MATH 205A HOMEWORK 7 (FALL 2018)

1. Suppose f and g are Lebesgue measurable functions on $\mathbf{R}^n$. If $x \in \mathbf{R}^n$, we let

$$(f * g)(x) = \int_{y \in \mathbf{R}^n} f(x-y)g(y)\, d\lambda y$$

provided the integral exists. Suppose $\phi : \mathbf{R}^n \to \mathbf{R}$ is a bounded measurable function such that $\phi(x) = 0$ for $|x| > 1$ and such that $\int \phi = 1$. For $r > 0$, define $\phi_r : \mathbf{R}^n \to \mathbf{R}$ by $\phi_r(x) = r^{-n}\phi(x/r)$. Prove that if $f \in \mathcal{L}^1(\mathbf{R}^n)$, then

$$\lim_{r \to 0} (\phi_r * f)(x) = f(x)$$

for almost every $x \in \mathbf{R}^n$.

[Hint: to understand what's going on, you might first consider the case when ϕ is a constant times the indicator function of a ball centered at the origin.]

2. Suppose that A and B are Lebesgue measurable subsets of $\mathbf{R}$ with $\lambda(A) > 0$ and $\lambda(B) > 0$. Prove that the set $A - B := \{x - y : x \in A, y \in B\}$ contains an interval.

3. Suppose F and $F_1, F_2, \dots$ are increasing functions from $[a,b]$ to $\mathbf{R}$ such that $F(x) = \sum_k F_k(x)$ for all $x \in [a,b]$. Prove that

$$F'(x) = \sum_k F_k'(x)$$

for λ almost every $x \in [a,b]$. (Remark: if you wish, you may assume that $F_k(a) = 0$ for all k since the general case follows easily.)

4. (a). Suppose $\overline{\mathbf{B}}(x_i, r_i)$, $i = 1, \dots, n$ are closed balls in $\mathbf{R}^d$ with $r_i > 0$ and with $|x_i - x_j| > r_i$ for all i and j. (Thus no ball contains the center of another ball.) Suppose also that $r_1 \le 2r_i$ for all i and that each of the balls intersects $\overline{\mathbf{B}}(x_1, r_1)$. Prove that n is bounded above by a number $C(d) < \infty$ depending only on the dimension d. **Hint**: Show that given a collection of n such balls, you can find a collection of n such balls with $\overline{\mathbf{B}}(x_1, r_1) = \overline{\mathbf{B}}(0,1)$ and with $r_i \le 2$ for all i. **Alternate hint**: Fix some $R > 1$ (say $R = 3/2$ or $R = 2$). You can assume that $\overline{\mathbf{B}}(x_1, r_1)$ is $\overline{\mathbf{B}}(0,1)$. Divide the collection of balls into those (the "small" balls) with centers in $\overline{\mathbf{B}}(0,R)$, and those (the "large" balls) with centers outside of $\overline{\mathbf{B}}(0,R)$. (Use one argument to bound the number of small balls, and another argument to bound the number of large balls.)

(b). Suppose $\mathcal{F}$ is a FINITE collection of closed balls in $\mathbf{R}^d$. Let X be the set of their centers. Show that there exist families $\mathcal{F}_1, \dots, \mathcal{F}_{C(d)}$ such that

1. $\mathcal{F}_i \subset \mathcal{F}$ for each i.
2. For each i, the balls in $\mathcal{F}_i$ are disjoint.
3. $X \subset \cup_i \cup_{\overline{\mathbf{B}} \in \mathcal{F}_i} \overline{\mathbf{B}}$.

5. Let $S \subset \mathbf{R}^n$ be a Lebesgue measurable such that if x and y are distinct elements of S, then their midpoint $(x+y)/2$ is not in S. Prove that $\lambda(S) = 0$.

6. Suppose $(X, \mathcal{A}, \mu)$ be a measure space and $F : X \to [0, \infty)$ be an $\mathcal{A}$-measurable function. Prove that

$$\int F\, d\mu = \int_{y=0}^{\infty} \mu\{x : F(x) > y\}\, d\lambda(y).$$