MATH 205A HOMEWORK 8 (FALL 2018)

1. Let X be $\mathbf{R}^2$ with the following topology: S is open if and only if $S_x := \{y : (x,y) \in S\}$ is an open subset of $\mathbf{R}$ for each x (with respect to the usual topology on $\mathbf{R}$.) (This topology is given by the metric $d((x,y),(x',y'))$ that is $|y-y'|$ if $x = x'$ and that is $|y-y'|+1$ if $x \neq x'$.) Note that X is locally compact. Note also that if $f \in \mathcal{K}(X)$, i.e, if f is continuous and compactly supported in X, then $f(x,\cdot) = 0$ except for a finite set of x's. Define

$$L : \mathcal{K}(X) \to \mathbf{R},$$
$$Lf = \sum_x \int f(x,y)\,dy$$

(a). Find a regular Borel measure μ on X such that

(*) $$Lf = \int f\,d\mu$$

for all $f \in \mathcal{K}(X)$.

Solution: By the Riesz Representation Theorem, we know existence and uniqueness of μ. Now for any continuous, compactly supported function $f : \mathbf{R} \to \mathbf{R}$

$$\int f(y)\,d\mu(x,y) = \int f\,d\lambda.$$

From this it follows that for every Borel set $T \subset \mathbf{R}$ and every $x \in \mathbf{R}$, we have

(†) $$\mu(\{x\} \times T) = \lambda(T).$$

If S is a Borel subset of X, i.e, if S_x is a Borel subset of $\mathbf{R}$ for every x, let $I(S) = \{x \in \mathbf{R} : S_x \neq \emptyset\}$. From (†), it follows that for every Borel set $S \subset X$,

(**) $$\mu(S) \geq \sum_{x \in I(S)} \lambda(S_x) \quad \text{with equality if } I(S) \text{ is countable.}$$

Now suppose that $I(S)$ is uncountable. Let $U \subset X$ be an open set containing S. Then $I(U)$ is uncountable. Now for each $x \in I(U)$, $\lambda(U_x) > 0$. Thus $\sum_{x \in I(U)} \lambda(U_x) = \infty$ (see hw 1, problem 7), so by (**) (with U in place of S),

$$\mu(U) = \infty.$$

Taking the infimum among all such U gives $\mu(S) = \infty$.

We have shown that if $S \subset X$ is a Borel set, then

$$\mu(S) = \begin{cases} \sum_{x \in I(S)} \lambda(S_x) & \text{if } I(S) \text{ is countable,} \\ \infty & \text{if } I(S) \text{ is uncountable.} \end{cases}$$

(b). Find a Borel set S such that $\lambda(S_x) = 0$ for every x, but $\mu(S) \neq 0$, where $S_x = \{y : (x,y) \in S\}$.

Solution: For example, let $S = \{(x,0) : x \in \mathbf{R}\}$. Then $\mu(S) = \infty$ but $\lambda(S_x) = 0$ for every x.

(c). Show that X contains a Borel set T such that $\mu(T)$ is not equal to the supremum of $\mu(K)$ among compact subsets K of T.

Solution: The set S in part (b) is an example. Note (for this S) that if $K \subset S$ is compact, then K is finite, and hence $\mu(K) = 0$.

2. Let $\prec$ be a well-ordering of $[0,1]$. Thus

(1) $x \prec y$ and $y \prec z$ imply $x \prec z$.
(2) For each $x \neq y$, either $x \prec y$ or $y \prec x$.
(3) $x \not\prec x$.
(4) If $S \subset [0,1]$ is nonempty, then S has a least element, i.e, an element x such that $x \prec y$ for every $y \in S \setminus \{x\}$.

(The axiom of choice implies that such well-orderings exist.) Let $A = \{(x,y) \in [0,1] \times [0,1] : x \prec y\}$. Prove that A is **not** a Borel set.

Hint: Fubini theorem. You might first try to prove it if $\prec$ has the following property: for each $x \in [0,1]$, the set $\{y \in [0,1] : y \prec x\}$ is countable. (Remark: existence of such a well-ordering is equivalent to the continuum hypothesis.)

Solution: By hypothesis, $A^y = \{x : x \prec y\}$ is countable for every y, so

$$\lambda(A^y) = 0.$$

Also,

$$A_x = \{y : x \prec y\} = [0,1] \setminus (A^y \cup \{x\})$$

so

$$\lambda(A_x) = \lambda([0,1]) - \lambda(A^y \cup \{x\}) = 1 - 0 = 1.$$

Thus

$$\int_{y \in [0,1]} \lambda(A^y)\, d\lambda(y) = 0,$$

$$\int_{x \in [0,1]} \lambda(A_x)\, d\lambda(x) = 1.$$

If A were a Borel set, then these two integrals would be equal by Fubini's theorem. Hence A is not Borel.

3. Let A be a Lebesgue measurable set with the following property: if $x - y$ is rational and if $x \in A$, then $y \in A$. Prove that $\lambda(A) = 0$ or $\lambda(\mathbf{R} \setminus A) = 0$.

Solution: The hypothesis implies that if q is rational, then $A \cap [q - r, q + r] = (A \cap [-r, r]) + q$, and thus

$$\lambda(A \cap [q - r, q + r]) = \lambda(A \cap [-r, r]) \tag{*}$$

for all r and for all rationals q. Also, $\lambda([A \cap [x - r, x + r])$ is a continuous function of x. Thus by (*) it is constant:

$$\lambda(A \cap [x - r, x + r]) \equiv c(r). \tag{†}$$

By Lebesgue's Theorem,

$$\lim_{r \to 0} \frac{\lambda(A \cap [x - r, x + r])}{2r} = 1_A(x)$$

for almost every x. By (†), the left side does not depend on x. Thus there is a constant c such that $c = 1_A(x)$ for almost every x.