Hyper-elastoplastic/damage modeling of rock with application to porous limestone

K.C. Bennett a,b, R.I. Borja a,*

a Department of Civil and Environmental Engineering, Stanford University, Stanford, CA 94305, USA
b Fluid Dynamics and Solid Mechanics Group, Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

1. Introduction

Porous rocks exhibit microfracture and changes in porosity under inelastic deformation, which affect their strength and elastic properties. Describing their inelastic deformation by means of classical plasticity theory typically requires attributing changes in the relative void space (porosity) to bulk plasticity, representing dilatant and compactive volume changes (e.g., Drucker, 1957; Borja and Tamagnini, 1998; Possam et al., 2000; Borja and Choo, 2016, among others). Such models often incorporate phenomenological hardening/softening rules (e.g., Borja and Lee, 1990; Simo and Meschke, 1993; Borja and Tamagnini, 1998; Tamagnini and Ciantia, 2016). Including the evolution of elastic properties with microfracture and changes in porosity typically requires introducing either empirical dependence of elastic moduli (and hardening moduli) on the extent of plastic strain (cf. Borja and Lee, 1998, among others) or, alternatively, a ductile damage law which relates the damaged state of the material to the extent of plastic strain (cf. Hueckel, 1976; Ortiz, 1985; Ju, 1989, among others).

Continuum damage mechanics (CDM), generally attributed to being first described by Kachanov (1958), provides a description of elastic softening ascribed to material degradation by microcracking and/or void growth. The coupling of CDM with elastoplasticity, i.e., ductile damage theory, further provides a representation of CDM as an energy-dissipative process, allowing for a thermodynamically consistent representation of the material’s elastic constitutive response dependent on its damaged state (cf. Lemaitre, 1985a, 1985b; Ju, 1989). In ductile damage theory, material damage affects both the elastic response and the material’s strength by identifying the damage-effective stress, which provides a physically meaningful description of how a material becomes weakened under microcrack/void growth (cf. Lemaitre and Desmorat, 2005). Combining hyper-elastoplasticity theory with a thermodynamically consistent description of damage for geomaterials poses longstanding challenges, especially with respect to dilation and compaction (cf. Simo and Ju, 1987; Krajcinovic, 1996; Arson, 2014).

Relations between Eshelby's stress tensor (Eshelby, 1951; 1956; 1973) and changes to a material's free energy state during damage have long been recognized (e.g., Rice, 1968; Maugin, 1994; Brünig, 2004). Recently, a description of the role of the Eshelby stress specifically in describing the hyper-elastoplasticity of materials that undergo large inelastic changes in volume (such as geomaterials) was provided by Bennett et al. (2016). The implications of recognizing this are examined herein with respect to ductile damage theory of porous rocks, and a novel hyper-elastoplastic ductile damage framework is provided. This work, contrary to that of Bennett et al. (2016), explores the relevance of the Eshelby stress in formulating ductile damage constitutive models. Notably, the constitutive theory unifies concepts of volumetric (also known as "dilative" or "adhesive") damage and dilation/compaction in a...
thermodynamically consistent framework. The present work is motivated also by scanning electron microscope (SEM) observations by the present authors of extensive micro-fracture and changes in micro-void space induced by nanoindentation of a Woodford shale, as described by Bennett et al. (2015) and shown in Fig. 1.

Observations of microfracture in cohesive geomaterials are well documented in the literature (Cuss et al., 2003; Wong and Baud, 2012; Bennett et al., 2015; Abousoleiman et al., 2016; Hu et al., 2017; Sennani and Borja, 2017). Over the years, numerous continuum damage models have been proposed to model this type of damage in rocks, both isotropic (e.g., Hajabdolmajid et al., 2002; Ricard and Bercovici, 2003; Hamiel et al., 2004; Liu et al., 2016; Tran-Manh et al., 2016) and anisotropic damage (e.g., Chen et al., 2010; Voyiadjis et al., 2012; Arson and Gatzmini, 2012), including also application to damage zones at faults and joints (White, 2014; Johri et al., 2014). The description of porosity changes as a damage and/or healing phenomena has, further, particularly been a topic of recent interest (Ju et al., 2012; Zhu and Arson, 2015; Arson and Pereira, 2013; Arson and Vanorio, 2015; Le Pense et al., 2016). Micromechanical descriptions of damage have been provided for (non-cohesive) granular (Zhu et al., 2010) and cohesive (Chazallon and Hicher, 1998; Zhu et al., 2008; Gu et al., 2008; Zhu et al., 2011; Shen et al., 2012) geomaterials, with micromechanical descriptions of porosity and damage provided in the context of Modified Cam-Clay material model (Bignonnet et al., 2016) and the degradation of cemented clays (Ju et al., 2012; Nguyen et al., 2014). The instability of granular assemblies (Andrade et al., 2008; Chang and Meidani, 2013; Misra and Poorsolhjouy, 2015b; Wang et al., 2016) can also be described as a damage phenomenon (cf. Misra and Poorsolhjouy, 2015a).

The equating of damage with pore growth appears to have been first proposed by Gurson (1977), and has since found much utility by others (Salari et al., 2004; Arson and Pereira, 2013; Lebensohn et al., 2013; Bronkhorst et al., 2016; Bennett et al., 2018). Contrarily, some have ascribed damage purely to microfracture without induced void space (e.g., Zhu et al., 2008; Chen et al., 2010; Buechler and Luscher, 2014). Another approach that effectually combines some volumetric damage with deviatoric damage contributions was proposed by Ortiz (1985) for concrete, and has since been widely used for describing damage of rocks (e.g., Chiarelli et al., 2003; Parisio et al., 2015, among others), where it is assumed only the positive principal stresses contribute to damage. Various alternative approaches have been proposed to account for both volumetric and isotropic damage separately (Lee and Fenves, 1998; Bažant et al., 2000; Clayton, 2006; Bakhttary et al., 2014; White, 2014; Shojai et al., 2014; Clayton and Tonge, 2015; Grigic, 2016, among others).

Herein, a kinematic and thermodynamic description is provided relating phenomenological damage parameters to both volumetric and isotropic deformation attributable to changes in porosity and micro-fracture, respectively. The role of a spatial Eshelby-like stress tensor in the damage-constitutive equations is described, which is shown to be equivalent to the “Eshelby-zeta” stress tensor described by Bennett et al. (2016), but who made no connection with damage mechanics. The coupling of critical state and ductile damage theory is provided in terms of a novel Drucker-Prager/Damage (DP-D) material model that consists of a smooth transition into a pressure cap of the yield surface. Two distinct mechanisms of damage are considered: (1) volumetric damage attributed to changes in void ratio (i.e., porosity), and (2) isotropic damage attributed to the development of microcracks—with or without an associated void space opening (i.e., exclusive of their crack aperture). This unifies two common but disparate descriptions of continuum damage, where the former description does not lend itself well to describing a statistical distribution of microcrack orientations, e.g., as necessary for describing anisotropic damage (c.f. Kenlchi, 1984; Krajcinovic, 1996; Voyiadjis et al., 2012), and the latter does not account for crack openings along their length (apertures). The two types of damage are developed with attention to thermodynamic restrictions, in particular the evolution of volumetric damage is shown to arise directly from thermodynamic principles, where damage is predicted to increase under dilation and reduce in compression. Considering that this provides the connection between ductile damage and inelastic volumetric strain, the dependence of the state of damage on isochoric plastic deformation is then introduced through a phenomenological isotropic (also called deviatoric or shear) damage parameter, which has a clear physical meaning as a measure of the shear induced microfracture. In order to examine the ability of the proposed framework to represent hardening/softening behavior through the evolution of damage, no other phenomenological hardening parameters are introduced, i.e., the proposed framework is not a mixed hardening/damage model but a purely ductile damage one.

The DP-D material model is developed for finite deformations, and the model is implemented with Abaqus non-linear finite element modeling software within a UMAT material subroutine. Calibration to the triaxial compression (TC) tests of Vajdova et al. (2004) on Tavel limestone are provided to show how the model is capable of capturing observed stress-strain and hardening/softening behavior, in particular the observed pressure dependence of the so-called brittle-ductile transition described in that work. Simulation of a horizontal wellbore in porous limestone as described by Coelho et al. (2005) is also conducted, and the model predictions of damage around the wellbore are examined. We note that issues of objectivity as they relate to mesh size dependence and localization of damage are not addressed herein, as the present model is meant to address a (local) continuum description of damage. The need to integrate global descriptions of damage and fracture (e.g., Clayton and Knap, 2015; Li et al., 2015; Zhang et al., 2016) with continuum damage theory (cf. Chen et al., 2000; Gao and Huang, 2003; Bazant, 2010) remains an issue of much present research effort (e.g., Verhoosel et al., 2011; Li et al., 2015; Heyden et al., 2015; Weed et al., 2017; Tjoe and Borja, 2016), and is beyond the scope of this present work. We also note that the present work is restricted to material isotropy, while extension to anisotropic ductile damage is part of ongoing research efforts. We refer to the works of Nova (1980); Cazacu et al. (1998); Sennani et al. (2016); Jiang et al. (2017) among others for a description of the anisotropic plasticity of geomaterials and Al-Rub and Voyiadjis (2003); Kondo et al. (2010); Chen et al. (2010) among others for a description of anisotropic damage coupled with anisotropic plasticity.
2. Theory describing damage, porosity, and micro-fracture

In the following, a description of the kinematics and adopted notation is first given. The description is made in as concise a manner as possible since the emphasis of the work is the development of the damage-constitutive model. We refer to Bennett et al. (2016) for further kinematic and thermodynamic background relevant to the current work. Throughout, tensors are written in boldface and their components are specified by subscript index notation. Summation convention is assumed for repeated indices. Matrix notation for the inner product is assumed, whereas dyadic products are indicated by \( \otimes \). The double contraction of indices is indicated by double dots (\( \cdot \cdot \cdot \)). For example, the second order identity tensor is represented as \( I \) and its components by the Kronecker delta \( \delta_{ij} \), such that the trace of a second order tensor is given by \( \text{tr}(\mathbf{a}) := (\mathbf{a} \cdot \mathbf{1}) = \delta_{ij} a_{ij} \). We make use of similar standard notions for the natural log \( \ln \cdot \), the deviatoric operator \( \text{dev}(\mathbf{a}) := (\mathbf{a} - 1/3 \text{tr}(\mathbf{a}) \mathbf{1}) \), and the symmetric operator \( \text{sym}(\mathbf{a}) := 1/2(\mathbf{a} + \mathbf{a}^T) \), where the \( T \) superscript denotes the transpose.

The concept of a representative elementary volume (REV) is assumed, such that the material properties at a point describe the microstructure, including herein microcracks and also void space. Typically, fluids may fill the voids with time dependent dissipation giving rise to effective stresses (Biot, 1941; Terzaghi, 1944; Borja, 2006). All stresses described herein are always considered to be in this regard effective stresses, and we note that the model presented here could be applied to coupled fluid/solid problems with the addition of fluid phase diffusion laws (e.g., Choo and Borja, 2015; Wang and Sun, 2016).

We adopt standard mechanics sign convention, where, for example, positive mean stress in compressive; however, when comparing to measurements where opposite (geomechanics) convention is used we sometimes report negative values in order to match those figures and also adopt, for example, the convention for positive “axial” strain to imply compression. Conventional notation for the first stress invariant \( I_1^\sigma := \text{tr}(\sigma) \) and the second deviatoric stress invariant \( I_2^\sigma := 1/2 \text{dev}(\sigma) : \text{dev}(\sigma) \) is adopted, where \( (\sigma) \) denotes a stress measure. The mean stress (of any particular stress measure) is then defined by \( p^\sigma := 1/3 I_1^\sigma \), and the Mises stress as \( q^\sigma := \sqrt{2 I_2^\sigma} \). Consistent with convention, unspecified description of “mean stress” or “Mises stress” implies with respect to Cauchy stress.

The deformation gradient is a function of position and time, given by

\[
F := \dot{F}(X, t) = \frac{\partial \varphi(X, t)}{\partial X},
\]

where \( \varphi(X, t) \) describes a deformation from the reference to current configurations of the material, i.e. \( \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \). The multiplicative split of the deformation gradient into elastic and plastic parts is assumed, such that \( F = F^E F^p \). The left Cauchy-Green deformation tensor (finger tensor) of \( F \) is given by

\[
b := FF^T; \quad b_{ij} := F_{ik} F_{jk},
\]

and the logarithmic strain (Hencky strain) by

\[
\varepsilon := \frac{1}{2} \ln[b]; \quad e_{ij} := \frac{1}{2} \ln[b_{ij}].
\]

The natural volume strain is then defined as \( \varepsilon_v := \varepsilon : 1 = \ln[J] = \varepsilon_v^p + \varepsilon_v^e \) such that the natural deviatoric strain is \( \varepsilon := \varepsilon - 1/3 \varepsilon_v^e 1 \), where \( J := \det[F] = f^p \) is the Jacobian determinant. This allows for a representation of large strain elastoplasticity that follows closely the small strain version, as has been described by Simo (1992, 1996) and others. Note that intermediate configuration quantities are represented with an over-bar, current configuration in lower-case, and reference configuration in upper-case, whenever possible. For example infinitesimal volumes are mapped by the Jacobian determinant, \( f \) by \( df = f^p d\bar{V} \) and \( d\bar{V} = J^p dV \), where \( f = f^p \).

Two scalar variables of damage, \( d^e \) and \( d^\ell \), are distinguished, associated with volumetric and isochoric damage, respectively. The volumetric part \( d^e \), which we call also the comprehensive damage parameter, is isotropic because it is directly related to porosity, which cannot be given a statistically preferential direction or alignment according to standard REV theory (cf. Hill, 1963; Borja, 2006).

An advantage of the present model is that because it separates the volumetric from the isochoric contributions to the overall damage, it allows for future work to consider the isochoric damage to be anisotropic. This is significant because an anisotropic damage tensor can then be assembled from the statistical distribution of microcracks when the cracks are considered to have no opening by the method first proposed by Lubarda and Krajcinovic (1993) (see also Voyiadis et al., 2012), which is analogous to the development of the micro-mechanical fabric tensor used to represent the distribution of particle contact planes in granular assemblies (e.g., as described in Chang and Bennett, 2015). We emphasize, however, that the present work considers only an isotropic distribution of microcracks, allowing for the isochoric damage to be represented by a scalar variable. Subsequently, it will be shown that \( d^\ell \) is taken as a non-linear function of its conjugate energy release rate \( \gamma \), similar to the general form proposed by Lemaître and Chaboche (1994), \( d^\ell = \hat{d}^\ell(\gamma) \). The terms “deviatoric damage” and “shear damage” are variably used in the literature for what we call here “isochoric damage,” and we use these terms interchangeably as well.

The porosity \( n \) is the ratio of the volume of the voids to the total volume in the current configuration, i.e. \( n := dV^v / dV \), where the total volume is the sum of the solid and void volumes, i.e., \( dV = dV^s + dV^v \). Changes in porosity result from both plastic and elastic deformation; however, it is convenient to define the porosity relative to the plastically-deformed/elastically-unloaded state, i.e., the intermediate configuration \( \bar{\varepsilon} \). In this case, the definition of porosity becomes \( \bar{n} := d\bar{V}^v / d\bar{V} \). This is the definition of porosity used in this work, which is the assumption that change in porosity is attributed to plastic volumetric deformation, and plastic volumetric deformation is likewise attributed solely to changes in porosity. This assumption is analogous to the assumption that \( f^p \approx 1 \), which is reasonable if the elastic volume change is negligible in relation to inelastic volume change, for which case \( n \approx \bar{n} \). We note that this assumption is commonly justifiable for geomaterials. For example, consider triaxial compression tests where initial and final specimen porosities (before loading and after unloading) can differ significantly relative to the elastic volume strain exhibited during the test. A further justification for this assumption is that the solid grains are expected to exhibit elastic volumetric deformation but not plastic. Hence, all of the plastic deformation is reasonably attributed to changes in porosity, but the separation of bulk elastic deformation into that of the solid and that of the void fractions is not obvious without making some other assumptions (or perhaps higher resolution modeling, e.g., micromechanical modeling). With this definition of porosity, the plastic Jacobian determinant, \( J^p \), is related to porosity by

\[
J^p = \frac{\rho_0}{\rho} = \frac{d\bar{V}}{dV} = \frac{dV^s + dV^v}{dV} = \frac{dV - dV^s + d\bar{V}^v}{dV} = 1 - n_0 + d\bar{V}^v / dV,
\]

where because of near-incompressibility of the solid constituent, \( dV = dV^s = d\bar{V}^s \), and the subscript 0 is used throughout to denote initial (or reference) values. Similarly, the definition of porosity provides
\[ n := \frac{d\rho}{dV} \approx \frac{dV^\ast}{dV} = \frac{d\bar{V} - dV^\ast}{d\bar{V}} = \frac{d\bar{V} - dV + dV^\ast}{dV} = \frac{dV}{dV} + \frac{f_0 dV}{dV} = 1 - \frac{1 - n_0}{J_0^p} . \]  

(5)

Since we should expect that damage is at least in part attributable to changes in porosity (see references in the introduction), we identify that there is a volumetric damage parameter proportional to porosity by \( \alpha \), i.e., \( d^c = \alpha n \). It is noted that this relationship could in general be nonlinear, where, for example, \( \alpha \) could depend on the plastic strain history. However, we do not need to specify \( \alpha \) explicitly here nor make any assumptions about how \( d^c \) is related to the total damage, only recognize the proportionality for consideration in Section 2; the precise form will be determined by the choice of constitutive equations in Section 4, where an expression for \( d^c \) as a function of volumetric plastic deformation and model parameters is obtained. The expression for the proportionality between volumetric damage and \( J_0^p \) can thus be written as

\[ d^c = \alpha n = \alpha \left(1 - \frac{1 - n_0}{J_0^p}\right) , \]  

(6)

or in other words,

\[ J_0^p = \frac{1 - n_0}{1 - \frac{d^c}{\alpha \epsilon}} . \]  

(7)

The relation between \( J_0^p \) and \( d^c \) in Eq. (7) is remarkably similar to the relation often used in modeling spall damage of metals at large strains, which appears to be first obtained by Davison et al. (1977), the difference being in only the scaling factor \( \alpha \) and the initial porosity. The relation between changes in damage and porosity is then found to be given by

\[ \Delta d^c = \alpha \Delta n = \alpha (n - n_0) = \alpha (1 - n_0) \left(1 - \frac{1}{J_0^p}\right) . \]  

(8)

Eq. (8) provides an explicit expression for the plastic Jacobian determinant in terms of the damage increment when volumetric damage is related to volumetric plasticity,

\[ J_0^p = \frac{1}{1 - \frac{\Delta d^c}{\alpha (1 - n_0)}} . \]  

(9)

We emphasize that this relationship is based only on the classical assumption that there is some proportionality between damage and porosity and that we do not require \( \alpha \) to be a fixed material parameter in the derivation of our material model in what follows. It will be shown in Section 5 that the proposed model can be used to solve for \( \alpha \) in Eq. (8) by providing \( [p^s, d^s] \).

### 3. Thermodynamic consistency

The second law of thermodynamics, neglecting temperature effects, can be expressed in the form of the Clausius-Planck inequality (cf. Truesdell and Noll, 2004; Simo, 1998),

\[ \int_{\partial\Omega} (d : \sigma - \rho \Psi) \, dV \geq 0 , \]  

(10)

where \( d = \text{sym}(\mathbf{F}\mathbf{F}^{-1}) \) is the deformation rate, \( \sigma \) is the Cauchy stress, \( \rho \) is density, and \( \Psi \) is the specific Helmholtz free-energy. Bennett et al. (2016) have shown that for a material undergoing bulk plasticity this implies that the hyperelastic constitutive equation for the stress is given for the current configuration representation of elastoplasticity in terms of the so-called “zeta” stress, \( \zeta \), by

\[ \zeta := f^s = 2 \frac{\partial (\hat{\rho} \Psi)}{\partial \mathbf{b}} \mathbf{b}^s . \]  

(11)

In order for a material model to be thermodynamically consistent, it was shown in Bennett et al. (2016) that the plastic dissipation \( (D) \) must satisfy

\[ D = \xi : \mathbf{d}^p - \mathbf{q} : \mathbf{z} \geq 0 , \]  

(12)

where \( \mathbf{d}^p \) is the plastic part of the deformation rate \( \mathbf{d} = \mathbf{d}^p + \mathbf{d}^s \), \( \mathbf{z} \) and \( \mathbf{q} \) are respectively vectors of internal state variables (ISVs) and their conjugate thermodynamic forces, and

\[ \xi := f^s - (\hat{\rho} \Psi) \mathbf{1} = \zeta - (\hat{\rho} \Psi) \mathbf{1} \]  

(13)

is an Eshelby-like stress measure we call the Eshelby-zeta stress.

In this work, the free-energy \( \Psi \) is assumed to consist of only the elastic stored energy and internal state variables associated with damage (e.g., there is no additional hardening potential). It is taken to be of the functional form,

\[ \Psi = \Psi(\mathbf{b}^s, d^s, d^c) , \]  

(14)

where \( \mathbf{b}^s \) is the elastic Finger tensor, and \( d^s \) and \( d^c \) are ISV’s describing the damaged state of the material. The dissipation inequality hence takes the form

\[ D = \xi : \mathbf{d}^p - \mathbf{q} : \mathbf{d}^c - \mathbf{y}^s d^s \geq 0 , \]  

(15)

where \( \mathbf{y}^s = \hat{\partial}_\mathbf{d} (\hat{\rho} \Psi) \) and \( \mathbf{y}^c = \hat{\partial}_d (\hat{\rho} \Psi) \) are the energy release rates associated with comprehensive (i.e., reduction of the total free-energy) and purely deviatoric damage (i.e., microfractures), respectively.

Under the supposition of non-associated flow, the dissipation potential \( \mathcal{G} \) and yield function \( \mathcal{F} \) are taken to be of the forms, respectively,

\[ \mathcal{G} := \hat{\mathcal{G}}(\xi, y^c, y^s, d^s, d^c) ; \quad \mathcal{F} := \hat{\mathcal{F}}(\xi, y^c, y^s, d^s, d^c) , \]  

(16)

with the dissipation potential taken to be of the additive form

\[ \mathcal{G} := \mathcal{G}^p(\xi, y^c, d^s, d^c) + \mathcal{G}^d(\xi, y^s, d^s, d^c) = \mathcal{G}^p(\xi, y^c, d^s, d^c) + \mathcal{G}^d(\xi, y^s, d^s, d^c) , \]  

(17)

where \( \mathcal{G}^p \) is the (non-associative) plastic potential and the \( \mathcal{G}^d(\xi, y^c, d^s, d^c) \) is the damage potential, assumed to be additively decomposed into comprehensive and deviatoric parts, \( \mathcal{G}^c(\xi, y^c, d^s, d^c) \) and \( \mathcal{G}^d(\xi, y^d, d^c) \), respectively. Although additive decomposition into plastic and damage parts is typical (cf. Lemaître and Desmorat, 2005), the assumption that the damage potential can be further additively decomposed is justified by recognizing the functional dependence of the plastic potential on \( y^c \) due to the Eshelby-zeta stress (but not \( y^c \)). This motivates the concept adopted here that \( \mathcal{G}^c \) can be somehow related to \( \mathcal{G}^p \). In anticipation of providing a damage evolution equation consistent with Eq. (10), we take the form of this relation to be

\[ \mathcal{G}^c : = \mathcal{C}_0 \mathcal{G}^p . \]  

(18)

This provides a convenient canonical form of the comprehensive damage evolution equation in terms of the plastic potential,

\[ \dot{d}^c = -\lambda \frac{\partial \mathcal{G}^c}{\partial y^c} = -\lambda \mathcal{C}_0 \frac{\partial \mathcal{G}^p}{\partial y^c} , \]  

(19)

where \( \lambda \) is the plastic multiplier. Similarly, the shear damage evolution equation is given by

\[ \dot{d}^{s} = -\lambda \frac{\partial \mathcal{G}^s}{\partial y^s} . \]  

(20)

---

1 Equivalence to the spall damage expression of Davison et al. (1977) is obtained by setting \( \alpha = 1 \) in the absence of initial porosity.
and the flow rule by
\[ d^\iota = \lambda \frac{\partial \mathcal{G}^p}{\partial \mathbf{\varepsilon}}. \]  

(21)

Along with the standard so called KKT and consistency conditions (cf. Borja, 2013), this provides the complete canonical form of the (hyper-elastoplastic/damage) constitutive and evolution equations.

4. Modeling choices and implementation

The material model presented in this section can be called a Drucker-Prager/Damage (DP-D) model. It is loosely based on the Drucker-Prager type model first presented by Regueiro and Ebrahimi (2010) and further modified by Bennett et al. (2016). This model incorporates a two-invariant Drucker-Prager yield surface with a smooth transition to a pressure cap (DP-C), and has been further developed herein (incorporating damage) especially for the purpose of modeling the constitutive behavior of rocks. It is noted, however, that Drucker-Prager type models inclusive of a pressure cap have found broad application in the modeling of porous materials in general, and the model as presented here may be similarly extendable (with proper selection of model parameters). It is emphasized that the relationship that is described in the following between the damage-energy release rate and the Eshelby-zeta stress of Eq. (13) provides a useful connection between enforcing thermodynamic consistency and providing a description of the hardening/softening behavior of a porous material.

4.1. Helmholtz free energy

The free-energy function is taken to depend on the state of damage described by the scalar damage variables \( d^\iota \) and \( d^s \) through the damage functions \( \Omega^\iota = \Omega^\iota (d^\iota) \) and \( \Omega^s = \Omega^s (d^s) \), respectively. It is considered to be additively composed of volumetric and deviatoric elastic stored energy potentials in the functional form
\[ \Psi = \tilde{\Psi} (\mathbf{e}_\iota, \mathbf{\varepsilon}_\iota, d^\iota, d^s) = \Omega^\iota (d^\iota) \cdot \left( \tilde{\Psi}^\iota (\mathbf{e}_\iota) + \Omega^s (d^s) \cdot \tilde{\Psi}^s (\mathbf{\varepsilon}_\iota) \right). \]  

(22)

The damage functions \( 0 < \Omega^\iota \leq 1 \) and \( 0 < \Omega^s \leq 1 \) range between 1 for the undamaged state and approach 0 for the completely damaged state of the material. Consistent with the principle of strain equivalence (Lemaitre, 1985a, 1985b), damage-effective stress and the corresponding damage-effective free-energy are identified. Damage-effective quantities are represented with tilde notation, such that the per unit volume damage-effective volumetric and deviatoric elastic stored energy potentials are given respectively by
\[ \tilde{\rho} \tilde{\Psi}^\iota (\mathbf{e}_\iota) := \frac{K_0}{2} \mathbf{e}_\iota^2; \quad \tilde{\rho} \tilde{\Psi}^s (\mathbf{\varepsilon}_\iota) := \mu_0 \mathbf{\varepsilon}_\iota : \mathbf{\varepsilon}_\iota. \]  

(23)

where \( K_0 \) and \( \mu_0 \) are the damage-effective bulk and shear moduli, respectively. The damage-effective free-energy per unit volume is then expressed by
\[ \tilde{\rho} \tilde{\Psi} = \frac{K_0}{2} \mathbf{e}_\iota^2 + \mu_0 \mathbf{\varepsilon}_\iota : \mathbf{\varepsilon}_\iota. \]  

(24)

The damage functions could in general be non-linear, but are taken for the sake of simplicity and clarity here as linear,
\[ \Omega^\iota := (1 - d^\iota); \quad \Omega^s := (1 - d^s), \]  

(25)

for \( 0 < d^\iota < 1 \) and \( 0 < d^s < 1 \). It is convenient to also define (consistent with standard notation)
\[ \omega^\iota := \Omega^{-1} = \frac{1}{(1 - d^\iota)}; \quad \omega^s := \Omega^{-1} = \frac{1}{(1 - d^s)}. \]  

(26)

The comprehensive damage-effective free-energy is defined as
\[ \tilde{\Psi}^\iota := \omega^\iota \tilde{\Psi}. \]  

(27)

The stress is then found from the hyperelastic constitutive relation of Eq. (11) to be given by
\[ \mathbf{\xi} = (1 - d^\iota) \cdot \left( K_0 (1 + (1 - d^\iota) / 2) \mu_0 \mathbf{\varepsilon}_\iota \right), \]  

(28)

with the damage-effective stress thus given by
\[ \tilde{\mathbf{\xi}} = \omega^\iota \omega^s \text{dev} [\mathbf{\xi}] + \omega^s \mathbf{p} \mathbf{1} = K_0 \mathbf{d}_\iota + 2 \mu_0 \mathbf{\varepsilon}_\iota. \]  

(29)

Note too that the expression for the damage-effective Eshelby-zeta stress is
\[ \tilde{\mathbf{\xi}} = \omega^\iota \omega^s \text{dev} [\mathbf{\xi}] + \omega^s \mathbf{p} \mathbf{1} = \text{dev}[\tilde{\mathbf{\xi}}] + (\tilde{\mathbf{p}} - \tilde{\mathbf{\xi}}) \mathbf{1}. \]  

(30)

The expressions for the energy release rates are found according to their definitions within Eq. (15) to be
\[ y^\iota = \frac{\partial (\tilde{\rho} \tilde{\Psi})}{\partial d^\iota} = -\tilde{\rho} \tilde{\Psi}^\iota; \quad y^s = \frac{\partial (\tilde{\rho} \tilde{\Psi})}{\partial d^s} = -(1 - d^\iota) \tilde{\rho} \tilde{\Psi}^s, \]  

(31)

which completes the definition of all damage state variables and potentials. A list of the damage state variables and their definitions is provided in Table 1. Having defined the damage state variables, the expressions for the stress and damage-effective stress can be identified. Making use of the definition in Eq. (13) with Eqs. (27) and (31), the Eshelby-zeta stress is expressed as
\[ \tilde{\mathbf{\xi}} = \mathbf{\xi} + (1 - d^\iota) y^\iota \mathbf{1}. \]  

(32)

such that
\[ \tilde{\mathbf{\xi}} = \tilde{\mathbf{\xi}} + y^\iota \mathbf{1}, \]  

(33)

which implies
\[ \tilde{\mathbf{\xi}} = \tilde{\mathbf{\xi}} - y^\iota \mathbf{1}. \]  

(34)

Within the described framework, we note the following relations:
\[ \tilde{\mathbf{p}}^\iota = \tilde{\mathbf{p}} - y^\iota; \quad \tilde{\mathbf{q}}^\iota \equiv \tilde{\mathbf{q}}. \]  

(35)

4.2. Yield function and plastic potential

The Drucker-Prager/cap (DP-C) type yield function is taken as a function of the hyperelastic damage-effective stress \( \tilde{\mathbf{\xi}} = f^\iota \mathbf{d}_\iota \), i.e.,
\[ f := f^\iota (\tilde{\mathbf{\xi}}). \]  

(36)

This form is convenient because it both provides a bound for the plastic domain, i.e., \( f \leq f^\iota (\tilde{\mathbf{\xi}}) \leq 0 \), and is tenable in the sense that its parameters can be associated with material properties obtained from measurements (e.g., friction angle, preconsolidation stress, etc.). However, the yield function is required to be of the functional form described by Eq. (16) in order to find the necessary derivatives described within Section 3. Fortuitously, we can
see by examining Eqs. (34) and (35) that this expression is readily available, since \( F(\kappa) = F(\tilde{\kappa} - y^*) \). Performing the substitution of \( p^* = y^* \) provides the alternative but equivalent expression of the DP-C yield criteria, which we call a Drucker-Prager/Damage (DP-D) yield criteria:

\[
F := \| \text{dev}(\kappa)^2 \| - F_{\text{cap}}(A^\phi \tilde{\beta}_1 - B^\phi p^*)^2 = \| \text{dev}(\tilde{\kappa})^2 \| - F_{\text{cap}}(A^\phi \tilde{\beta}_1 - B^\phi (\tilde{p}^* - y^*))^2 \leq 0,
\]

where

\[
A^\phi := \frac{2 \sqrt{\sin \phi}}{3 + r \sin \phi}, \quad B^\phi := \frac{2 \sqrt{\sin \phi}}{3 + r \sin \phi}, \quad -1 \leq r \leq 1,
\]

\[
F_{\text{cap}} := 1 - (\tilde{\beta}_2 - 3(\tilde{p}^* - y^*) \cdot (\tilde{X}^* - \tilde{p}^*)^2),
\]

\[
\tilde{\beta}_2 := \frac{1}{2} \left[ \tilde{\beta}_2 - 3(\tilde{p}^* - y^*) \right] + (\tilde{\beta}_2 - 3(\tilde{p}^* - y^*))
\]

\[
\tilde{X}^* := \tilde{p}^* - R (A^\phi \tilde{\beta}_1 - B^\phi \tilde{\beta}_2).
\]

The yield function is defined in damage-effective stress space (see Fig. 2). The material parameters of the yield function are hence damage-effective parameters, and so denoted with tilde notation. The parameters \( \phi \) and \( \beta_1 \) are the (damage-effective) friction angle and the cohesion parameter, respectively. The parameter \( \tilde{\beta}_2 \) is associated with the preconsolidation stress, the position of the cap along the \( \tilde{X}^* = \tilde{p}^* \) axis being given by \( X^\phi \). The parameter \( R \) controls the ellipticity of the cap, and the shape of the yield surface on the octahedral plane is controlled by \(-1 \leq r \leq 1\), such that for \( r = 1 \) and \( r = -1 \) coincide with the intersection of the triaxial extension (TE) and triaxial compression (TC) corners of the Mohr-Coulomb yield surface, respectively. The Frobenius norm is denoted by \( \| \cdot \| \), \( | \cdot | \) is the absolute value, and \( ( \cdot ) \) is the Macaulay bracket. For non-associative plasticity, we have similar functional form for the plastic potential function \( \Psi^p \) as for the yield function \( F \), the only difference being that the damage-effective dilatation angle \( \tilde{\psi} \) replaces the damage-effective friction angle \( \phi \) in Eq. (37) such that

\[
\Psi^p := \| \text{dev}(\tilde{\kappa})^2 \| - F_{\text{cap}} (A^\psi \tilde{\beta}_1 - B^\psi (\tilde{p}^* - y^*))^2.
\]
where

\[
C^\psi := 2(A^p \ddot{B} - B^p \ddot{B}^2) \left[ \frac{F_{\text{cap}}^p B^\psi}{3} - \frac{(A^p \ddot{B} - B^p \ddot{B})}{(X^\psi - \ddot{B})^2} \right].
\]

The flow rule of Eq. (21) is then expressed as

\[
d^\psi = \dot{\lambda} \dot{m}.
\]

The deviatoric and volumetric parts of the plastic deformation rate can be distinguished as

\[
\text{dev}[d^\psi] = 2\lambda \omega^c \omega^\psi \text{dev}[\xi], \quad \text{vol}[d^\psi] = 3\lambda \omega^\psi C^\psi.
\]

Similarly, the damage rates of Eqs. (19) and (20) are found to be expressed by

\[
\dot{d}^\epsilon = 3c_0 \lambda C^\psi, \quad \dot{d}^\sigma = \lambda \omega^c \omega^\psi \frac{(-c_0 \omega^c) \psi^a}{S}.
\]

The intermediate equations necessary for finding these solutions are provided in Appendix A.

4.3. Implicit integration

Backward Euler implicit integration making use of the exponential map is utilized in order to obtain the integrated flow rule in terms of the Hencky strain (cf. Simo, 1992; 1998, among others). This allows the classical predictor/corrector return mapping algorithm of the classical small strain theory to be preserved (cf. Borja and Tamagnini, 1998; de Souza Neto et al., 2011, among others). The detailed procedure involving the exponential map is not repeated here, as it has been well documented in the literature, and we refer also to the details provided in Bennett et al. (2016) for a description of the kinematics relevant to the current work.

The integrated flow rule is hence given by

\[
\dot{\epsilon}^\psi_{\text{int}+1} = \epsilon^\psi_{\text{int}+1} + \Delta \lambda D m_{\text{int}+1},
\]

which provides the elastic strain at a given increment of total strain \(\Delta e\) (i.e., the predictor/corrector equation),

\[
\dot{\epsilon}^\psi_{\text{int}+1} = \epsilon^\psi_{\text{int}+1} - \Delta \lambda D m_{\text{int}+1},
\]

where the superscript \((\cdot)^{\text{int}}\) denotes a trial value, and is for the elastic strain taken to be at each time step as \(\epsilon^\psi_{\text{int}+1} = \epsilon^\psi_{\text{int}+1} + \Delta e\). The expressions for the volumetric and deviatoric elastic strains can also be respectively expressed by

\[
\dot{\epsilon}^\psi_{\text{int}+1} = \epsilon^\psi_{\text{int}+1} - 3\Delta \lambda C^\psi_{\text{int}+1}.
\]

and

\[
\dot{\epsilon}^\psi_{\text{int}+1} = \epsilon^\psi_{\text{int}+1} - 2\Delta \lambda C^\psi_{\text{int}+1} \omega^\psi_{\text{int}+1},
\]

such that

\[
\dot{\epsilon}^\psi_{\text{int}+1} = \epsilon^\psi_{\text{int}+1} - 2\Delta \lambda C^\psi_{\text{int}+1} \omega^\psi_{\text{int}+1} \text{dev}[\xi].
\]

The update equations for the damage variables are found from Eq. (47), making use of Eqs. (50) and (53) to find

\[
\dot{d}^\sigma_{\text{int}+1} = c_0 \frac{\dot{d}^\epsilon_{\text{int}+1} + \Delta \epsilon^\psi_{\text{int}+1}}{\Delta \epsilon^\psi_{\text{int}+1}} ; \quad \dot{d}^\epsilon_{\text{int}+1} = d^\epsilon_{\text{int}+1} + 3 \frac{\Delta \epsilon^\psi_{\text{int}+1}}{2\sqrt{6} \omega^\psi_{\text{int}+1}} \left( -c_0 \omega^c_{\text{int}+1} \psi^a_{\text{int}+1} \right)^\frac{1}{2}.
\]

Having the updated elastic strains allows the updated stress to be calculated according to the constitutive equation (28). We note here also the update equations for the mean and deviatoric stress measures. The mean stress update is given by

\[
\begin{align*}
p^\prime_{\text{int}+1} &= (1 - d^\sigma_{\text{int}+1}) \cdot (K_0 \epsilon^\psi_{\text{int}+1}) \\
&= (1 - d^\sigma_{\text{int}+1}) \cdot K_0 \left( \text{tr}[\epsilon^\psi_{\text{int}+1}] - 3\Delta \lambda C^\psi_{\text{int}+1} \right) \\
&= \bar{p}^\prime_{\text{int}+1} / \omega^\psi_{\text{int}+1} - 3K_0 \Delta \lambda C^\psi_{\text{int}+1}.
\end{align*}
\]

The deviatoric stress update is similarly found as

\[
\begin{align*}
\text{dev}[\dot{\epsilon}^\psi_{\text{int}+1}] &= \text{dev}[\epsilon^\psi_{\text{int}+1}] \\
&= (1 - d^\sigma_{\text{int}+1}) \cdot \left( 1 - d^\sigma_{\text{int}+1} \right) 2\mu \epsilon^\psi_{\text{int}+1} \\
&= (1 - d^\sigma_{\text{int}+1}) \cdot \left( 1 - d^\sigma_{\text{int}+1} \right) 2\mu \epsilon^\psi_{\text{int}+1} - 2\Delta \lambda C^\psi_{\text{int}+1} \omega^\psi_{\text{int}+1} \text{dev}[\epsilon^\psi_{\text{int}+1}] \\
&= \omega^\psi_{\text{int}+1} \omega^\psi_{\text{int}+1} \frac{1 + 4\mu K_0 \Delta \lambda C^\psi_{\text{int}+1} \omega^\psi_{\text{int}+1}}{1}
\end{align*}
\]

which provides (see also further details provided in Appendix A),

\[
q^\prime_{\text{int}+1} = (1 - d^\sigma_{\text{int}+1}) \cdot \bar{q}^\psi_{\text{int}+1} \left( 1 + 4\Delta \lambda C^\psi_{\text{int}+1} \omega^\psi_{\text{int}+1} \right).
\]

An iterative solution of the integrated equations is required. In this work we employ a Newton–Raphson iteration.

5. Comparison of model predictions with measurements

The model is evaluated by calibration and comparison to laboratory and in-situ measurements. Two different materials and corresponding sets of measurements are considered: (1) Tavel limestone after Vajdova et al. (2004), and (2) a horizontal wellbore through 30% porosity limestone off the coast of Brazil as described by Coelho et al. (2005). The experimental data of Vajdova et al. (2004) were chosen as a first example for the DP-D model calibration for a number of reasons: it consists of triaxial compression (TC) at a wide range of confining pressures, it includes isotropic compression data, and it is widely referenced as representative for describing the pressure dependent so-called “brittle–ductile transition” and “shear enhanced compaction/dilation” behavior of rocks (Rudnicki and Rice, 1975; Borja and Requeiro, 2001; Wong and Baud, 2012; Tijoe and Borja, 2015). The wellbore study was chosen because it is accompanied by substantial site-characterization measurements and is through high porosity limestone. The model parameters were calibrated independently to each of the materials and corresponding data sets, and the calibration methods along with calibrated values are presented independently in each of the following sub-sections.

It is noted that the shear stress intercept in \(q\) vs. \(p\) space is given by \(q^\prime = 2c^\prime\), where \(c^\prime\) is the Mohr–Coulomb (MC) intercept. Although the MC friction angle is a parameter of the model, it needs to be evaluated in damage-effective space to do so. Examining Eq. (29), it can be seen that the damage-effective friction angle is related to the (measured) friction angle by \(\phi = \tan^{-1} (c^\prime \tan \phi)\). For each calibration procedure, the initial yield surface is provided along with the corresponding damage-effective yield surface.

5.1. Tavel limestone triaxial compression simulations

The DP-D model was calibrated to the triaxial compression (TC) test data on Tavel limestone presented by Vajdova et al. (2004). The isotropic compression data allowed the damage-effective bulk modulus \(K_0\), the preconsolidation stress parameter \(\beta_2\) and the compressive hardening parameter \(c_0\) to be determined from the test data (see Fig. 3). The TC measurements also allowed for the determination of the Mohr–Coulomb (MC) friction angle \(\phi\) and cohesion intercept \(c^\prime\) (corresponding to \(\beta_1\) parameter), which were directly reported by the authors as \(c^\prime = 60\) MPa and \(\phi = 27\).
respectively. The damage parameter \( c_t \) was calibrated to best fit the TC measurements that were on the dilative side (at low confining pressure), and the \( c_t \) parameter calibrated to the high confining pressure curves. Fig. 4 shows the initial DP-D yield surface alongside the damage-effective yield surface. A critical aspect of calibrating the model was to adjust the ellipticity parameter \( R \) in order to keep the brittle-ductile transition point (peak of the plastic potential surface in \( p-q \) space) near the intersection of the 100 MPa confining pressure stress paths, consistent with the measurements and description provided by the authors (op. cit.). A low damage-effective dilation angle was chosen as typical of rock \( \psi = 0.16 \). Some minor adjustments of parameters were needed to best fit the measurements, while attempting to keep as many fixed as possible.

A comparison of the model simulations with the measurements is provided in Fig. 5. All calibrated model parameters are reported in Tables 3 and 4 and reports variations of model parameters for individual tests necessary to best fit the measurements. The results show that the model is able to simulate well both the distinctive hardening and softening evident in the measurements, as well as the pressure dependent transition between these regimes, the so-called brittle-ductile transition. We emphasize that this behavior is modeled as changes in porosity and microfracture through evolution of the damage state variables in a thermodynamically consistent way made possible by recognizing the role of the Eshelby stress tensor as described in Sections 2 and 3.

### Table 4
Variation of parameters from those reported in Table 3 required to best fit measurements of Vajdova et al. (2004).

<table>
<thead>
<tr>
<th>Confining pressure</th>
<th>Parameter(s)</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 MPa</td>
<td>( K_0, \beta_2 )</td>
<td>+1.4%</td>
</tr>
<tr>
<td></td>
<td>( v_\psi, \phi, \psi )</td>
<td>+1.4%</td>
</tr>
<tr>
<td>20 MPa</td>
<td>( K_0, \beta_2 )</td>
<td>-3.0%</td>
</tr>
<tr>
<td></td>
<td>( v_\psi, \phi, \psi )</td>
<td>-1.4%</td>
</tr>
<tr>
<td>30 MPa</td>
<td>( K_0, \beta_2 )</td>
<td>+3.0%</td>
</tr>
<tr>
<td></td>
<td>( v_\psi, \phi, \psi )</td>
<td>+2.9%</td>
</tr>
<tr>
<td>50 MPa</td>
<td>( K_0, \beta_2 )</td>
<td>-9.1%</td>
</tr>
<tr>
<td></td>
<td>( v_\psi, \phi, \psi )</td>
<td>-7.1%</td>
</tr>
<tr>
<td>100 MPa</td>
<td>( K_0, \beta_2 )</td>
<td>-2.9%</td>
</tr>
<tr>
<td></td>
<td>( v_\psi, \phi, \psi )</td>
<td>+20%</td>
</tr>
<tr>
<td>150 MPa</td>
<td>( K_0, \beta_2 )</td>
<td>+36%</td>
</tr>
<tr>
<td></td>
<td>( v_\psi, \phi, \psi )</td>
<td>+2.9%</td>
</tr>
<tr>
<td>200 MPa</td>
<td>( K_0, \beta_2 )</td>
<td>+2.9%</td>
</tr>
<tr>
<td></td>
<td>( v_\psi, \phi, \psi )</td>
<td>+2.9%</td>
</tr>
</tbody>
</table>

### Table 6
DP-D model parameters established from measurements of Coelho et al. (2005).

<table>
<thead>
<tr>
<th>Type</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic</td>
<td>( K_0 )</td>
<td>571.0</td>
<td>MPa</td>
</tr>
<tr>
<td></td>
<td>( v_\psi )</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td>Yield Surface</td>
<td>( \beta_2 )</td>
<td>25.0</td>
<td>MPa</td>
</tr>
<tr>
<td></td>
<td>( \phi )</td>
<td>0.42</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>( \psi )</td>
<td>0.1</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>( R )</td>
<td>2.0</td>
<td>-</td>
</tr>
<tr>
<td>Damage</td>
<td>( c_t )</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>( c_t )</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>( c_t )</td>
<td>40</td>
<td>-</td>
</tr>
</tbody>
</table>

Evolution of damage parameters is provided for the 50 MPa and 100 MPa confining pressure simulations in Fig. 6. These tests mark the transition from brittle to ductile behavior as described by the authors (op. cit.) and apparent in Fig. 5. The comprehensive damage parameter, \( d^* \), in the 50 MPa simulation transitions from decreasing to increasing, which roughly corresponds to the transition from hardening to softening apparent in the 50 MPa stress ratio \( q/p \) curve on the right side (Fig. 6b); whereas, the 100 MPa simulation predicts only increasing \( d^* \). Note that negative total volume strain is plotted, showing that the 50 MPa specimen is predicted to dilate during shear and the 100 MPa specimen is predicted to contract. Recalling that \( d^* \) is proportional to the porosity through the plastic part of volume strain only (see Eqs. (6) and (54)), the 50 MPa \( d^* \) curve exhibits the phenomenon described by Vajdova et al. (2004) (see also Lubarda et al., 1996, among others) of an initial decrease in porosity followed by a subsequent increase under dilation. The figure shows that the shear damage parameter, \( d^* \), is predicted to monotonically increase in both simulations, but significantly more in the 50 MPa simulation, where softening is ev-
Fig. 5. Comparison of model simulations (solid lines) with measurements (symbols) of Tavel limestone in triaxial compression (Vajdova et al., 2004) with confining pressures indicated on the figures.

Fig. 6. Model predictions of change in damage during simulated TC at confining pressures of 50 MPa and 100 MPa: (a) change in comprehensive damage, $\Delta d_c$, alongside (negative) volume strain, $e_v$, and (b) change in shear damage, $d_s$, alongside (negative) stress ratio, $q/p$.

Fig. 7. Model predictions with parameters calibrated to measurements of Vajdova et al. (2004) for relation between changes in $d'$ and plastic volumetric deformation as described in Eq. (8). The slope of each line corresponds to $\alpha(1 - n_0) = 0.896\alpha$ in Eq. (8), and linear regression values are reported in Table 5.

We emphasize that both damage parameters contribute to the overall amount of damage as apparent for example in Eq. (28).

The evolution of $d'$ for all simulations is shown in Fig. 7 plotted against plastic volumetric deformation as $1 - 1/J_p$. This plot corre-

Fig. 8. Plane strain finite element model mesh and boundary conditions for borehole simulation.
sponds to the relation of Eq. (8), where the slope of each curve is \( \alpha(1 - n_0) \). The average initial porosity was reported by the authors as 10.4%, allowing for the model predictions of the proportionality factor \( \alpha \) at each confining pressure to be calculated. The predicted linearized values for \( \alpha \) calculated in this way are reported in Table 5. Notably, \( \alpha \) is predicted to increase approximately by an order of magnitude on the dilative (brittle) side of the brittle-ductile transition (i.e., for the 10, 20, 30, and 50 MPa confining pressures). As can be seen in Fig. 7 and also by the coefficient of determination for the least squares linear regression reported in Table 5, the model predicts a near-linear relationship in Eq. (8) (i.e., near-constant alpha) at any specific confining pressure. Furthermore, the results suggest that the proportionality between volumetric damage and porosity (\( \alpha \)) is binary, with approximately constant distinct values for dilative and compactive regimes.

It should be emphasized that the material model considers only ductile damage and does not address the onset of brittle fracture observed by the authors at low confining pressures and evident in the measurements of Fig. 5(a) as the cataclysmic failure of the low confining pressure (below 100 MPa in this case) specimens. However, the dilative volumetric damage predicted at low confining pressures may be considered to be indicative of such a cataclysmic failure, i.e., comparison of the model predictions to the measurements suggests that the threshold for an increase in \( \alpha \) may be very low for this material. As is evidenced by the comparison with the measurements and is perhaps what we may intuitively expect, consistent with a description of the rock as being able to sustain relatively little shear induced dilation in comparison to shear induced compaction.

It should also be pointed out that shear band bifurcation analysis was not conducted on the specimen responses. The softening responses predicted by the constitutive model resulted from material softening, and not from localized deformation. Evidently, localized deformation in the form of a deformation band could have dominated most of the softening responses exhibited by the specimens. However, material and geometric imperfections on these specimens were so pervasive that any bifurcation analysis from an initially homogeneous response would not be meaningful, since initial specimen responses were not truly homogeneous to begin with. Nevertheless, we have checked the constitutive tangent tensor for the 10 MPa simulation and observed that this tensor lost positive-definiteness right at the onset of plasticity.

5.2. Simulated borehole through limestone

The model was implemented in an Abaqus UMAT material subroutine. A horizontal borehole in a porous limestone reservoir off
the coast of Brazil was simulated in order to examine the model predictions of stress and damage around the borehole. The material properties from laboratory measurements and the far-field stress field from in situ measurements, along with a more detailed description of the reservoir are reported by Coelho et al. (2005).

The elastic material properties were taken from those reported by Coelho et al. (2005), as were the MC friction angle, cohesion intercept, and yield point on the hydrostatic axis ($\chi^0$ determining $\beta_1$ and $\beta_2$). The intermediate principal stress model parameter $r$ was taken as the median value of zero, and the ellipticity parameter $R$ was assigned a typical value of 2. The only remaining model parameters that needed to be calibrated were the damage parameters, $c_t$, $c_s$, and $c_r$, which were kept equivalent and calibrated to best mimic the Drucker-Prager/Cap model simulated stress field results of Spiezia et al. (2016), while reaching final damage states within reasonable threshold values (i.e., to not predict breakout at the walls, which is consistent with the observations). The mean stress value for evaluating Eq. (41) was taken as the average mean stress value of 18 MPa. All model parameters are reported in Table 6.

The borehole radius is 4.25 in. = 107.95 mm. The in-situ geostatic stresses were reported by Coelho et al. (2005) as vertical stress $\sigma_z = 32.1$ MPa and major and minor horizontal stresses, respectively, $\sigma_H = 9.0$ MPa. The finite element model was constructed as plane strain, making use of the two-fold symmetry (see Fig. 8). The analysis simulated the drilling of the borehole by applying the far-field stresses and pressure at the borehole wall simultaneously.

Fig. 9 shows the predicted stress distribution around the borehole with (excess) pressure on the borehole wall absent. Fig. 10 shows the corresponding model predictions of damage distribution. Fig. 10(a) plots the volumetric damage parameter $d^v$, and Fig. 10(b) plots the shear damage parameter $d^s$. Localization of both volumetric and shear damage occur at the borehole wall in the orientation coinciding with the direction of least principal stress, but with opposite trends, i.e., shear damage accompanied by volumetric hardening (healing). In other words, the simulation predicts shear induced compaction in this region, where overall hardening is accompanied by microfracture. This is indicative of the well known tendency for “borehole breakout,” where stress concentrations at the borehole wall cause microfracture and eventually collapse of sections of the wall, effectively elongating the borehole in that direction (cf. Read and Martin, 1996; Zoback, 2010). This is consistent also with the Drucker-Prager/cap model predictions of Spiezia et al. (2016), who predicted similar trends of volumetric and shear strain localization around the borehole wall (although their model did not include damage). We emphasize that the inclusion of volumetric and shear damage in the model not only provides for a prediction of damage around the borehole wall, but also provides for a prediction of the stress field associated with the state of damage, i.e., the stress field prediction of Fig. 9 is influenced by the damage predicted in Fig. 10.

6. Conclusion

Relations between inelastic volume change (bulk plasticity), changes in porosity, and evolution of damage have been examined and incorporated into a unified hyper-elastoplasticity and continuum damage framework. This has allowed for a thermodynamically consistent description of the damage of porous rocks that separates volumetric and isochoric contributions to damage, such that the isochoric damage part describes distributed microcracks exclusive of their associated apertures. An Eshelby-like stress tensor we call the Eshelby-zeta stress, which was identified by Bennett et al. (2016) as being energy-conjugate to the plastic deformation rate, has been shown herein to play a valuable role in distinguishing between volumetric and isochoric contributions to the overall damaged state of the material while ensuring thermodynamic consistency.

This theory has been developed and implemented in a novel hyper-elastoplastic damage constitutive model appropriate for porous rocks, which we call a Drucker-Prager/Damage (DP-D) model. The model is based on a Drucker-Prager/Cap plasticity model previously presented by Regueiro and Ebrahimi (2010) and further modified by Bennett et al. (2016), but the addition of damage and associated damage potentials is novel. Furthermore, previously phenomenological hardening/softening rules are excluded, such that hardening/softening behavior is modeled entirely through the described damage evolution making use of the damage-effective stress concept. The DP-D model is provided in terms of a current configuration representation of finite deformation hyper-elastoplasticity, and an implicit integration scheme for numerical implementation has also been provided.

The DP-D model has been applied to simulation of confined TC measurements on Taval limestone exemplifying the so called brittle-ductile transition behavior (Vajdova et al., 2004). The simulations show that the model is capable of simulating the pressure
dependent hardening/softening behavior exhibited in these measures entirely through damage and damage-effective stress concepts related to porosity and microfracture changes and made possible by recognizing the role of the Eschelby stress tensor in ensuring thermodynamic consistency. In particular, marked softening at low confining pressures is captured well by modeling the evolution of the damaged state with dilative strain. The model results further allow for evaluation of the relationship between damage and porosity established on kinematic grounds, where an identified proportionality factor (\(\alpha\)) has been examined. The results suggest that for the measurements considered, such proportionality is essentially binary, with the factor \(\alpha\) increasing by approximately an order of magnitude between compressive and dilative regimes, while remaining approximately constant in each.

The model has been implemented within an Abaqus UMAT material subroutine, and nonlinear finite element simulations of a reservoir borehole through limestone have also been provided. These simulations were carried out in order to demonstrate the robustness of the model and to investigate the predictions of stress and damage distribution around the borehole wall. They demonstrate the usefulness of the model for the analysis of borehole stability and the prediction of both damaged zones and resulting stress fields around the borehole wall.

The separation of volumetric and isochoric damage variables described in this work is especially significant because it allows for the possibility of a thermodynamically consistent representational isochoric damage to be constructed from the continuum distribution of microcracks without the need to disregard (or otherwise make any simplifying assumptions about) their apertures. Incorporating a second order isochoric damage tensor into the DP-D model presented here is currently underway as part of future work with the goal of providing a description of elastoplastic anisotropy that ascribes the relative weakness of the bedding plane observed in many geomaterials to the anisotropy produced by the microcracks. We note that further applications of the proposed model could also include extension to coupled hydro-mechanical problems where hydraulic conductivity is dependent on the damaged state. The phenomenological distinction emphasized in this work between the contribution of porosity and microfracture to damaged could be advantageous in this regard, especially in distinguishing changes (possibly anisotropic) in hydraulic conductivity attributed to microfracture from those attributed to porosity.

Acknowledgments

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Basic Energy Sciences, Geosciences Research Program, under Award number DE-FG02-03ER15454. Support for materials and additional student hours were provided by the National Science Foundation under Award number CMML-1462231, and by the Stanford University James M. Gere Research Fellowship.

Appendix A

The evolution of the damage variables are found from

\[
\begin{align*}
\frac{\partial G^c}{\partial \epsilon^{ab}} &= 2\epsilon_0 \delta^{ab} \left( A^{\psi} \hat{b}_1 - B^{\psi} \hat{b}_2 \right) \left( \frac{3}{\sqrt{2}} \hat{n} + \frac{1}{\sqrt{2}} \hat{m} \right) \left( \frac{3}{\sqrt{2}} \hat{n} - \frac{1}{\sqrt{2}} \hat{m} \right) \\
&= -3\epsilon_0 C^{\psi}.
\end{align*}
\] (58)

\[
\frac{\partial G^p}{\partial \epsilon^{tr}} = 2\epsilon_0 \frac{\partial G^p}{\partial q} \left( \frac{3}{\sqrt{2}} \hat{n} + \frac{1}{\sqrt{2}} \hat{m} \right) \left( \frac{3}{\sqrt{2}} \hat{n} - \frac{1}{\sqrt{2}} \hat{m} \right)
\]

\[
\frac{\partial G^p}{\partial q} = -\alpha \omega^p \left( -\frac{c_{\alpha} \omega^p \epsilon^{tr}}{S} \right).
\] (59)

The detailed solution of \(q_{n+1}\) is provided by first noting that

\[
s := \text{dev} \left( \frac{2}{3} \hat{n} \right) = 2\mu \epsilon^{tr}.
\] (60)

where \(\epsilon^{tr} \parallel \hat{e}^{tr} \parallel \hat{e}^{tr} =: \hat{n}^{tr}\). Eq. (60) also reveals that

\[
\frac{s}{||s||} = \hat{n}.
\]

\[
\frac{\epsilon^{tr}}{||\epsilon^{tr}||} = \left( \frac{2}{3} q \right) = 2\mu \epsilon^{tr}
\]

\[
\Rightarrow \left( \frac{s}{||s||} \right) = \hat{n}
\]

The plastic potential function can then be written as

\[
G^p := \frac{2}{3} q^2 - K_{cap} \left( A^{\psi} \hat{b}_1 - B^{\psi} \hat{b}_2 \right)^2 \leq 0,
\] (62)

such that

\[
\hat{m} := \frac{\partial G^p}{\partial \tilde{\epsilon}^{tr}} = \frac{3}{2} \frac{\partial G^p}{\partial q} \left( \frac{3}{\sqrt{2}} \hat{n} + \frac{1}{\sqrt{2}} \hat{m} \right) = 2\sqrt{6} \frac{3}{\sqrt{2}} \hat{n} + \alpha \omega^p \hat{m}.
\] (63)

The evolution and update equation for \(\epsilon^{tr}_{n+1}\) and \(q_{n+1}\) are then found by noticing

\[
\epsilon^{tr}_{n+1} = \epsilon^{tr}_{n} + \Delta \lambda \frac{3}{2} \frac{\partial G^p}{\partial q} \left( \frac{3}{\sqrt{2}} \hat{n} + \frac{1}{\sqrt{2}} \hat{m} \right)
\]

\[
\Rightarrow \epsilon^{tr}_{n+1} = \epsilon^{tr}_{n} - \Delta \lambda \frac{3}{2} \frac{\partial G^p}{\partial q} \left( \frac{3}{\sqrt{2}} \hat{n} + \frac{1}{\sqrt{2}} \hat{m} \right)
\]

\[
\epsilon^{tr}_{n+1} = \epsilon^{tr}_{n+1} - \Delta \lambda \frac{3}{\sqrt{2}} \frac{\partial G^p}{\partial q} \left( \frac{3}{\sqrt{2}} \hat{n} + \frac{1}{\sqrt{2}} \hat{m} \right)
\]

\[
\Rightarrow \epsilon^{tr}_{n+1} = \epsilon^{tr}_{n+1} - \Delta \lambda \frac{3}{\sqrt{2}} \frac{\partial G^p}{\partial q} \left( \frac{3}{\sqrt{2}} \hat{n} + \frac{1}{\sqrt{2}} \hat{m} \right)
\]

And then from

\[
\tilde{q}_{n+1} = 2 \sqrt{\frac{3}{2}} \mu \epsilon^{tr}_{n+1}
\]

\[
\Rightarrow \tilde{q}_{n+1} = \tilde{q}^{tr}_{n+1} \left[ 1 + 4 \Delta \lambda \alpha \omega^p \hat{m} \hat{n} \right]
\] (65)

References



