MATH 205A HOMEWORK 4 (FALL 2018)

0. (Not to turn in.) **(a)**. Make examples to show that in the "big" theorems about integration we've proved (monontone convergence, dominated convergence, fatou, egorov), the hypotheses really are necessary. For example, the mononotone convergence theorem and Fatou's theorem are false if we don't assume that the functions are nonnegative, the dominated convergence theorem is false if we don't assume the functions are dominated, and egorov's theorem is false if we don't assume the measure space is finite.) Also, make up examples to show that we may get strict inequality in Fatou's theorem (even if the f_n converge to f everywhere. **(b)**. Make examples of functions $f_n : [0,1] \to \mathbf{R}$ that converge to 0 in $\mathcal{L}^p$ but not almost everywhere. **(c)**. Make examples that converge to 0 almost everywhere but not in $\mathcal{L}^p$.

1. Suppose $(X, \mathcal{A}, \mu)$ is a finite measure space and $f : X \to \mathbf{R}$ is a nonnegative μ-measurable function. Prove that $\int f \, du$ is finite if and only

$$\sum_{n=1}^{\infty} \mu\{x : f(x) \geq n\} < \infty.$$

2. Let $(X, \mathcal{A}, \mu)$ be a measure space where X is a metric space and where $\mathcal{A}$ includes the Borel sets. Suppose also that balls (of finite radius) have finite μ-measure. (a) Prove that for each fixed $r > 0$, $\mu(\mathbf{B}(x,r))$ is lower semicontinuous (as a function of x.) Here $\mathbf{B}(x,r)$ is the **open** ball of radius r about x:

$$\mathbf{B}(x,r) = \{y : d(y,x) < r\}.$$

(b) Let

$$\overline{\mathbf{B}}(x,r) = \{y : d(y,x) \leq r\}.$$

Prove that $\mu(\overline{\mathbf{B}}(x,r))$ is upper semicontinuous as a function of x.

WARNING: If we let $\overline{\mathbf{B}(x,r)}$ be the closure of $\mathbf{B}(x,r)$, then (in a general metric space), $\overline{\mathbf{B}}(x,r)$ and $\overline{\mathbf{B}(x,r)}$ can differ, and $\mu(\overline{\mathbf{B}(x,r)})$ need not be an upper semicontinuous function of x.

3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is an increasing function (i.e., a function such that $s < t$ implies $f(s) \leq f(t)$.) Suppose also that f is differentiable λ almost everywhere. Prove that

$$\int_0^1 f'(t) \, dt \leq f(1) - f(0).$$

(Hint: you may find it convenient to assume that $f(x) = f(1)$ for all $x \geq 1$. This assumption is OK since changing f on the complement of $[0,1]$ does not change either side of the inequality to be proved.)

4. Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and that f_n, f are nonnegative $\mathcal{A}$ measurable functions from X to $\mathbf{R}$ such that $f_n(x) \to f(x)$ for all x. Suppose

$$\int_X f_n \, d\mu \to \int_X f \, d\mu < \infty.$$

Prove that $\int_S f_n \, d\mu \to \int_S f \, d\mu$ for every $S \in \mathcal{A}$.

5. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be any continuous function. Show that the function

$$g(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is a borel function. (b) Let D be the set of points at which f is differentiable. Prove that D is a borel set.

6. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $f : X \to [0, \infty)$ is a $\mathcal{A}$-measurable function with $\int f \, d\mu < \infty$. Prove that for every $\epsilon > 0$, there is a $\delta > 0$ such that for $E \in \mathcal{A}$, $\mu(E) < \delta$ implies $\int_E f \, d\mu < \epsilon$. (Note: $\int_E f \, d\mu$ is defined to be $\int f \, 1_E \, d\mu$.)

7. Let X be a metric space, and let $\mu : \mathcal{P}(X) \to [0, \infty)$ be a function such that $A \subset B$ implies $\mu(A) \le \mu(B)$. Suppose that $0 \le m \le \infty$ and that $0 < \delta \le \infty$. Define $f_\delta : X \to [0, \infty]$ by

$$f_\delta(a) = \sup_{0 < r < \delta} r^{-m} \mu(\mathbf{B}(a, r)).$$

Here $\mathbf{B}(a, r) := \{x : d(x, a) < r\}$ is the open ball of radius r about a.

(a) Show that f_δ is a Borel function. (b) Show that the function $f := \lim_{\delta \to 0} f_\delta$ is also a Borel function.

8. [Not to turn in] Let X be a metric space. Define families of subsets $\Sigma_n = \Sigma_n(X)$ and $\Pi_n = \Pi_n(X)$ of X inductively as follows:

(1) Σ_0 consists of all open subsets of X.
(2) For $n \ge 0$, $\Pi_n = \{S^c : S \in \Sigma_n\}$.
(3) For $n > 0$, Σ_n consists of all countable unions of sets in $\cup_{i<n} \Pi_i$.

Note that each Σ_n is clearly closed under countable unions, and therefore that each Π_n is closed under countable intersections. (a). Show that for $\alpha < \beta$, $\Sigma_\alpha \cup \Pi_\alpha \subset \Sigma_\beta \cap \Pi_\beta$. [It might be helpful to think of this as four statements: $\Sigma_\alpha \subset \Sigma_\beta$, etc.](b). Show that Σ_n is closed under finite intersections (and therefore that Π_n is closed under finite unions.) (c). Show that Σ_{n+1} consists of all countable unions of sets in Π_n.

Remarks (i) If $\mathcal{S}$ is a family of sets, let $\mathcal{S}^\sigma$ be the family of sets that are countable unions of sets in $\mathcal{S}$, and let $\mathcal{S}^\delta$ be the family of sets that are countable intersections of sets in $\mathcal{S}$. Let G be the family of open subsets of X and F be the family of closed subsets of X. It follows from (c) that

$$\Sigma_0, \Pi_0, \Sigma_1, \Pi_1, \Sigma_2, \Pi_2, \ldots$$

are the same as

$$G, F, F^{\sigma}, G^{\delta}, G^{\delta\sigma}, F^{\sigma\delta}, \ldots.$$

(ii) For those of you familiar with ordinals, the definitions in problem 7 make sense for all ordinals n. The assertions are also true for all ordinals.