MATH 205A HOMEWORK 5 (FALL 2018)

0. (Not to turn in). Let $1 \le q < r < \infty$. Show that $\mathcal{L}^q(\mathbf{R}) \not\subset \mathcal{L}^r(\mathbf{R})$ and $\mathcal{L}^r(\mathbf{R}) \not\subset \mathcal{L}^q(\mathbf{R})$.

1. Suppose $1 \le p < \infty$. Let $f : \mathbf{R} \to \mathbf{R}$ be a Lebesgue measurable function such that $\int |f|^p \, d\lambda < \infty$. Prove that for every $\epsilon > 0$, there is a **step function** g such that

$$\|f - g\|_p < \epsilon.$$

(A step function is a function that can be written as a linear combination of characteristic functions of intervals, i.e., a function of the form $\sum_{k=1}^n a_k 1_{A(k)}$ where each $A(k)$ is an interval.)

2. Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e., a measure space with $\mu(X) = 1$. Let $f : X \to \mathbf{R}$ be a $\mathcal{A}$-measurable function. Prove that if $1 \le p \le r$, then $\|f\|_p \le \|f\|_r$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \le p < q < r < \infty$. (Here p and q need not be conjugate exponents.) Show that if $f \in L^p$ and if $f \in L^r$, then $f \in L^q$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a Lebesgue measurable function with $\int |f| \, d\lambda < \infty$. Prove that

$$\lim_{a \to \infty} \int f(t) \sin(at) \, dt = 0.$$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose f_n $(n = 1, 2, \dots))$ and f are functions in $\mathcal{L}^p(X, \mathcal{A}, \mu)$ such that

$$\int f_n g \, d\mu \to \int f g \, d\mu$$

for every $g \in \mathcal{L}^q(X, \mathcal{A}, \mu)$, where $p, q \in (1, \infty)$ are conjugate exponents. Prove that $\|f\|_p \le \liminf_{n \to \infty} \|f_n\|_p$.

6. Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space and that f is a μ-integrable function. Let $\mathcal{F}$ be a σ-algebra contained in $\mathcal{A}$. Prove that there is an $\mathcal{F}$-measurable function g such that

$$\int_S f \, d\mu = \int_S g \, d\mu$$

for every $S \in \mathcal{F}$.