MATH 205A HOMEWORK 8 (FALL 2018)

1. Let X be $\mathbf{R}^2$ with the following topology: S is open if and only if $S_x := \{y : (x, y) \in S\}$ is an open subset of $\mathbf{R}$ for each x (with respect to the usual topology on $\mathbf{R}$.)

Note that X is locally compact. Note also that if $f \in \mathcal{K}(X)$, i.e, if f is continuous and compactly supported in X, then $f(x, \cdot) = 0$ except for a finite set of x's. Define

$$L : \mathcal{K}(X) \to \mathbf{R},$$
$$Lf = \sum_x \int f(x, y)\, dy$$

(a). Find a regular Borel measure μ on X such that

$$Lf = \int f\, d\mu$$

for all $f \in \mathcal{K}(X)$.

(b). Find a Borel set S such that $\lambda(S_x) = 0$ for every x, but $\mu(S) \neq 0$, where $S_x = \{y : (x, y) \in S\}$.

(c). Show that X contains a Borel set T such that $\mu(T)$ is not equal to the supremum of $\mu(K)$ among compact subsets K of T.

2. Let $\prec$ be an ordering of $[0, 1]$ with the following properties:

(1) $x \prec y$ and $y \prec z$ imply $x \prec z$.
(2) For each $x \neq y$, either $x \prec y$ or $y \prec x$.
(3) $x \not\prec x$.
(4) If $S \subset [0, 1]$ is nonempty, then S has a least element, i.e, an element x such that $x \prec y$ for every $y \in S \setminus \{x\}$.
(5) If $x \in [0, 1]$, then $\{y : y \prec x\}$ is countable.

(The existence of such an ordering is equivalent to the continuuum hypothesis.)

Let $A = \{(x, y) \in [0, 1] \times [0, 1] : x \prec y\}$. Prove that A is **not** a Borel set.

Hint: Fubini theorem.

3. Let A be a Lebesgue measurable set with the following property: if $x - y$ is rational and if $x \in A$, then $y \in A$. Prove that $\lambda(A) = 0$ or $\lambda(\mathbf{R} \setminus A) = 0$.