MATH 205A HOMEWORK 6 SOLUTIONS (FALL 2018)

1. Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space and that $f : X \to \mathbf{R}$ is an $\mathcal{A}$-measurable function. Prove that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. Recall that $\|f\|_\infty$ is the infimum of the set of nonnegative essential upper bounds for $|f(\cdot)|$. An **essential upper bound** for a function $g : X \to [-\infty, \infty]$ is a number c such that $\mu\{x : g(x) > c\} = 0$. (The definition of $\|f\|_\infty$ given in the book is more complicated. It agrees with the definition above for finite and for σ-finite measure spaces.)

Solution: We may assume that $0 < \|f\|_\infty$ since the result is trivially true if $f = 0$ a.e. Let $0 < c < \|f\|_\infty$. Then the set

$$S = \{x : |f(x)| > c\}$$

has positive measure (by definition of $\|f\|_\infty$.) Thus

$$\left(\int |f|^p\right)^{1/p} \geq \left(\int_S |f(x)|^p\right)^{1/p} \geq \left(\int_S c^p\right)^{1/p} = c\mu(S)^{1/p}$$

which tends to c as $p \to \infty$ (since $0 < \mu(S) < \infty$.) Thus $\liminf_{p\to\infty} \|f\|_p \geq c$. Since this holds for every $0 < c < \|f\|_\infty$, we have

$$(*) \qquad \liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty.$$

On the other hand, if $c > \|f\|_\infty$, then $|f(x)| \leq c$ for almost every x, so $\left(\int |f|^p\right)^{1/p} \leq \left(\int c^p\right)^{1/p} = (c^p\mu(X))^{1/p} = c(\mu(X))^{1/p}$ which tends to c as $p \to \infty$, so

$$(\dagger) \qquad \limsup_{p\to\infty} \|f\|_p \leq c.$$

Since this holds for all $c < \|f\|_\infty$, we have $\limsup_{p\to\infty} \|f\|_p \geq \|f\|_\infty$. By (*), equality must hold.

2. Let $f \in \mathcal{L}^p(\mathbf{R})$ where $1 \leq p < \infty$. Prove that for every ϵ, there is a continuous, compactly supported function $g : \mathbf{R} \to \mathbf{R}$ such that $\|f - g\|_p < \epsilon$.

(The *support* of g is the closure of the set of points where $g \neq 0$. We say that g is compactly supported if the support of g is compact.)

Solution: Suppose first that $f = 1_I$ is the indicator function of an interval I. Let J be a closed interval contained in the interior of I, and let $g : \mathbf{R} \to [0, 1]$ be a continuous function that is equal to 0 on the complement of I and that is equal to 1 on J. (For instance, we can interpolate linearly on $I \setminus J$.) Then

$$\|f - g\|_p \leq (\lambda(I \setminus J))^{1/p}$$

which we can make arbitrarily small by suitable choice of J. Thus the result is true for $f = 1_I$.

It follows immediately that it is true for any step function f. (For such an f can be written as $\sum_{j=1}^n c_j 1_{I(j)}$ where $c_j \neq 0$ and $I(j)$ is a bounded interval. For each j, there is a continuous, compactly supported g_j such that $\|1_{I(j)} - g_j\|_p < \epsilon/(nc_j)$. Thus $\|f - g\|_p < \epsilon$, where g is the continuous, compactly supported function $\sum c_j g_j$.)

Now for any $f \in \mathcal{L}^p$, we showed in an earlier hw that there is a step function $h : \mathbf{R} \to \mathbf{R}$ such that $\|f - h\|_p < \epsilon/2$. We just showed that there is a continuous, compactly supported function g such that $\|h - g\|_p < \epsilon/2$. Thus $\|f - g\|_p < \epsilon$.

3. Suppose $f \in \mathcal{L}^p(\mathbf{R})$, where $1 \le p < \infty$. Let f_h be the function given by $f_h(x) = f(x + h)$. (a) Prove that

$$\lim_{h \to 0} \|f_h - f\|_p = 0.$$

Solution: Suppose first that $f = c1_J$ where J is an interval and $c \neq 0$. Then for all h sufficiently near 0,

$$\|f_h - f\|_p = |c|\,(2h)^{1/p}$$

Thus the result is true for such an f. It follows immediately (by Minkowski's inequality) that it holds for any step function $f \in \mathcal{L}^p$.

Now let f be an arbitrary function in $\mathcal{L}^p$. Let $\epsilon > 0$. Then there is a step function $g \in \mathcal{L}^p$ such that

$$\|f - g\|_p < \epsilon.$$

Now

$$\begin{aligned}
\|f - f_h\|_p &\le \|f - g\|_p + \|g - g_h\|_p + \|f_h - g_h\|_p \\
&= 2\|f - g\|_p + \|g - g_h\|_p \qquad \text{(by translation invariance of } \lambda) \\
&< 2\epsilon + \|g - g_h\|_p
\end{aligned}$$

Thus since the result is true for the step function g,

$$\limsup_{h \to 0} \|f - f_h\|_p \le 2\epsilon.$$

Since this is true for all $\epsilon > 0$, in fact the lim sup must be 0. □

(b) Suppose $g \in \mathcal{L}^q(\mathbf{R})$ (where $\frac{1}{p} + \frac{1}{q} = 1$). Define $\phi : \mathbf{R} \to \mathbf{R}$ by

$$\phi(x) = \int_{t \in \mathbf{R}} f(x - t) g(t)\, dt.$$

Prove that ϕ is bounded and uniformly continuous.

Remark: The function ϕ obtained from f and g in this way is written $f * g$ and is called the **convolution** of f and g.

Solution: Note that f and $x \mapsto f(t - x)$ have the same $\mathcal{L}^p$ norm (since Lebesgue measure is invariant under $x \mapsto t - x$.) Thus by Holder's inequality,

$$(*) \qquad |(f * g)(x)| \le \|f\|_p \|g\|_q$$

for each x. Thus ϕ is bounded.

To show uniform continuity, note that

$$(f*g)(x+h)-(f*g)(x) = \int f(x+h-t)g(t)\,dt - \int f(x-t)g(t)\,dt \tag{1}$$

$$= \int (f(x+h-t)-f(x-t))g(t)\,dt = ((f_h - f)*g)(x). \tag{2}$$

Thus

$$\begin{aligned}\sup_{x\in\mathbf{R}} |(f*g)(x+h)-(f*g)(x)| &= \sup x \in \mathbf{R} |((f_h - f)*g)(x)| \\ &\le \|f_h - f\|_p \, \|g\|_q \qquad \text{(by (*))}\end{aligned}$$

which tends to 0 as $h \to 0$ by part (a). Thus $f * g$ is uniformly continuous.

4. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is Lebesgue measurable. Define a function $Mf : \mathbf{R}^n \to [0, \infty]$ by

$$Mf(x) = \sup_{r>0} \frac{1}{\lambda \mathbf{B}(x,r)} \int_{\mathbf{B}(x,r)} |f| \, d\lambda$$

Let $A = \{x : Mf(x) > a\}$. Prove that

$$\lambda(A) \le \frac{5^n}{a} \|f\|_1.$$

Solution: We may assume that $\|f\|_1 < \infty$. Let $\mathcal{F}$ be the family of closed balls $\overline{\mathbf{B}}(x,r)$ such that

$$\frac{1}{\lambda \overline{\mathbf{B}}(x,r)} \int_{\overline{\mathbf{B}}(x,r)} |f| \, d\lambda > a.$$

Note that A is covered by the balls in $\mathcal{F}$. Note also that if $\overline{\mathbf{B}}_1, \overline{\mathbf{B}}_2, \dots$ are disjoint balls in $\mathcal{F}$, then

$$\infty > \|f\|_1 \ge \sum_i \int_{\overline{\mathbf{B}}_i} |f| \, d\lambda > a \sum_i \lambda \overline{\mathbf{B}}_i.$$

Thus we see that $\mathcal{F}$ does not contain an infinite sequence of balls whose diameters are bounded away from 0.

Thus by the five-times covering lemma, there is an infinite sequence (finite or infinite) $\overline{\mathbf{B}}_i$ of disjoint balls in $\mathcal{F}$ such that

$$A \subset (\cup_{i<m} \overline{\mathbf{B}}_i) \cup (\cup_{i \ge m} \widehat{\overline{\mathbf{B}}}_i).$$

For each i. In particular,

$$A \subset \cup_i \widehat{\overline{\mathbf{B}}}_i.$$

Thus

$$\begin{aligned}
\lambda(A) &\le \sum_i \lambda\widehat{\overline{\mathbf{B}}_i} \\
&= 5^n \sum_i \lambda\overline{\mathbf{B}}_i \\
&\le \frac{5^n}{a} \sum_i \int_{\overline{\mathbf{B}}_i} |f|\, d\lambda \\
&\le \frac{5^n}{a} \|f\|_1.
\end{aligned}$$

5. [Not to turn in] Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $0 < r < s < t < \infty$. Prove that

$$(\|f\|_s)^s \le (\|f\|_r)^{\alpha r} (\|f\|_t)^{(1-\alpha)t}$$

where $\alpha \in (0,1)$ is the number such that $s = \alpha r + (1-\alpha)t$.

(For $0 < p < 1$, we define $\|f\|_p$ to be $\left(\int |f|^p\right)^{1/p}$, just as we did for $p \ge 1$. However, for $0 < p < 1$, this quantity is *not* a norm: the triangle inequality fails.)

Solution: Let $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, so $1 = \frac{1}{p} + \frac{1}{q}$ and $s = \frac{r}{p} + \frac{t}{q}$. Then

$$\begin{aligned}
(\|f\|_s)^s &= \int |f|^s \\
&= \int |f|^{r/p} |f|^{t/q} \\
&\le \left(\int (|f|^{r/p})^p\right)^{1/p} \left(\int (|f|^{t/q})^q\right)^{1/q} \\
&= \left(\int |f|^r\right)^{1/p} \left(\int |f|^t\right)^{1/q} \\
&= (\|f\|_r)^{r/p} (\|f\|_t)^{t/q} \\
&= (\|f\|_r)^{\alpha r} (\|f\|_t)^{(1-\alpha)t}.
\end{aligned}$$