Delay-Optimal Burst Erasure Codes for Parallel Links

Zhi Li[1], Ali C. Begen[2] and Bernd Girod[1]
[1]Department of Electrical Engineering, Stanford University, Stanford, CA 94305, USA
[2]Cisco Systems, 170 West Tasman Drive, San Jose, CA 95134, USA
Email: [1]{leeoz,bgirod}@stanford.edu, [2]abegen@cisco.com

***Abstract*—Motivated by live streaming applications using peer assistance, we study burst erasure correction codes with optimal decoding delay for parallel-link networks. We consider a new model for network error correction, where each source symbol is decoded at the sink on-the-fly with a strict decoding deadline. We distinguish two types of errors which have very different implications on the code design – erasure burst and link outage. For a class of codes that achieve the Singleton bound, we state a theorem that characterizes the fundamental tradeoff among the coding rate, decoding delay and error correction performance, whose achievability is illustrated by a practical code construction.**

***Keywords*—low delay, burst erasure code, network error correction, peer-to-peer**

I. Introduction

Designing a live streaming system over the Internet is often challenging, since one needs to meet multiple conflicting goals, including high throughput, low latency and high robustness to errors. In a practical network architecture, errors often come from multiple sources – some of them are long-term, such as a node or link outage; some are temporary, such as random or bursty packet losses.

We study in this work erasure correction codes for simple parallel-link networks, under the constraint that *each packet/symbol* is subject to a decoding deadline. This work falls into the framework of coherent[1] network error correction [1], [2]. However, we introduce a model that distinguishes spatial and temporal coding redundancy[2], thus allowing us to consider two types of errors differently – erasure burst and link outage. Focusing on a class of codes that achieve the Singleton bound, we study the fundamental tradeoff among the coding rate, decoding delay and its error correction performance.

A number of applications could benefit from our proposed codes, including peer-assisted IPTV packet loss repair [3], multiple-sender distributed video streaming [4] and peer-to-peer live multicast with multiple complementary trees [5].

This paper is structured as follows. Section II discusses related work. We formulate the problem in Section III, and present the main results in Section IV. The achievability and the converse part of the main theorem are proven in Section V by construction and in Section VI, respectively.

[1]By coherent coding, we can carefully select the coding structure to meet the decoding deadline (potentially before the coding block ends). In contrast, incoherent coding eliminates the possibility of decoding a symbol earlier than the end of a coding block.

[2]A way to introduce a time dimension to the model in [1] is through a time-expanded graph. In this work, we do not follow this method since it hardly gives any insight on how to construct the codes with optimal decoding delay.

II. Related Work

Our study is inspired by [6], [7], [8], where Martinian et al. studied delay-optimal systematic codes for a single erasure burst. They first design a family of block codes, then use diagonal interleaving [7] to convert the block codes into streaming codes.

For single-link block codes that correct one B-burst with delay T, it is sufficient to use a $(T, T-B)$ cyclic code over finite field Q, of generator matrix

$$G_C = \begin{pmatrix} I_{T-B} & P_{(T-B)\times B} \end{pmatrix} \quad (1)$$

where $I_M \in Q^{M\times M}$ denotes identity matrix of dimension $M\times M$ and $P_{(T-B)\times B} \in Q^{(T-B)\times B}$ is the parity sub-matrix. Martinian et al. propose a construction of $(T+B, T)$ code with the same delay and burst correction performance, but of better rate, having the form:

$$G_D = \begin{pmatrix} I_T & \begin{matrix} I_B \\ P_{(T-B)\times B} \end{matrix} \end{pmatrix} \quad (2)$$

where $P_{(T-B)\times B}$ is the same as in (1). An intuition behind this construction is that since it does not correct wrap-around bursts, the rate can be improved. A converse result based on a periodic erasure channel argument shows that the constructed streaming code achieves the rate-delay-burst bound [6]:

$$T \geq \max\left(\frac{R}{1-R}\,,\,1\right)\cdot B. \quad (3)$$

The single-link code is extended to parallel links in [8], but is limited to a single burst. Delay-optimal erasure codes for broadcast channel is studied by Khisti et al. in [9].

This work differs from [6], [7], [8], [9] as follows. i) Block codes are considered instead of streaming codes. The conversion from block codes to streaming codes can be achieved trivially [7]. ii) A broader class of causal codes are considered instead of systematic codes. iii) More general erasure patterns are considered, including erasure bursts and link outages. In particular, we consider multiple erasure bursts instead of a single burst as in [8].

Our code construction has some similarity with product codes, which is a special class of concatenated codes [10].

III. Problem Formulation

Throughout the paper, random variables are denoted by sans-serif letters (e.g., s, x, y). The entropy of random variable x is denoted by $H(\mathsf{x})$, and the conditional entropy of x given y is denoted by $H(\mathsf{x}\,|\,\mathsf{y})$. (m,n) is used to index the position of a symbol within the parallel-link coding block, where m indexes the link and n indexes the time. $a,...,b$ is sometimes denoted by $a:b$ for brevity.

An M-parallel-link network contains a source node s connected to a sink node t via M parallel links. In this work we consider homogeneous links and assume each link has unit capacity, i.e., it can transmit one symbol within a unit time. Multiple channel symbols can be transmitted over a link sequentially over time[3]. The channel symbol can either be erased (denoted by symbol $\star$) or passed to the receiver intact. Consider a length-N channel symbol block for each link m, $m=1,...,M$. Denote by $\mathcal{B}\subseteq\{(m,n)\}_{m=1,...,M,\,n=1,...,N}$ an *erasure pattern*, i.e., the set of channel symbol positions where erasures occur. When $\mathsf{x}[m,n]$ is transmitted over the link, the received channel symbol is

$$\mathsf{y}[m,n]=\begin{cases}\star, & (m,n)\in\mathcal{B}\\ \mathsf{x}[m,n], & \text{otherwise.}\end{cases}$$

We care about two types of practically relevant errors. A *link outage* occurs when all the symbols transmitted over a link are erased. A B-*burst* is B consecutive symbol erasures on a link. In this work, we assume that within a coding block, at most one burst occurs in a link. An *erasure pattern collection* is a set of erasure patterns $\{\mathcal{B}\}$. For example, the set of all patterns with L link outages and Z B-bursts, each occurs on a separate link, is an erasure pattern collection.

A code C over an M-parallel-link network consists of a set of M encoding functions $\{E_m\}_{m=1}^M$ and a decoding function D. The operation of the code is described as follows.

Encoding. At time n, the source node observes a vector of source symbols $\mathsf{S}[n]=(\mathsf{s}[1,n],...,\mathsf{s}[K_n,n])^{\mathsf{T}}\in Q^{K_n\times 1}$, where each source symbol is over finite field Q, and K_n is the number of source symbols injected at time n [4]. To encode the source symbols into channel symbols, for each link m, $m=1,...,M$, we consider an encoding function $E_m: Q^K\mapsto Q^N$that takes in a vector of $K:=\sum_{n=1}^N K_n$ source symbols $\mathsf{S}\in Q^K$ [5] and maps them into a block of N channel symbols. The generated channel symbols $\mathsf{X}_m=E_m(\mathsf{S})$ are then transmitted sequentially over the link. The rate of the code is defined as:

$$R:=\frac{K}{N}\text{ symbols/unit time.}\tag{4}$$

[3]Note that in this formulation, we ignore the link delay.

[4]In our setting, K_n varies over time. This is akin to the single-link (N,K) block code case, where $K_n=1$ for $n=1,...,K$ and $K_n=0$ for $n=K+1,...,N$. Note that the source symbol rate can be smoothed out by packetizing symbols through diagonal interleaving [7].

[5]S and $\{\mathsf{S}[n]\}_{n=1,...,N}$ are one-one, but the symbols could be re-ordered at the convenience of the algorithm description. See Section V-A for more details.

The code is *causal*, if in the encoding function the current channel symbol is a function of the current and previous source symbols. That is, let $E_m=\{E_{m,n}\}_{n=1}^N$, where $E_{m,n}$ is the encoding function for output symbol at time n, then $\mathsf{x}[m,n]=E_{m,n}\left(\{\mathsf{S}[n']\}_{n'=1}^n\right)$, $m=1,...,M$, $n=1,...,N$.

If the code is linear, the encoding functions can be represented in matrix multiplication as

$$\mathsf{X}_m=E_m(\mathsf{S})=\mathsf{S}\cdot G_m\tag{5}$$

where $\mathsf{S}\in Q^{1\times K}$, $\mathsf{X}_m\in Q^{1\times N}$ and $G_m\in Q^{K\times N}$.

Decoding. At the sink, the decoding function $D:\{Q,\star\}^{M\times N}\mapsto Q^K$ takes in the received channels symbols $\{\mathsf{Y}_m\}_{m=1}^M$, where $\mathsf{Y}_m=(\mathsf{y}[m,1]\ \dots\ \mathsf{y}[m,N])$, and maps them into the reconstructed source symbols $\widehat{\mathsf{S}}\in Q^K$. If $\widehat{\mathsf{S}}=\mathsf{S}$ for every $\mathcal{B}\in\{\mathcal{B}\}$, we say code C is *feasible* for the erasure pattern collection $\{\mathcal{B}\}$. Furthermore, each source symbol is subject to a decoding delay of T symbols. That is, let $D=\{D_n\}_{n=1}^N$, where D_n is the decoding function that outputs the reconstructed source vector $\widehat{\mathsf{S}}[n]\in Q^{K_n\times 1}$, then $\widehat{\mathsf{S}}[n]=D_n(\{\mathsf{y}[m,n']\}_{m=1,...,M,\,n'=1,...,n+T})$, $i=1,...,N$. We then say the code is *delay*-T.

The canonical Singleton bound, in the context of erasure codes, assures that the number of source symbols is no larger than the number of unerased symbols. For an M-parallel-link block erasure code feasible for L link losses and Z B-bursts, it means $MN\geq K+LN+ZB$. If $MN=K+LN+ZB$, we refer to the code as *Singleton-achieving*.

IV. The Main Result

In this section, we state the main theorem of this paper.

Theorem 1. *It is possible to construct a delay-T rate-R Singleton-achieving M-parallel-link causal block erasure code feasible for L link outages and Z B-bursts, each occurring on a separate link, if*

$$R\leq R^*:=\begin{cases}M-L-\frac{ZB}{T+B}, & T\geq B\\ M-L-Z, & T<B,\end{cases}\tag{6}$$

Conversely, if $R>R^*$, no feasible code can be constructed.

Proof: The achievability of the code is proven by construction in Section V. The converse is proven in Section VI. ∎

Theorem 1 assures the best rate that can be achieved for the case when the decoding of the source takes place either before ($T<B$) or after ($T\geq B$) the end of a single burst. An alternative view of the theorem is by obtaining the expression for T from (6):

$$T\geq\begin{cases}\max\left(\frac{Z}{M-L-R}-1,1\right)B, & R>M-L-Z\\ 0, & R\leq M-L-Z,\end{cases}\tag{7}$$

which suggests that when $R>M-L-Z$, the lowest delay that can be achieved is $T=B$, which occurs when $R=M-L-\frac{Z}{2}$. The delay T cannot be reduced further when

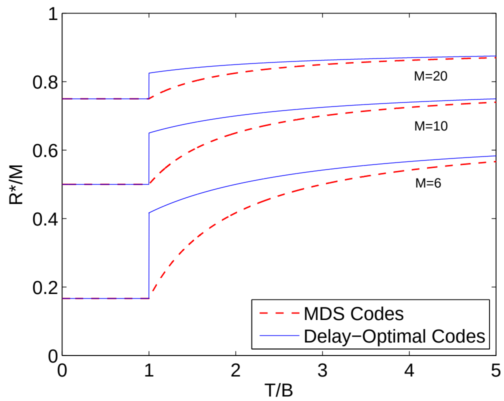

Fig. 1. Numerical comparison of the rate-delay bound for delay-optimal codes and MDS codes. In this plot, we fix $L = 2$ and $Z = 3$.

$M - L - Z < R < M - L - \frac{Z}{2}$. However, when $R \le M - L - Z$, we could have $T = 0$.

For $M = 1$ and $L = 0$, (7) degenerates to (3), which is the single-link bound in [6]. As expected, in this case it is not possible to recover an erasure before delay B. However, notice that the result in [6] applies to systematic codes only, whereas Theorem 1 applies to more general causal codes.

We plot in Fig. 1 the rate-delay bound derived in Theorem 1 for a few parameters. The result is benchmarked against the bound for codes that use symbols within delay T from different links forming a long coding block, and apply a $K \times (MN)$ generator matrix for maximum distance separable (MDS) codes. We observe that the bounds are the same at $T < B$. At $T \ge B$, the delay-optimal bound have higher rate than the bound for MDS codes, with the largest gain in low-rate region. The gain diminishes as $\frac{T}{B}$ becomes large. Furthermore, codes of larger M have a higher per-link rate $\frac{R}{M}$ than codes of smaller M. This is expected, as path diversity introduces more robustness to errors, in turn improves coding efficiency.

V. Code Construction

We consider the achievability part of Theorem 1 by constructing a code to meet the rate-delay bound in (7).

A. Encoding

Proposition 2 (Encoding)**.** *The encoding matrices G_m, $m = 1, \ldots, M$, of an delay-T rate-R^* (as in (6)) Singleton-achieving M-parallel-link causal linear block erasure code feasible for L link outages and Z B-bursts, each occurs on a separate link, can be constructed as follows. First select a field size q^m such that a $(T, T - B)$ cyclic code, an $(M, M - L)$ MDS code and an $(M, M - L - Z)$ MDS code can be constructed in the finite field $Q = \mathrm{GF}(q^m)$. Create a systematic $(T, T - B)$ cyclic code with generator matrix G_C and the corresponding single-link inner code generator matrix G_D (as in (2)). Create an $(M, M - L)$ MDS outer code with generator matrix $U \in Q^{(M-L)\times M}$ and an $(M, M - L - Z)$ MDS outer code with generator matrix $V \in Q^{(M-L-Z)\times M}$. Create a generator matrix $G \in Q^{[(M-L)T+(M-L-Z)B]\times[(T+B)M]}$ as*

$$G = \begin{pmatrix} G^U \\ G^L \end{pmatrix} := \begin{pmatrix} U \otimes G_D \\ V \otimes \begin{pmatrix} 0_{B\times T} & I_B \end{pmatrix} \end{pmatrix}$$

where $\otimes$ denotes Kronecker product, $0_{B\times T} \in Q^{B\times T}$ is an all-0 matrix and $I_B \in Q^{B\times B}$ is an identity matrix. $G^U \in Q^{[(M-L)T]\times[(T+B)M]}$ and $G^L \in Q^{[(M-L-Z)B]\times[(T+B)M]}$ are the upper and lower part of G, respectively. Divide G into M equal-size sub-matrices, i.e., $G = \begin{pmatrix} G_1 & G_2 & \ldots & G_M \end{pmatrix}$ and use $G_m \in Q^{[(M-L)T+(M-L-Z)B]\times[T+B]}$ as the generator matrix for link m, $m = 1, \ldots, M$.

Let the source vector (as in (5)) be $\mathsf{S} =: (\mathsf{s}_1\ \mathsf{s}_2\ \ldots\ \mathsf{s}_K)$. $\mathsf{S}[n]$, the source symbols injected at time n, is specified in (8).

Example. To construct a 4-parallel-link code that corrects one link outage and two 2-bursts with delay 3, we choose a $(5, 3)$ single-burst block erasure code in $\mathrm{GF}(2)$ with generator matrix G_D as

$$G_D = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

a $(4, 3)$ MDS code in $\mathrm{GF}(2)$ with generator matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and a $(4, 1)$ MDS code in $\mathrm{GF}(2)$ with generator matrix $V = (1\ 1\ 1\ 1)$. $G = (G_1\ G_2\ G_3\ G_4)$ is then constructed as in (9). Let $\mathsf{S} = (\mathsf{s}_1\ \mathsf{s}_2\ \ldots\ \mathsf{s}_{11})$. The source symbols injected at time 1 to 5 is specified as $\mathsf{S}[1] = (\mathsf{s}_1\ \mathsf{s}_4\ \mathsf{s}_7)^\intercal$, $\mathsf{S}[2] = (\mathsf{s}_2\ \mathsf{s}_5\ \mathsf{s}_8)^\intercal$, $\mathsf{S}[3] = (\mathsf{s}_3\ \mathsf{s}_6\ \mathsf{s}_9)^\intercal$, $\mathsf{S}[4] = (\mathsf{s}_{10})^\intercal$ and $\mathsf{S}[5] = (\mathsf{s}_{11})^\intercal$.

The constructed generator matrix G consists of two parts. The upper part G^U uses the single-link code G_D as the inner code at each link, and is further coded by an outer code U of rate $\frac{M-L}{M}$ applied across the links to resist up to L link outages. The upper part, if used alone, provides resistance to up to one B-burst at each link and up to L link losses. Given that the B-bursts occur only in Z links, the lower part G^L inserts more source symbols while hoping to recover them all. The inserted source symbols (corresponding to the I_B part) share the channel symbols with the parity part of G_D. Since they have to resist both L link losses and Z bursts, they are coded by the outer code V of rate $\frac{M-L-Z}{M}$.

One can verify that the constructed code is causal by observing the source symbols $\mathsf{S}[n]$ injected at each time n and the corresponding coefficients in G_m, $m = 1, \ldots, M$.

One can also verify that the constructed code has rate $R = M - L - \frac{ZB}{B+T}$. As a special case, when $T < B$, (6) suggests that we rather consider codes with $R = M - L - Z$, which would give us $T = 0$. In this case, the generator matrix degenerates to $G = V \otimes I_N$, i.e., the source symbols are coded via generator matrix V and then interleaved by N. For the single-link code with $M = 1$, $L = 0$ and $Z = 1$, G simply degenerates to G_D. For codes that only correct link outages, i.e., $Z = 0$ (or equivalently, $B = 0$), the generator matrix degenerates to $G = U \otimes I_N$.

$$\mathsf{S}[n] = \begin{cases} (\{\mathsf{s}_{(i-1)T+n}\}_{i=1}^{M-L})^{\mathtt{T}}, & n = 1, ..., T \\ (\{\mathsf{s}_{(M-L-1)T+(i-1)B+n}\}_{i=1}^{M-L-Z})^{\mathtt{T}}, & n = T+1, ..., T+B. \end{cases} \tag{8}$$

$$\begin{pmatrix} 1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0\\ 0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\ 0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&1&1\\ 0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1 \end{pmatrix} \tag{9}$$

In a slight modification to this construction, we can also construct an burst-correction-only $(M-L)$-link code correcting Z B-bursts as the inner code and an $(M, M-L)$ MDS outer code across the links. This concatenated construction, after simplification, gives a generator matrix very similar (but not identical) to the one in Proposition 2. This type of construction often requires a larger field size Q, but is useful when extending the parallel-link code to single-source multicast networks.

B. Decoding

Define ℓ_L as the set of link outages and ℓ_B the set of links where erasure bursts occur. Divide the source vector into

$$\mathsf{S} = (\ \mathsf{A}_1 \quad ... \quad \mathsf{A}_{M-L} \quad \mathsf{B}_1 \quad ... \quad \mathsf{B}_{M-L-Z}\ )$$

where $\mathsf{A}_i \in Q^T$ and $\mathsf{B}_i \in Q^B$ are the sub-vectors corresponding to G^U and G^L, respectively. Denote by $E_{G_D} : Q^T \to Q^{T+B}$ the encoding function of the single-link code with generator matrix G_D, and $D_{G_D} : \{Q, \star\}^{T+B} \to Q^T \times Q^{T+B}$ the corresponding decoding function that outputs the source symbol vector and the clean channel symbol vector. The encoding and decoding functions for U and V are defined similarly.

Proposition 3 (Decoding)**.** *The M-parallel-link causal linear block erasure code described in Proposition 2 can be decoded in the following steps:*

- *For each link $m \notin \ell_L \cup \ell_B$, mark the last B symbols of Y_m as erased and perform decoding $(\mathsf{C}_m, \mathsf{X}_m^U) = D_{G_D}(\mathsf{Y}_m)$. Compute the difference $\mathsf{X}_m^L = \mathsf{Y}_m - \mathsf{X}_m^U$, $m \notin \ell_L \cup \ell_B$ where X_m^L corresponds to the information encoded by G^L.*
- *Perform decoding $(\{\widehat{\mathsf{B}}_i\}_{i=1}^{M-L-S}, \{\mathsf{X}_m^L\}_{m=1}^M) = D_V(\{\mathsf{X}_m^L\}_{m \notin \ell_L \cup \ell_B})$. Compute the difference $\mathsf{Y}_m^U = \mathsf{Y}_m - \mathsf{X}_m^L$, $m \in \ell_B$ where Y_m^U corresponds to the information encoded by G^U (but still corrupted by the bursts).*
- *Perform decoding $(\mathsf{C}_m, \mathsf{X}_m^U) = D_{G_D}(\mathsf{Y}_m^U)$, $m \in \ell_B$.*
- *Perform decoding $(\{\widehat{\mathsf{A}}_i\}_{i=1}^{M-L}, \{\mathsf{X}_m^U\}_{m=1}^M) = D_U(\{\mathsf{X}_m^U\}_{m \notin \ell_L})$.*
- *Output $\widehat{\mathsf{S}} = (\ \widehat{\mathsf{A}}_1 \quad ... \quad \widehat{\mathsf{A}}_{M-L} \quad \widehat{\mathsf{B}}_1 \quad ... \quad \widehat{\mathsf{B}}_{M-L-Z}\ )$.*

Example. Consider the 4-parallel-link code example correcting one link outage and two 2-bursts with delay 3, described in Section V-A. Assume Link 1 is lost (i.e., $\ell_L = \{1\}$), Link 2 has $\mathsf{y}[2,1]$, $\mathsf{y}[2,2]$ erased, and Link 3 has $\mathsf{y}[3,2]$, $\mathsf{y}[3,3]$ erased (thus $\ell_B = \{2,3\}$). To decode, start with decoding channel symbols on Link 4. Mark $\mathsf{y}[4,4]$, $\mathsf{y}[4,5]$ as erased (since they are "corrupted" by the last two source symbols) and perform decoding to recover X_4^U. Note that since the code is linear, if we linearly combine three codewords, the resulting sum is still a codeword, thus the decoding can be performed successfully. After that, we recover $\mathsf{X}_4^L = \mathsf{Y}_4 - \mathsf{X}_4^U$ and perform decoding on X_4^L, which allows us to completely recover the source information corresponding to the lower part G^L. After removing these information, we are left with Y_2^U and Y_3^U, and then we can proceed to recover X_2^U, X_3^U and the corresponding source information within delay constraint. Finally, performing decoding across the links, we recover X_1^U and the corresponding source information.

VI. The Converse

In this section, we prove the converse part of Theorem 1. The converse result in [6], [8] based on the periodic erasure channel argument is applicable when the code is systematic. For the more general causal codes, a different proof technique is required. The entropy argument used in this section is based on the following reasoning. Since the feasibility of a code must be irrespective of the source's prior distribution, if we can assign the source symbols some distribution, and use the entropy inequalities to show that there exists a set of conditions (i.e. erasure patterns) that the code cannot possibly meet all at the same time, we thereby show the infeasibility of the code. We first state a useful entropy lemma:

Lemma 4. *Let* $\mathsf{X} = (\mathsf{x}_1, ..., \mathsf{x}_n)$*,* $n \geq 2$*, and* $\mathsf{X}_{\Omega\backslash i} = (\mathsf{x}_1, ...\mathsf{x}_{i-1}, \mathsf{x}_{i+1}, ..., \mathsf{x}_n)$*. If* $H(\mathsf{s}) > 0$ *and*

$$H(\mathsf{s}|\mathsf{X}_{\Omega\backslash i}) = 0,\ i = 1, ..., n, \tag{10}$$

then $H(\mathsf{X}) < \sum_{i=1}^n H(\mathsf{x}_i)$.

Proof: Consider

$$\begin{aligned} H(\mathsf{x}_1 \,|\, \mathsf{x}_2^n) &\leq H(\mathsf{s}, \mathsf{x}_1 \,|\, \mathsf{x}_2^n) \\ &= H(\mathsf{s} \,|\, \mathsf{x}_2^n) + H(\mathsf{x}_1 \,|\, \mathsf{s}, \mathsf{x}_2^n) \\ &= H(\mathsf{x}_1 \,|\, \mathsf{s}, \mathsf{x}_2^n) & (11) \\ &\leq H(\mathsf{x}_1 \,|\, \mathsf{s}, \mathsf{x}_3^n) \\ &= H(\mathsf{s} \,|\, \mathsf{x}_1, \mathsf{x}_3^n) + H(\mathsf{x}_1 \,|\, \mathsf{x}_3^n) - H(\mathsf{s} \,|\, \mathsf{x}_3^n) \\ &= H(\mathsf{x}_1 \,|\, \mathsf{x}_3^n) - H(\mathsf{s} \,|\, \mathsf{x}_3^n) & (12) \\ &\leq H(\mathsf{x}_1) - H(\mathsf{s} \,|\, \mathsf{x}_3^n) & (13) \end{aligned}$$

where x_i^j denotes $(\mathsf{x}_i, ..., \mathsf{x}_j)$. In (11) and (12) we apply (10) for $i = 1$ and $i = 2$, respectively. Apply (13), we have:

$$\begin{aligned} H(\mathsf{X}) &= H(\mathsf{x}_1 \,|\, \mathsf{x}_2^n) + H(\mathsf{x}_2^n) \\ &\leq H(\mathsf{x}_1) - H(\mathsf{s} \,|\, \mathsf{x}_3^n) + H(\mathsf{x}_2^n) \\ &\leq \sum_{i=1}^n H(\mathsf{x}_i) - H(\mathsf{s} \,|\, \mathsf{x}_3^n). \end{aligned}$$

If $H(\mathsf{s} \,|\, \mathsf{x}_3^n) > 0$, the conclusion follows; otherwise we have

$$H(\mathsf{s} \,|\, \mathsf{x}_3^n) = 0. \tag{14}$$

Similarly, apply (14), together with (10) for $i = 3$ to upper bound $H(\mathsf{x}_1^2 \,|\, \mathsf{x}_3^n)$, we end up with

$$H(\mathsf{x}_1^2 \,|\, \mathsf{x}_3^n) \leq H(\mathsf{x}_1^2) - H(\mathsf{s} \,|\, \mathsf{x}_4^n).$$

If $H(\mathsf{s} \,|\, \mathsf{x}_4^n) > 0$, the conclusion follows; otherwise we have $H(\mathsf{s} \,|\, \mathsf{x}_4^n) = 0$. Repeat the same argument until we exhaust all the cases. ∎

Let the source symbols $\mathsf{s}[m, n]$, $m = 1, ..., K_n$, $n = 1, ..., n$ be distributed i.i.d. uniform over Q. Thus

$$H(\mathsf{s}[m, n]) = H(\mathsf{s}) := \log Q. \tag{15}$$

Furthermore, since each $\mathsf{x}[m, n]$ is over Q, we have

$$H(\mathsf{x}[m, n]) \leq H(\mathsf{s}), \quad \forall m, n. \tag{16}$$

Let $W := M - L$ denote the number of links without link outage. By the Singleton-achieving assumption,

$$K = WN - BZ. \tag{17}$$

In the sequel, we discuss two cases. Note that the causality assumption is used in proving Case II, and the proof for Case I holds for general non-causal codes.

Case I: $T \geq B$, where we want to show

$$T \geq N - B. \tag{18}$$

Assume the opposite is true, for example, $T = N - B - 1$. Then $\mathsf{S}[1]$ must be recovered at time $N - B$.

Let $\mathsf{X}_{W\times(N-B)} := \mathsf{x}[1 : W, 1 : (N - B)]$ be divided into non-overlapping segments of length B except the segments in the last columns, then the number of segments in each row is $N_B := \lceil \frac{N-B}{B} \rceil$. Precisely, let the index sets be $\mathcal{I}_{m,1} = \{(1, 1), ..., (1, B)\}$, $\mathcal{I}_{m,2} = \{(1, B + 1), ..., (1, 2B)\}$, ..., $\mathcal{I}_{m,N_B} = \{(1, N_B B - B + 1), ..., (1, N - B)\}$, $m = 1, ..., W$, then $\mathsf{X}_{W\times(N-B)} = \{\mathsf{x}[\mathcal{I}_{m,i}]\}_{m=1,...,W,\, i=1,...,N_B}$. Overall, there are WN_B segments. Let them be indexed as $\mathcal{I}_{(1)} = \mathcal{I}_{1,1}$, $\mathcal{I}_{(2)} = \mathcal{I}_{2,1}$,..., $\mathcal{I}_{(W+1)} = \mathcal{I}_{1,2}$,..., $\mathcal{I}_{(WN_B)} = \mathcal{I}_{W,N_B}$.

Consider a set of $WN_B - Z + 1$ erasure patterns (refer to Table I(a) for an example):

$$\mathcal{B}_i = \{\mathcal{I}_{(j)}\}_{j=1}^{Z-1} \bigcup \mathcal{I}_{(Z-1+i)}, \quad i = 1, ..., WN_B - Z + 1.$$

That $\mathsf{S}[1]$ is recovered at $N - B$ implies

$$H(\mathsf{S}[1] \,|\, \mathsf{X}_{W\times(N-B)} \setminus \mathsf{x}[\mathcal{B}_i]) = 0, \quad i = 1, ..., WN_B - Z + 1.$$

Applying Lemma 4 with $\mathsf{s} = \mathsf{S}[1]$ and $\mathsf{x}_i = \mathsf{x}[\mathcal{I}_{(Z-1+i)}]$, $i = 1, ..., WN_B - Z + 1$, we have

$$\begin{aligned} & H(\mathsf{X}_{W\times(N-B)} \setminus \mathsf{x}[\{\mathcal{I}_{(j)}\}_{j=1}^{Z-1}]) \\ < & \sum_{i=1}^{WN_B-Z+1} H(\mathsf{x}[\mathcal{I}_{(Z-1+i)}]) & (19) \\ \leq & \sum_{\substack{\mathsf{x}[m,n]\in\mathsf{X}_{W\times(N-B)} \\ \setminus\mathsf{x}[\{\mathcal{I}_{(j)}\}_{j=1}^{Z-1}]}} H(\mathsf{x}[m, n]) \\ \leq & (K - WB + B) \cdot H(\mathsf{s}) & (20) \end{aligned}$$

where (20) follows (17) and (16).

Now, consider the erasure pattern (refer to Table I(b) for an example):

$$\mathcal{B}_0 = \{\mathcal{I}_{(j)}\}_{j=1}^{Z-1} \bigcup \{(W, N - B + 1), ..., (W, N)\}.$$

The feasibility of the code implies that all the source symbols S must be recovered from the remaining symbols $\mathsf{X}_{W\times(N-B)} \setminus \mathsf{x}[\{\mathcal{I}_{(j)}\}_{j=1}^{Z-1}]$ and $\mathsf{x}[1 : W - 1, N - B + 1 : N]$, or

$$H(\mathsf{S} \,|\, \mathsf{X}_{W\times(N-B)} \setminus \mathsf{x}[\{\mathcal{I}_{(j)}\}_{j=1}^{Z-1}], \mathsf{x}[1 : W-1, N-B+1 : N]) = 0.$$

But

$$\begin{aligned} 0 &= H(\mathsf{S} \,|\, \mathsf{X}_{W\times(N-B)} \setminus \mathsf{x}[\{\mathcal{I}_{(j)}\}_{j=1}^{Z-1}], \\ &\quad \mathsf{x}[1 : W - 1, N - B + 1 : N]) \\ &\geq H(\mathsf{S}) - H(\mathsf{X}_{W\times(N-B)} \setminus \mathsf{x}[\{\mathcal{I}_{(j)}\}_{j=1}^{Z-1}]) & (21) \\ &\quad - H(\mathsf{x}[1 : W - 1, N - B + 1 : N])) \\ &> K \cdot H(\mathsf{s}) - (K - WB + B) \cdot H(\mathsf{s}) \\ &\quad - (W - 1)B \cdot H(\mathsf{s}) & (22) \\ &= 0 \end{aligned}$$

where in (21) we use the fact that $\mathsf{x}[m, n]$ must be a function of S, implying $H(\mathsf{X}_{W\times(N-B)} \setminus \mathsf{x}[\{\mathcal{I}_{(j)}\}_{j=1}^{Z-1}], \mathsf{x}[1 : W - 1, (N - B + 1) : N] \,|\, \mathsf{S}) = 0$; (22) follows (15), (16), (20) and that the source symbols are independent. The above equations lead to contradiction; thus, the assumption is false and (18) is true.

Combining (4), (17) and (18), we obtain

$$R \leq W - \frac{BZ}{T + B}, \quad T \geq B.$$

Case II: $T < B$, where we want to show

$$N \leq B. \tag{23}$$

Assume the opposite is true, for example, let $N = B + 1$. Define two index sets

$$\begin{aligned} \mathcal{I}_{(Z-1)\times B} &:= \{(m,n)\}_{m=W-Z+2,...,W,\, n=1,...,B}, \\ \mathcal{I}_{(Z-1)\times 1} &:= \{(m,n)\}_{m=W-Z+2,...,W,\, n=N}. \end{aligned}$$

Consider the erasure pattern (refer to Table I(c) for an example):

$$\mathcal{B}_1 = \mathcal{I}_{(Z-1)\times B} \bigcup \{(1,1),...,(1,B)\}.$$

Since $\mathsf{S}[1]$ is recovered at $T+1$, where $T+1 \leq B$, we must have $H(\mathsf{S}[1] \,|\, \mathsf{x}[2:(W-Z+1), 1:(T+1)]) = 0$, or

$$\mathsf{S}[1] = f\left(\mathsf{x}[2:W-Z+1, 1:T+1]\right). \tag{24}$$

Now consider the erasure pattern (refer to Table I(d) for an example):

$$\mathcal{B}_2 = \mathcal{I}_{(Z-1)\times B} \bigcup \{(1,2),...,(1,N)\}.$$

That the code is feasible implies

$$\begin{aligned} 0 &= H(\mathsf{S} \,|\, \mathsf{x}[1:W-Z+1, 1], \\ &\quad \mathsf{x}[2:W-Z+1, 2:N], \mathsf{x}[\mathcal{I}_{(Z-1)\times 1}]) \\ &= H(\mathsf{S} \,|\, \mathsf{x}[1:W-Z+1, 1], \\ &\quad \mathsf{x}[2:W-Z+1, 1:N], \mathsf{x}[\mathcal{I}_{(Z-1)\times 1}]) & (25) \\ &\geq H(\mathsf{S} \,|\, \mathsf{S}[1], \\ &\quad \mathsf{x}[2:W-Z+1, 1:N], \mathsf{x}[\mathcal{I}_{(Z-1)\times 1}]) & (26) \\ &= H(\mathsf{S} \,|\, f\left(\mathsf{x}[2:W-Z+1, 1:T+1]\right), \\ &\quad \mathsf{x}[2:W-Z+1, 1:N], \mathsf{x}[\mathcal{I}_{(Z-1)\times 1}]) & (27) \\ &\geq H(\mathsf{S} \,|\, \mathsf{x}[2:W-Z+1, 1:N], \mathsf{x}[\mathcal{I}_{(Z-1)\times 1}]) & (28) \\ &= H(\mathsf{S}) \\ &\quad -H(\mathsf{x}[2:W-Z+1, 1:N], \mathsf{x}[\mathcal{I}_{(Z-1)\times 1}]) & (29) \\ &\geq [K-(W-Z)N-(Z-1)] \cdot H(\mathsf{s}) & (30) \\ &= H(\mathsf{s}) & (31) \end{aligned}$$

where in (25) we repeat the conditioning $\mathsf{x}[2:W-Z+1, 1]$, (26) follows the causality of code, thus $\mathsf{x}[1:W-Z+1, 1]$ must be a function of $\mathsf{S}[1]$, (27) follows (24), (28) follows that $T+1 \leq N$, (29) follows that $\mathsf{x}[m,n]$ is function of S for all m, n, (30) follows (15) and that the source symbols are independent, and (31) follows (17). This leads to contradiction. Therefore, the assumption is false, and we have (23). Combining (4), (17) and (23), we obtain

$$R \leq W - Z, \;\; T < B.$$

VII. Conclusion

We have studied the burst erasure correction problem for parallel-link networks with link outage. Our preliminary results show that this class of codes is both theoretically interesting and practically relevant. In the sequel, we outline a few future research problems. 1) Our discussion is restricted to Singleton-achieving codes. A natural question to raise is, can non-Singleton-achieving codes beat the rate-delay bounds in (6)? Our conjecture is no, but it remains a open problem for future research. 2) It will be interesting to extend the parallel-network code to more general scenarios, for example, single-source multicast. In fact, in, we have a simple example [11] to show that this is possible. 3) For practical implementation, it will be interesting to consider how to design the code when link delay is also taken into account.

TABLE I
Illustration of the erasure patterns used in proving the converse. $\star$ denotes an symbol erasure.

Time	1	2	3	4	5
Link 1	$\star$	$\star$			
Link 2	$\star$	$\star$			
Link 3					

(a) Case I: $\mathcal{B}_1$. $W=3$, $Z=2$, $B=2$, $N=5$.

Time	1	2	3	4	5
Link 1	$\star$	$\star$			
Link 2					
Link 3				$\star$	$\star$

(b) Case I: $\mathcal{B}_0$. $W=3$, $Z=2$, $B=2$, $N=5$.

Time	1	2	3	4	5
Link 1	$\star$	$\star$	$\star$	$\star$	
Link 2					
Link 3	$\star$	$\star$	$\star$	$\star$	

(c) Case II: $\mathcal{B}_1$. $W=3$, $Z=2$, $B=4$, $N=5$.

Time	1	2	3	4	5
Link 1		$\star$	$\star$	$\star$	$\star$
Link 2					
Link 3	$\star$	$\star$	$\star$	$\star$	

(d) Case II: $\mathcal{B}_2$. $W=3$, $Z=2$, $B=4$, $N=5$.

Acknowledgment

The authors would like to thank Ashish Khisti, Emin Martinian, Xiaoqing Zhu, Jinhua Jiang and Yuxin Chen for constructive discussions.

References

[1] R. W. Yeung and N. Cai, "Network error correction, part I: basic concepts and upper bounds," *Communications in Information and Systems*, vol. 6, no. 1, pp. 19–36, 2006.

[2] N. Cai and R. W. Yeung, "Network error correction, part II: lower bounds," *Communications in Information and Systems*, vol. 6, no. 1, pp. 37–54, 2006.

[3] Z. Li, X. Zhu, A. Begen, and B. Girod, "Peer-assisted packet loss repair for IPTV video multicast," in *Proc. ACM Multimedia*, Beijing, China, Oct. 2009.

[4] T. Nguyen and T. Zakhor, "Multiple sender distributed video streaming," *IEEE Transactions on Multimedia*, vol. 6, no. 2, pp. 315–326, 2004.

[5] E. Setton, P. Baccichet, and B. Girod, "Peer-to-peer live multicast: A video perspective," *Proceedings of the IEEE*, vol. 96, no. 1, pp. 25–38, 2008.

[6] C.-E. W. S. E. Martinian, "Burst erasure correction codes with low decoding delay," *IEEE Trans. on Information Theory*, vol. 50, pp. 2494–2502, October 2004.

[7] E. Martinian and M. Trott, "Delay-optimal burst erasure code construction," in *Proc. International Symposium on Information Theory*, July 2007.

[8] E. Martinian, "Dynamic information and constraints in source and channel coding," *Ph.D. Thesis, Massachusetts Inst. of Technology*, September 2004.

[9] A. Badr, A. Khisti, and E. Martinian, "Diversity embedded streaming codes: Construction and optimality," in *Proc. Globecom*, 2010.

[10] A. Neubauer, J. Freudenberg, and V. Kuhn, *Coding Theory Algorithms, Architectures and Applications*. John Wiley & Sons, 2007.

[11] Z. Li, A. Begen, and B. Girod, "Delay-optimal burst erasure codes for parallel links, extension to single-source multicast: An example," Available online: http://www.stanford.edu/$\sim$leeoz/multicastexample.pdf.