MATH 205A HOMEWORK 5 (FALL 2018)

0. (Not to turn in). Let $1 \le q < r < \infty$. Show that $\mathcal{L}^q(\mathbf{R}) \not\subset \mathcal{L}^r(\mathbf{R})$ and $\mathcal{L}^r(\mathbf{R}) \not\subset \mathcal{L}^q(\mathbf{R})$. **Solution**: Consider $f(x) = x^{-\alpha}1_{(0,1)}(x)$ and $g(x) = x^{-\alpha}1_{(1,\infty)}(x)$ where $q < \alpha < r$.

1. Suppose $1 \le p < \infty$. Let $f : \mathbf{R} \to \mathbf{R}$ be a Lebesgue measurable function such that $\int |f|^p \, d\lambda < \infty$. Prove that for every $\epsilon > 0$, there is a step function g such that

$$\|f - g\|_p < \epsilon.$$

(A step function is a function that can be written as a linear combination of characteristic functions of intervals, i.e., a function of the form $\sum_{k=1}^n a_k 1_{A(k)}$ where each $A(k)$ is an interval.)

Solution: Let $\mathcal{F}$ be the family of step functions in $\mathcal{L}^p$. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ in $\mathcal{L}^p$. That is, $\overline{\mathcal{F}}$ is the set of $f \in \mathcal{L}^p$ for which the assertion is true.

Claim 1: If f_1 and f_2 are in $\overline{\mathcal{F}}$ and if $c_1, c_2 \in \mathbf{R}$, then $c_1 f_1 + c_2 f_2 \in \overline{\mathcal{F}}$. **Proof**: Trivial.

Claim 2: If $U \subset \mathbf{R}$ is an open set with $\lambda(U) < \infty$, then $1_A \in \overline{\mathcal{F}}$.

Proof: Any such U is a countable union of disjoint open intervals: $U = \cup_i U(i)$. Note that $f_n := \sum_{i=1}^n 1_{U(i)}$ is a step function. Also

$$\int |1_U - f_n|^p \, d\lambda = \int |1_U - f_n| \, d\lambda = \lambda(\cup_{i>n} U(i)) = \sum_{i>n} \lambda(U(i))$$

which tends to 0 as $n \to \infty$ (since $\sum_n \lambda(U_n) = \lambda(U) < \infty$). This proves Claim 2.

Claim 3: If $A \subset \mathbf{R}$ is a Lebesgue measurable set with $\lambda(A) < \infty$, then $1_A \in \overline{\mathcal{F}}$.

Proof: Let $\delta > 0$. Let $U \subset \mathbf{R}$ be an open set such that $\lambda(U \setminus A) < \delta$. Then

$$\|1_A - 1_U\|_p = (\lambda(U \setminus A))^{1/p} < \delta^{1/p}. \tag{1}$$

Thus (by Claim 2), we see that $1_A \in \overline{\mathcal{F}}$.

***

Now let $f \in \mathcal{L}^p$. We showed in class that there are simple functions $f_n \in \mathcal{L}^p$ such that $\|f - f_n\|_p \to 0$. By claims 1 and ?, each f_n is in $\overline{\mathcal{F}}$. Thus $f \in \overline{\mathcal{F}}$.

[If you don't remember why the f_n exist: recall that there is a a sequence of nonnegative simple measurable functions p_i such that $p_i \uparrow f^+$. Likewise there is a sequence of nonnegative simple measurable functions n_i such that $n_i \uparrow f^-$. It then follows by the dominated convergence theorem that $\int |f - g_i|^p \to 0$ as $i \to \infty$, where $g_i = p_i - n_i$. (The dominating function is $|f|^p$.)]

2. Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e., a measure space with $\mu(X) = 1$. Let $f : X \to \mathbf{R}$ be a $\mathcal{A}$-measurable function. Prove that if $1 \le p \le r$, then $\|f\|_p \le \|f\|_r$.

Solution: Apply the Holder inequality to f and the constant function 1:

$$\|f\|_1 = \|f \cdot 1\|_1 \le \|f\|_p \|1\|_q$$

But

$$\|1\|_q = \left(\int 1^q \, d\mu \right)^{1/q} = (\mu(X))^{1/q} = 1.$$

Thus

$$\|f\|_1 \le \|f\|_p. \tag{*}$$

WLOG we may assume $f \ge 0$. Thus (substituting f^p for f and r/p for p in (*)) we get

$$\|f^p\|_1 \le \|f^p\|_{r/p} = \left(\int (f^p)^{r/p} \, d\mu \right)^{p/r}$$

so (taking the p^{th} root):

$$\|f\|_p \le \|f\|_r.$$

□

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \le p < q < r < \infty$. (Here p and q need not be conjugate exponents.) Show that if $f \in L^p$ and if $f \in L^r$, then $f \in L^q$.

Solution: Let $A = \{x : |f(x)| > 1\}$. Let $g = 1_A f$ and $h = 1_{X \setminus A} f$. Then

$$|f(x)|^q = |g(x)|^q + |h(x)|^q \le |g(x)|^r + |g(x)|^p \le |f(x)|^p + |f(x)|^r.$$

Now integrate. □

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a Lebesgue measurable function with $\int |f| \, d\lambda < \infty$. Prove that

$$\lim_{a \to \infty} \int f(t) \sin(at) \, dt = 0.$$

Solution: Suppose first that f is the indicator function of an bounded interval $[p, q]$. Then

$$\int f(t) \sin(at) \, dt = \int_p^q \sin(at) \, dt = \frac{1}{a}(\cos(ap) - \cos(aq))$$

which goes to 0 as $a \to \infty$.

It follows immediately that the assertion is true for every step function.

Now let f be an arbitrary Lebesgue integrable function and let $\epsilon > 0$. By problem 1, there is a step function g such that that

$$\|f - g\|_1 < \epsilon.$$

Thus

$$\int f(t) \sin(at) \, dt = \int (f - g) \sin(at) \, dt + \int g(t) \sin(at) \, dt.$$

The first term is at most $\|f - g\|_1$ in absolute value, and the second term tends to 0 as $a \to \infty$. Thus

$$\limsup_{a \to \infty} \left| \int f(t) \sin(at) \, dt \right| \le \|f - g\|_1 < \epsilon.$$

Since this holds for every ϵ, the limsup must be 0. □

5. Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose f_n $(n = 1, 2, \dots)$) and f are functions in $\mathcal{L}^p(X, \mathcal{A}, \mu)$ such that

$$\int f_n g \, d\mu \to \int f g \, d\mu$$

for every $g \in \mathcal{L}^q(X, \mathcal{A}, \mu)$, where $p, q \in (1, \infty)$ are conjugate exponents. Prove that $\|f\|_p \le \liminf_{n \to \infty} \|f_n\|_p$.

Solution: We showed in class that, given $f \in \mathcal{L}^p$, there is a $g \in \mathcal{L}^q$ such that $\|g\|_q = 1$ and such that

$$\int f g \, d\mu = \|f\|_p \|g\|_q = \|f\|_p.$$

Thus for this g,

$$\begin{aligned}\|f\|_p &= \int fg\,d\mu \\ &= \lim_n \int f_n g\,d\mu \\ &\le \liminf_n \|f_n\|_p \|g\|_q \qquad \text{(by Holder)} \\ &= \liminf_n \|f_n\|_p \qquad \text{(since } \|g\|_q = 1.)\end{aligned}$$

6. Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space and that f is a μ-integrable function. Let $\mathcal{F}$ be a σ-algebra contained in $\mathcal{A}$. Prove that there is an $\mathcal{F}$-measurable function g such that

$$\int_S f\,d\mu = \int_S g\,d\mu$$

for every $S \in \mathcal{F}$.

Solution: Note that

$$S \in \mathcal{A} \mapsto \int_S f\,d\mu \tag{*}$$

defines a finite signed measure on $(X, \mathcal{A})$ that is absolutely continuous with respect to μ.

Let ν be the restriction of the signed measure (*) to $\mathcal{F}$, and let $\tilde{\mu}$ be the restriction of μ to $\mathcal{F}$. Then ν is a finite signed measure, $\tilde{\mu}$ is a finite measure, and $\nu << \tilde{\mu}$.

By the Radon-Nykodym Theorem, there is a $\mathcal{F}$-measurable function $g : X \to \mathbf{R}$ such that for all $S \in \mathcal{F}$,

$$\nu(S) = \int_S g\,d\tilde{\mu}$$

i.e.,

$$\int_S f\,d\mu = \int_S g\,d\mu.$$

Remark: In probability theory, if $\mu(X) = 1$, then g is called the *conditional expectation* of f with respect to the σ-algebra $\mathcal{F}$.