# THE VITALI COVERING THEOREM

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As in the 5-Times Covering Theorem, if $B \subset \mathbf{R}^d$, we let

$$\widehat{B} = \{q \in \mathbf{R}^d : \operatorname{dist}(q, B) < 2\operatorname{diam}(B)\}.$$

In particular, if $B$ is a closed ball, then $\widehat{B}$ is the open ball with the same center and with 5 times the radius.

**Theorem** (The Vitali Covering Theorem)**.** *Suppose that $\mathcal{F}$ is a family of closed balls of positive radii in $\mathbf{R}^d$. Then there is a sequence $B_1, B_2, \dots$ (finite or infinite) of disjoint balls in $\mathcal{F}$ with the following property. If $\mathcal{F}$ covers $S$ finely, then*

$$S \setminus \cup_{i \le n} B_i \subset \cup_{i>n} \widehat{B}_i \tag{1}$$

*for every $n$, and*

$$\lambda(S \setminus \cup_i B_i) = 0. \tag{2}$$

*Proof.* Let $\mathcal{F}^*$ be the set of balls $B(a, r) \in \mathcal{F}$ such that $r < (|a| + 1)^{-1}$. Note that each ball in $\mathcal{F}^*$ has diameter $< 2$.

**Claim**: There is no infinite sequence of disjoint balls in $\mathcal{F}^*$ with diameters bounded away from 0.

To prove the claim, suppose to the contrary that $\mathbf{B}(a_i, r_i)$ is an infinite sequence of disjoint balls in $\mathcal{F}^*$ with $r_i > \delta > 0$. Then $a_i < 1/\delta$. Thus the $a_i$ would be an infinite sequence of points in a bounded subset of $\mathbf{R}^n$ and thus would have a convergent sequence. But that's impossible since $|a_i - a_j| \ge r_i + r_j > 2\delta$ for $i \ne j$. This proves the claim.

Thus the finiteness condition of the 5-Times Covering Lemma is satisfied, so by that lemma, there is a sequence $B_1, B_2, \dots$ of disjoint balls in $\mathcal{F}^*$ such that (1) holds.

It remains to show that (2) holds.

**Case 1**: $S$ lies in a bounded set $B(0, R) = \{x \in \mathbf{R}^m : |x| \le R\}$.

Let $I$ be the set indices $i$ such that $B_i \cap S$ is nonempty. Since each $B_i$ has diameter $\le 1$, each $\widehat{B_i}$ has diameter $\le 5$. Thus if $i \in I$, $\widehat{B_i}$ is contained in $\mathbf{B}(0, R + 10)$ and therefore $\mathbf{B}_i \in \mathbf{B}(0, R + 10)$. Thus

$$\begin{aligned} \sum_{i \in I} \lambda(B_i) &= \lambda(\cup_{i \in I} B_i) \\ &\le \lambda(\mathbf{B}(0, R + 10)) \\ &< \infty. \end{aligned} \tag{3}$$

By (1),

$$S \setminus \cup_i B_i \subset \cup_{i \in I, i \ge n} \widehat{B_i}$$

for each $n$, so

$$\begin{aligned}\lambda^*(S \setminus \cup_i B_i) &\le \sum_{i\in I, i\ge n} \lambda(\widehat{B_i}) \\ &= 5^d \sum_{i\in I, i\ge n} \lambda(B_i)\end{aligned}$$

By (3), this last expression tends to 0 as $n \to \infty$. This completes the proof of (2) in Case 1.

**Case 2**: $S$ is any subset of $\mathbf{R}^d$. Let $S_n = \{x \in S : |x| \le n\}$. By Case 1,

$$\lambda(S_n \setminus \cup_i B_i) = 0. \tag{4}$$

Since

$$S \setminus \cup_i B_i = (\cup_n B_n) \setminus \cup_i B_i = \cup_n (S_n \setminus \cup_i B_i),$$

it follows from (4) that $\lambda(S \setminus \cup_i B_i) = 0$. □