MATH 205A HOMEWORK 4 SOLUTIONS (FALL 2018)

0. Here are some examples to keep in mind. Let $f_n = 1_{[n,\infty)}$. Then $f_n \geq 0$, and $f_n(x) \to 0$ for all x. But $\int 0 = 0 < \infty = \lim_n \int f_n$. Thus strict inequality is possible in Fatou's theorem. Similarly, consider

$$\begin{aligned} g_n &: [0,1] \to \mathbf{R} \\ g_n &= n 1_{(0,1/n]}. \end{aligned}$$

Then $g_n(x) \to 0$ for all x and $\int g_n = 1$ for all n, so again the integral of the limit is strictly less than the limit of the integrals. **Egorov**: the functions f_n converge to 0 everywhere, but there is no set Z of finite measure such that that f_n converge uniformly on $\mathbf{R} \setminus Z$. **(b)**. Consider the characteristic functions of the following intervals:

$$[0, \tfrac{1}{2}],\ [\tfrac{1}{2}, 1], [0, \tfrac{1}{3}],\ [\tfrac{1}{3}, \tfrac{2}{3}],\ [\tfrac{2}{3}, 1], \dots, [0, \tfrac{1}{n}],\ [\tfrac{1}{n}, \tfrac{2}{n}], \dots, [\tfrac{n-1}{n}, 1], \dots$$

Those functions converge to 0 in L^p (for any $p \in [1,\infty)$, but there they do not converge at any point in $[0,1]$. (At every point, the limsup is 1 and the liminf is 0.) **(c)**. The functions g_n above converge to 0 almost everywhere (with respect to Lebesgue measure), but they do no converge to 0 in L^p: $\|g_n\|_p = n^p \cdot (1/n) = n^{p-1} \geq 1$. **(c)**.

1. Note that

$$\begin{aligned} \sum_{n=1}^{\infty} \mu\{x : f(x) \geq n\} &= \sum_{n=1}^{\infty} \int 1_{\{x: f(x) \geq n\}}\, d\mu x \\ &= \int \sum_{n=1}^{\infty} 1_{\{x: f(x) \geq n\}}\, d\mu x \\ &= \int \#\{n : 1 \leq n \leq f(x)\}\, d\mu x \\ &= \int [f]\, d\mu \end{aligned}$$

where for each x, $[f(x)]$ is the greatest integer less than or equal to $f(x)$. (We can interchange the sum and the integral by Beppo Levi's theorem.)

Now $[f] \leq f < [f] + 1$, so

$$\int [f] \leq \int f \leq \int ([f] + 1) = \int [f] + \int 1 = \int [f] + \mu(X).$$

Since $\mu(X) < \infty$, we see that $\int f < \infty$ if and only $\int [f]$ is finite. But we already showed that $\int [f]$ is equal to the sum in (*).

2. (a). Note $\mathbf{B}(x, r - \frac{1}{n})$ is an increasing sequence of nested measurable sets with union $\mathbf{B}(x,r)$. (This is where openness of the balls is used.) Thus

$$(*) \qquad \mu\mathbf{B}(x,r) = \lim_{n\to\infty} \mu\mathbf{B}(x, r - \tfrac{1}{n}).$$

Note that if $|y - x| < \frac{1}{n}$, then

$$\mathbf{B}(x, r - \tfrac{1}{n}) \subset \mathbf{B}(y, r)$$

so

$$\mu\mathbf{B}(x, r - \tfrac{1}{n}) \leq \mu\mathbf{B}(y, r).$$

Thus

$$\mu\mathbf{B}(x, r - \tfrac{1}{n}) \leq \liminf_{y\to x} \mu\mathbf{B}(y, r).$$

Thus letting $n \to \infty$ (see (*)) gives

$$\mu \mathbf{B}(x,r) \le \liminf_{y \to x} \mu \mathbf{B}(y,r).$$

2(b). Note that $\overline{\mathbf{B}}(x, r+\frac{1}{n})$ is a nested decreasing sequence of measurable sets whose intersection is $\overline{\mathbf{B}}(x,r)$. Also, by assumption these sets have finite measure. Thus

$$\mu\overline{\mathbf{B}}(x,r) = \lim_{n\to\infty} \mu\overline{\mathbf{B}}(x, r+\tfrac{1}{n}). \tag{$\dagger$}$$

If $|y-x| < \frac{1}{n}$, then

$$\overline{\mathbf{B}}(x, r+\tfrac{1}{n}) \supset \overline{\mathbf{B}}(y,r)$$

so

$$\mu\overline{\mathbf{B}}(x, r+\tfrac{1}{n}) \ge \mu\overline{\mathbf{B}}(y,r).$$

Letting $y \to x$ gives

$$\mu\overline{\mathbf{B}}(x, r+\tfrac{1}{n}) \ge \limsup_{y\to x} \mu\overline{\mathbf{B}}(y,r).$$

Now letting $n \to \infty$ (see (†)) gives

$$\mu\overline{\mathbf{B}}(x,r) \ge \limsup_{y\to x} \mu\overline{\mathbf{B}}(y,r).$$

3. Note that changing $f(x)$ for $x > 1$ does not change $f'(t)$ on $[0,1]$ except perhaps at $t = 1$. Thus it does not affect the integral in (*). Also it does not affect the right hand side of (*). Thus we may assume that $f(x) = f(1)$ for $x > 1$.

By definition of derivative and by Fatou's theorem,

$$\begin{aligned}
\int_0^1 f' &= \int_0^1 \lim_{n\to\infty} \frac{f(t+\frac{1}{n}) - f(t)}{\frac{1}{n}}\, dt \\
&\le \liminf_{n\to\infty} \int_0^1 \frac{f(t+\frac{1}{n}) - f(t)}{\frac{1}{n}}\, dt \\
&= \liminf_{n\to\infty} n \left(\int_0^1 f(t+\tfrac{1}{n})\, dt - \int_0^1 f(t)\, dt \right) \\
&= \liminf_{n\to\infty} n \left(\int_{1/n}^{1+1/n} f - \int_0^1 f \right) \qquad (\dagger) \\
&= \liminf_{n\to\infty} n \left(\int_1^{1+1/n} f - \int_0^{1/n} f \right) \\
&\le \liminf_{n\to\infty} n \left(\int_1^{1+1/n} f(1) - \int_0^{1/n} f(0) \right) \\
&= \liminf (f(1) - f(0)) \\
&= f(1) - f(0).
\end{aligned}$$

Note: In step (†), we used translation invariance of the Lebesgue integral. For simple functions, such translation invariance follows immediately from the translation invariance of lebesgue measure. Since the integral of an arbitrary function is defined in terms of integrals of simple functions, translation invariance for integrals of arbitrary (measurable) functions follows.

4. Solution 1: Note that $1_S f_n(x) \to 1_S f(x)$, so by Fatou

$$\int 1_S f \le \liminf_n \int 1_S f_n. \tag{*}$$

Likewise

$$\int 1_{S^c} f \le \liminf_n \int 1_{S^c} f_n.$$

Adding these gives

$$\begin{align} \int f &\le \liminf_n \int 1_S f_n + \liminf \int 1_{S^c} f_n \tag{1} \\ &\le \liminf_n \left(\int 1_S f_n + \int 1_{S^c} f_n \right) \tag{2} \\ &= \liminf_n \int (1_S f_n + 1_{S^c} f_n) \\ &= \liminf_n \int f_n \\ &= \int f \tag{3} \end{align}$$

If we had strict inequality in (*), we would have strict inequality in (1). But this is impossible since the left side of (1) and (3) are the same. Thus we must have equality in (*).

Note: The step from (1) to (2) is valid because the sum of infima is $\le$ the infimum of the sum and therefore the sum of liminfs is $\le$ the liminf of the sum. To see this, consider two sequences a_n and b_n of real numbers. Note that if $k \ge n$, then

$$\begin{gathered} \inf\{a_j : j \ge n\} \le a_k \\ \inf\{b_j : j \ge n\} \le b_k \end{gathered}$$

so

$$\inf\{a_j : j \ge n\} + \inf\{b_j : j \ge n\} \le a_k + b_k.$$

Therefore (taking the infimum over all $k \ge n$):

$$\inf\{a_j : j \ge n\} + \inf\{b_j : j \ge n\} \le \inf\{a_k + b_k : k \ge n\}.$$

Taking the limit as $n \to \infty$ gives

$$\liminf a_n + \liminf b_n \le \liminf(a_n + b_n).$$

(Incidentally, it is easy to make up examples in which equality fails.)

Solution 2: By passing to subsequences, we can often work with limits rather than with lim sups and lim infs. For this problem:

Proof by contradiction: assume that $\int_S f_n \, d\mu$ does not converge to $\int_S f \, d\mu$. By passing to a subsequence, we can the limit exists. Hence by Fatou,

$$\lim \int_S f_n \, d\mu > \int_S f \, d\mu.$$

Thus

$$\begin{aligned} \int_{X \setminus S} f_n \, d\mu &= \int_X f_n \, d\mu - \int_S f_n \, d\mu \\ &\to \int_X f \, d\mu - \int_S f \, d\mu \\ &= \int_{X \setminus S} f \, d\mu. \end{aligned}$$

But that contradicts Fatou.

5. (a) Note

$$g(x) = \lim_{r \to 0} G_r(x) \tag{*}$$

where

$$G_r(x) = \sup\{(f(x+h) - f(x))/h : 0 < |h| < r\}.$$

Now G_r is a supremum of continuous functions and is therefore lowersemicontinuous (by a previous hw problem), and thus borel. By (*),

$$g(x) = \lim_{n\to\infty} G_{1/n}(x).$$

Thus g is the limit of a sequence borel functions and is therefore borel.

5(b). Let $k(x) = \liminf_{h\to 0}(f(x+h) - f(x))/h$. As in (a), we can prove that $k(\cdot)$ is a borel function. (Or just deduce this by applying (a) to the function $-f$.) Thus

$$\begin{aligned} D &= \{x : g(x) - k(x) = 0\} \cap \{x : -\infty < g(x) < \infty\} \\ &= (g-k)^{-1}\{0\} \cap g^{-1}(\mathbf{R}) \end{aligned}$$

is borel.

6. Suppose it were false. Then there is an $\epsilon > 0$ and a sequence of sets E_n with $\mu(E_n) < 2^{-n}$ and

$$\int_{E_n} f\,d\mu > \epsilon.$$

Let $A_n = \cup_{k>n} E_k$. Then

(i) $$\mu(A_n) \le \sum_{k>n} \mu(E_k) < 2^{-n}$$

and

(ii) $$\int 1_{A_n} f\,d\mu = \int_{A_n} f\,d\mu \ge \int_{E_n} f\,d\mu > \epsilon.$$

Let A be the intersection of the A_ns. Then $\mu(A) \le \mu(A_n) \le 2^{-n}$ for every n (by (i)), so $\mu(A) = 0$. Hence

(iii) $$\int_A f\,d\mu = 0.$$

On the other hand, the A_n's are nested, so 1_{A_n} converges pointwise to 1_A. Thus by (ii) and the dominated convergence theorem,

(iv) $$\int 1_A f\,d\mu = \lim_n \int 1_{A_n} f\,d\mu \ge \epsilon.$$

[Note: this is where we use $\int f\,d\mu < \infty$: we need it to be able to apply the dominated convergence theorem.] This contradicts (iii), proving the result. $\square$

7(a). Suppose $f_\delta(a) > t$. Then there is an $r < \delta$ such that

$$r^{-2}\mu\mathbf{B}(a,r) > t.$$

Let $R(x) = r + d(x,a)$. Since $\mathbf{B}(x,R(x))$ contains $\mathbf{B}(a,r)$, we see that $\mu\mathbf{B}(x,R(x)) \ge \mu\mathbf{B}(a,r)$ and thus

$$R(x)^{-2}\mu\mathbf{B}(x,R(x)) \ge (r/R(x))^{-2} r^{-2}\mu\mathbf{B}(a,r) > (r/R(x))^{-2}t.$$

If $d(x,a) < \delta - r$, then $R(x) < \delta$, so

$$f_\delta(x) \ge R(x)^{-2}\mu\mathbf{B}(x,R(x)).$$

and thus

$$f_\delta(x) \ge (r/R(x))^{-2}t.$$

As $x \to a$, $R(x) \to r$, so

$$\liminf_{x\to a} f_\delta(x) \ge t.$$

Since this holds for each $t < f_\delta(a)$, we see that

$$\liminf_{x\to a} f_\delta(x) \ge f_\delta(a).$$

Thus f_δ is upper semicontinuous and therefore Borel.

7(b). Note that $f_\delta(a) \le f_{\delta'}(a)$ for $\delta \le \delta'$. Consequently, the limit $f(a) := \lim_{\delta\to 0} f_\delta(x)$ exists. Since f is the limit of the sequence of Borel functions $f_{1/n}$ as $n \to \infty$, f is also Borel.