MATH 205A HOMEWORK 6 (FALL 2018)

1. Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space and that $f : X \to \mathbf{R}$ is an $\mathcal{A}$-measurable function. Prove that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

Recall that $\|f\|_\infty$ is the infimum of the set of nonnegative essential upper bounds for $|f(\cdot)|$. An **essential upper bound** for a function $g : X \to [-\infty, \infty]$ is a number c such that $\mu\{x : g(x) > c\} = 0$.

(The definition of $\|f\|_\infty$ given in the book is more complicated. It agrees with the definition above for finite and for σ-finite measure spaces.)

2. Let $f \in \mathcal{L}^p(\mathbf{R})$ where $1 \le p < \infty$. Prove that for every ϵ, there is a continuous, compactly supported function $g : \mathbf{R} \to \mathbf{R}$ such that $\|f - g\|_p < \epsilon$.

(The *support* of g is the closure of the set of points where $g \neq 0$. We say that g is compactly supported if the support of g is compact.)

3. Suppose $f \in \mathcal{L}^p(\mathbf{R})$, where $1 \le p < \infty$. Let f_h be the function given by $f_h(x) = f(x + h)$. (a) Prove that

$$\lim_{h\to 0} \|f_h - f\|_p = 0.$$

(b) Suppose $g \in \mathcal{L}^q(\mathbf{R})$ (where $\frac{1}{p} + \frac{1}{q} = 1$). Define $\phi : \mathbf{R} \to \mathbf{R}$ by

$$\phi(x) = \int_{t\in\mathbf{R}} f(x - t)g(t)\, dt.$$

Prove that ϕ is bounded and uniformly continuous.

4. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is Lebesgue measurable. Define a function $Mf : \mathbf{R}^n \to [0, \infty]$ by

$$Mf(x) = \sup_{r>0} \frac{1}{\lambda\mathbf{B}(x, r)} \int_{\mathbf{B}(x,r)} |f|\, d\lambda$$

Let $A = \{x : Mf(x) > a\}$. Prove that

$$\lambda(A) \le \frac{5^n}{a} \|f\|_1.$$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $0 < r < s < t < \infty$. Prove that

$$(\|f\|_s)^s \le (\|f\|_r)^{\alpha r} (\|f\|_t)^{(1-\alpha)t}$$

where $\alpha \in (0, 1)$ is the number such that $s = \alpha r + (1 - \alpha)t$.

(For $0 < p < 1$, we define $\|f\|_p$ to be $\left(\int |f|^p\right)^{1/p}$, just as we did for $p \ge 1$. However, for $0 < p < 1$, this quantity is *not* a norm: the triangle inequality fails.)