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A log-normal distribution and two-sample tests for the full diffusion tensor

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PURPOSE

To develop formal testing tools for group comparisons of DTI maps using eigenvalues and eigenvectors, or both.

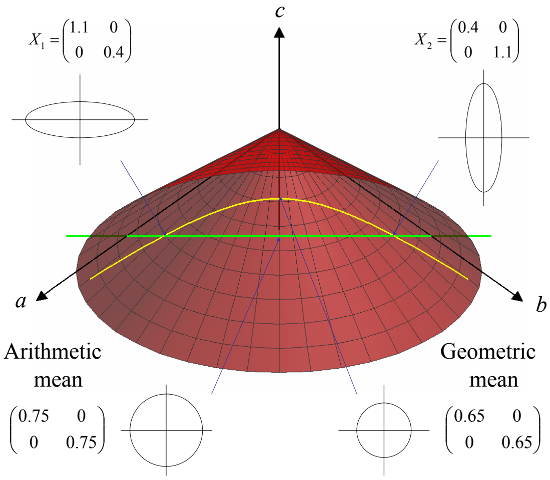
BACKGROUND

- DTI data: 3 × 3 positive definite matrix at each voxel.
- Previous DTI group studies use only scalar measures (e.g. FA – Deutsch et al.) and principal diffusion direction (Schwartzman et al.).
- New statistical methods needed for testing full set of eigenvalues and eigenvectors in tensor imaging data.

METHODS

The space of positive definite matrices:

- E.g.: 2 × 2 case: $\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0 \iff \begin{cases} a > 0, b > 0 \\ ab - c^2 > 0 \end{cases}$



- Straight line violates boundaries.
- Log-straight line stays within space.

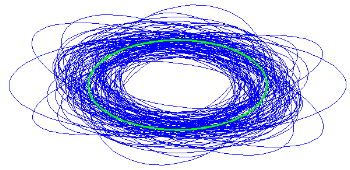
Log transform:

- Scalars: $x \in \mathbb{R}^+ \underset{\text{exp}}{\overset{\text{log}}{\rightleftarrows}} y \in \mathbb{R}$
- Matrices: $X \in \text{Sym}^+(p) \underset{\text{exp}}{\overset{\text{log}}{\rightleftarrows}} Y \in \text{Sym}(p)$
- Positive definite matrices are transformed into symmetric matrices with no restrictions on their coefficients.

$$X_i = V_i \Lambda_i V_i' \quad \Rightarrow \quad Y_i = \log(X_i) = V_i \log(\Lambda_i) V_i'$$

Normal distribution for symmetric matrices:

- Standard normal: $f(Z) = \frac{1}{(2\pi)^{q/2}} \exp\left(-\frac{1}{2}\text{tr}(Z^2)\right) \quad \Rightarrow \quad Z = \begin{pmatrix} N(0,1) & N(0,\frac{1}{2}) & N(0,\frac{1}{2}) \\ * & N(0,1) & N(0,\frac{1}{2}) \\ * & * & N(0,1) \end{pmatrix}$ $q = 6$ (# independent entries)
- Mean M, variance σ^2: $f(Y) = \frac{1}{(2\pi)^{q/2}\sigma^q} \exp\left(-\frac{1}{2\sigma^2}\text{tr}(Y - M)^2\right)$
- Simulated example: $\exp(M) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma = 0.5, \quad n = 100$



Two-sample testing:

- Estimates (assume common variance σ^2): $\bar{Y}_1 = V_1 \Lambda_1 V_1', \quad \bar{Y}_2 = V_2 \Lambda_2 V_2', \quad \bar{Y} = V \Lambda V'$, $\quad s^2 = \frac{1}{q(n-2)}\left(\sum_{i=1}^{n_1} \text{tr}(Y_i - \bar{Y}_1)^2 + \sum_{i=n_1+1}^{n} \text{tr}(Y_i - \bar{Y}_2)^2\right)$
- Test H_0: $M_1 = M_2$:

$$T = \frac{n_1 n_2}{qns^2}\text{tr}(\bar{Y}_1 - \bar{Y}_2)^2 \underset{H_0}{\sim} F(q, q(n-2))$$

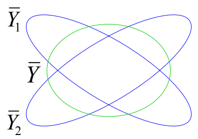
- Test H_0: M_1, M_2 have same eigenvalues (regardless of eigenvectors):

$$T = \frac{n_1 n_2}{pns^2}\text{tr}(\Lambda_1 - \Lambda_2)^2 \underset{H_0}{\overset{n\to\infty}{\sim}} F(p, q(n-2))$$

- Test H_0: M_1, M_2 have same eigenvectors (assume same eigenvalues):

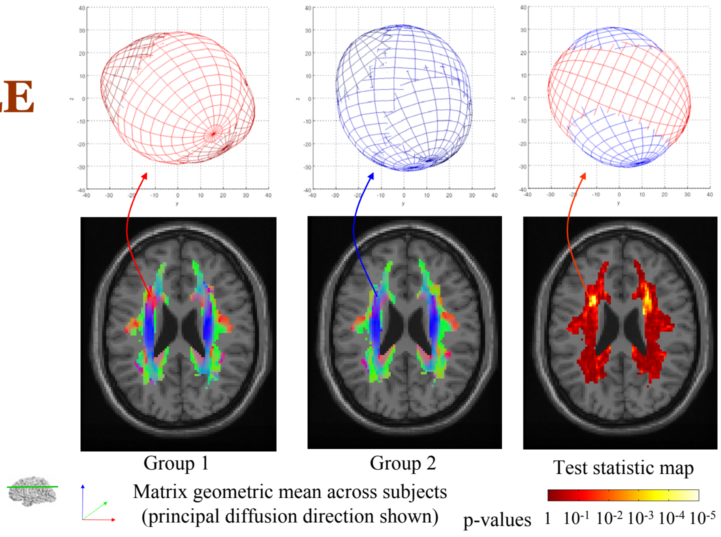
$$T = \frac{n}{(q-p)s^2}\text{tr}\left(\left(\frac{n_1\Lambda_1 + n_2\Lambda_2}{n}\right)^2 - \Lambda^2\right) \underset{H_0}{\overset{n\to\infty}{\sim}} F(q-p, q(n-2))$$

(↑ 3, ↑ 60)



DATA EXAMPLE

- Source Deutsch et al.: 2 groups, 6 subjects in each group.
- Voxelwise test for equality of eigenvectors between group means.
- E.g.: voxel with direction differences in corpus callosum / corona radiata area (Schwartzman et al.)



CONCLUSIONS

We have developed an integrated representation of diffusion tensors in log-space including:

- Removal of positive-definiteness constraints.
- A normal distribution for symmetric matrices.
- Formal two-sample tests of full tensor, eigenvalues or eigenvectors.

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References

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