x)$ does not converge to 0 for any $x \in [0,1]$.

7(c). Suppose there were such a pseudometric d. Let $f_1, f_2, \dots$ be the functions from 7(b). Let

$$\delta = \limsup_n d(f_n, 0).$$

Since the f_n do not converge to 0 almost everywhere, $\delta > 0$.

Thus $d(f_n, 0) > \delta/2$ for infinitely many n's. Let us label those functions as $g_1, g_2, \dots$. (Thus g_n is the nth of the functions $f_1, f_2, \dots$ for which $d(f_i, 0) > \delta/2$.)

Clearly $\int |g_n| \to 0$. Thus by 7(a), g_n has a subsequence $g_{n(k)}$ that converges to 0 almost everywhere. But then $d(g_{n(k)}, 0) \to 0$, which contradicts the fact that $d(g_n, 0) > \delta/2$ for all n.