Lecture 6
Qualitative properties of signals & Laplace transforms

- qualitative behavior from pole locations
- damping & quality factor
- dominant poles
- stability of autonomous LCCODEs
- initial value theorem, final value theorem

Inverse Laplace transform of rational F

suppose $F(s) = b(s)/a(s)$ is rational and strictly proper with $\mathcal{L}^{-1}(F) = f$

each term in partial fraction expansion of F gives a term in f:

- for single pole at $s = \lambda$,

$$\mathcal{L}^{-1}\left(\frac{1}{s-\lambda}\right) = e^{\lambda t}$$

- for pole at $s = \lambda$ of multiplicity k,

$$\mathcal{L}^{-1}\left(\frac{1}{(s-\lambda)^k}\right) = \frac{1}{(k-1)!}t^{k-1}e^{\lambda t}$$

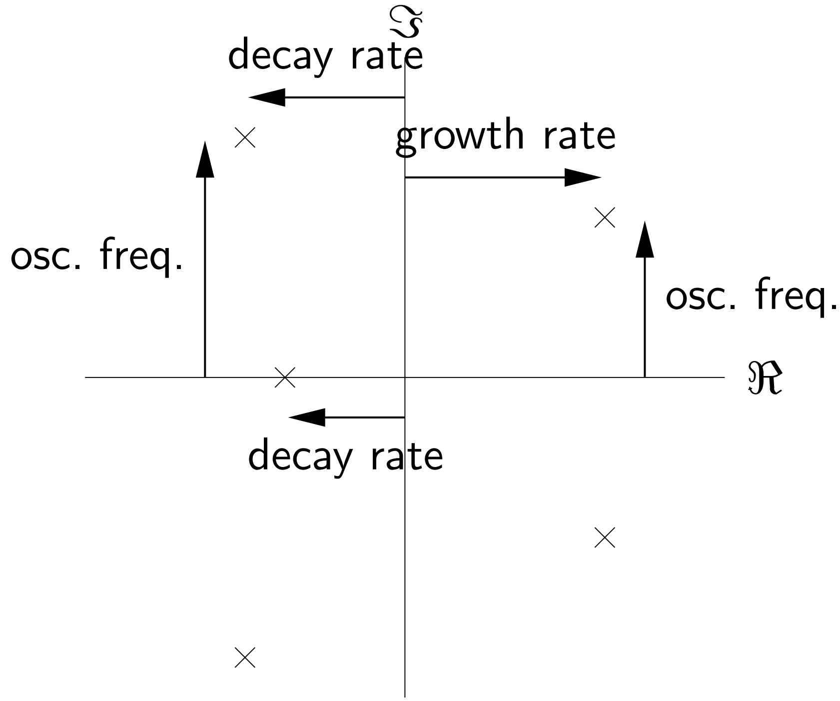
- the *poles* of F determine the types of terms that appear in f
- the *zeros* (or *residues*) of F determine the coefficients multiplying each term, or the amplitude & phase of oscillatory terms

Qualitative properties of terms

- real, positive poles correspond to growing exponential terms
- real, negative poles correspond to decaying exponential terms
- a pole at $s = 0$ corresponds to a constant term
- complex pole pairs with positive real part correspond to exponentially growing sinusoidal terms
- complex pole pairs with negative real part correspond to exponentially decaying sinusoidal terms
- pure imaginary pole pairs correspond to sinusoidal terms
- repeated poles yield same types of terms, multiplied by powers of t

Quantitative properties of terms

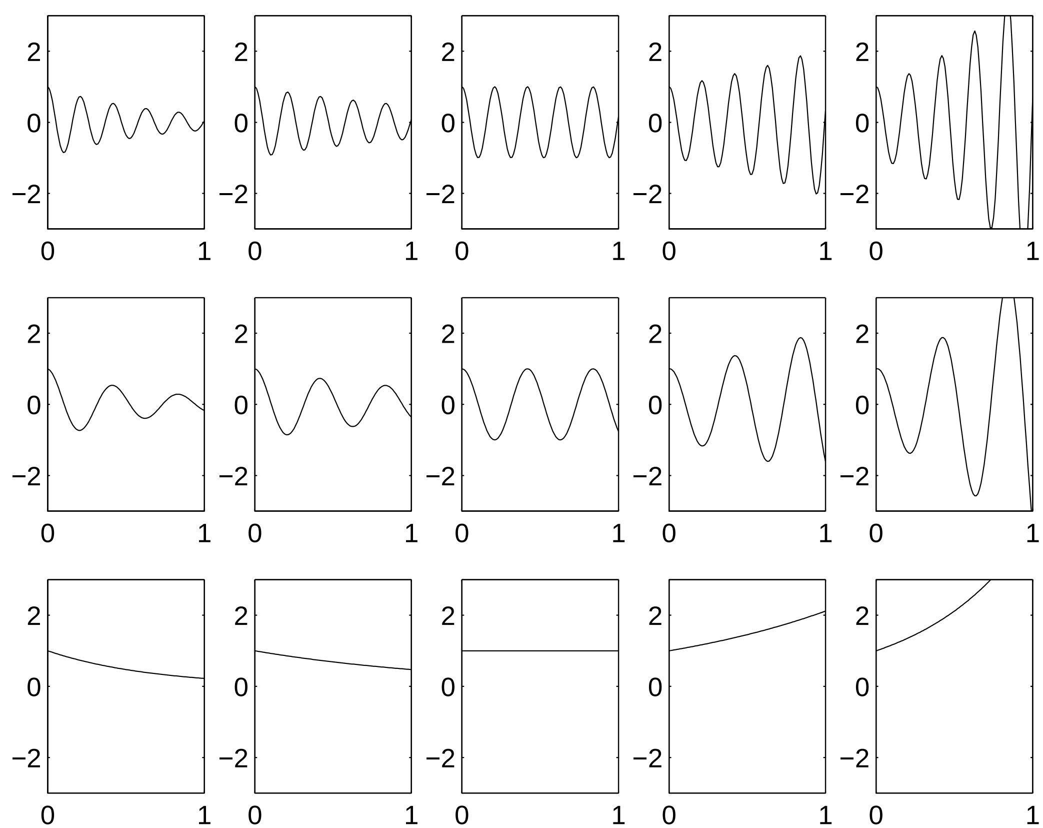
pole $\lambda = \sigma + j\omega$ (and $\overline{\lambda}$, if $\omega \neq 0$) gives time-domain term $ae^{\sigma t}\cos(\omega t + \phi)$



- the real part of a pole gives the *growth rate* (if positive) or decay rate (if negative) of the corresponding term in f
- the imaginary part gives the oscillation *frequency*

$$f(t) = e^{\sigma t}\cos(\omega t)$$

rows: $\omega = 30,\ 15,\ 0$; columns: $\sigma = -1.5, -0.75,\ 0,\ 0.75,\ 1.5$



Complex poles: damping ratio and Q

pole at $s = \lambda = \sigma + j\omega$ (hence also at $\overline{\lambda}$) with $\sigma < 0$

$$F(s) = \frac{r}{s-\lambda} + \frac{\overline{r}}{s-\overline{\lambda}}, \quad f(t) = ae^{\sigma t}\cos(\omega t + \phi)$$

two measures of decay rate per cycle of oscillation:

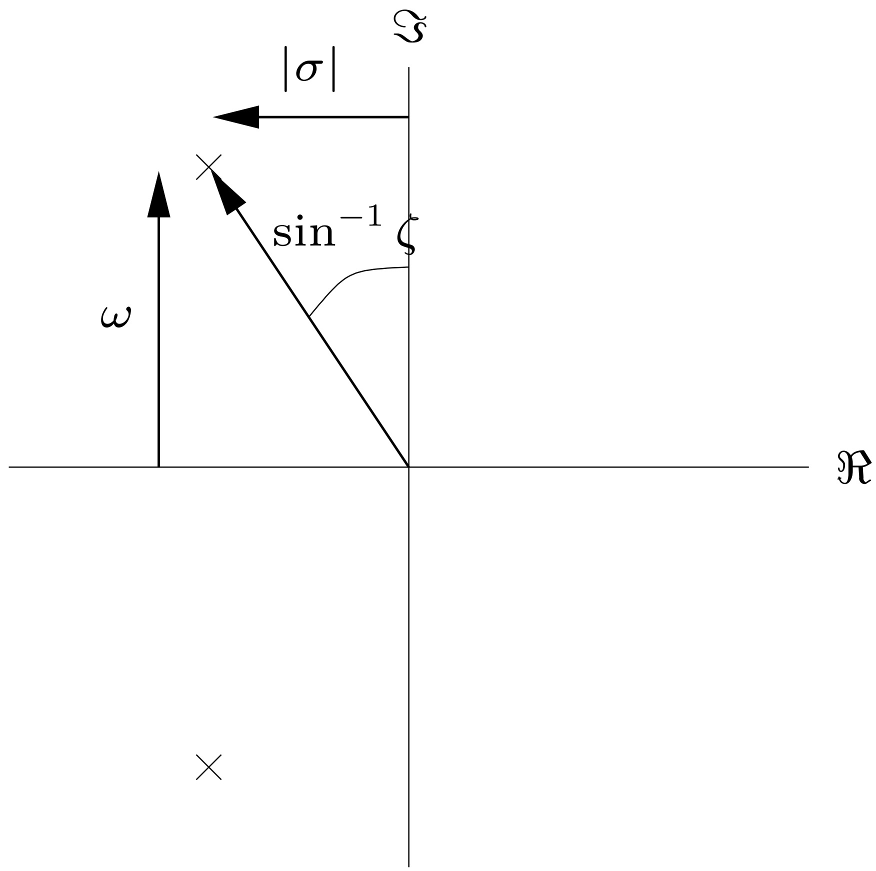
- **damping ratio**

$$\zeta = \frac{-\sigma}{\sqrt{\sigma^2 + \omega^2}}$$

- **quality factor**

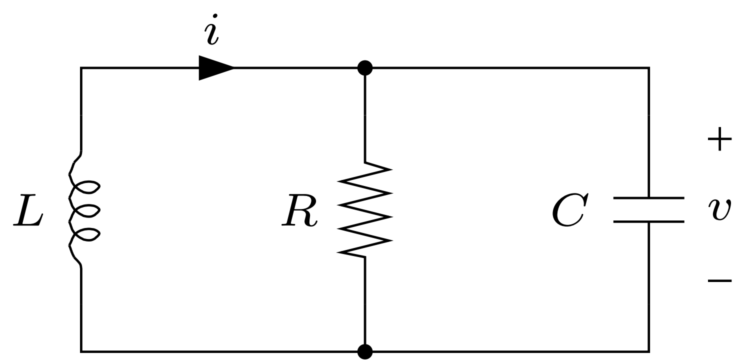
$$Q = \frac{1}{2}\sqrt{\frac{\sigma^2 + \omega^2}{\sigma^2}} = \frac{1}{2\zeta}$$

damping ratio (or Q) are related to *angle* of pole in complex plane:



- underdamped: $\zeta < 1$ ($Q > 1/2$)
- critically damped: $\zeta = 1$ ($Q = 1/2$)

example: underdamped parallel RLC circuit of page 4-31



$$\sigma = \frac{-1}{2RC}, \quad \omega = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

gives

$$Q = \frac{R}{\sqrt{L/C}}, \quad \zeta = \frac{\sqrt{L/C}}{2R}$$

interpretation: Q is a measure of number of cycles to decay

- time to decay to 1% amplitude is about $4.6/|\sigma|$
- period of oscillation: $2\pi/\omega$
- number of cycles to decay to 1% amplitude

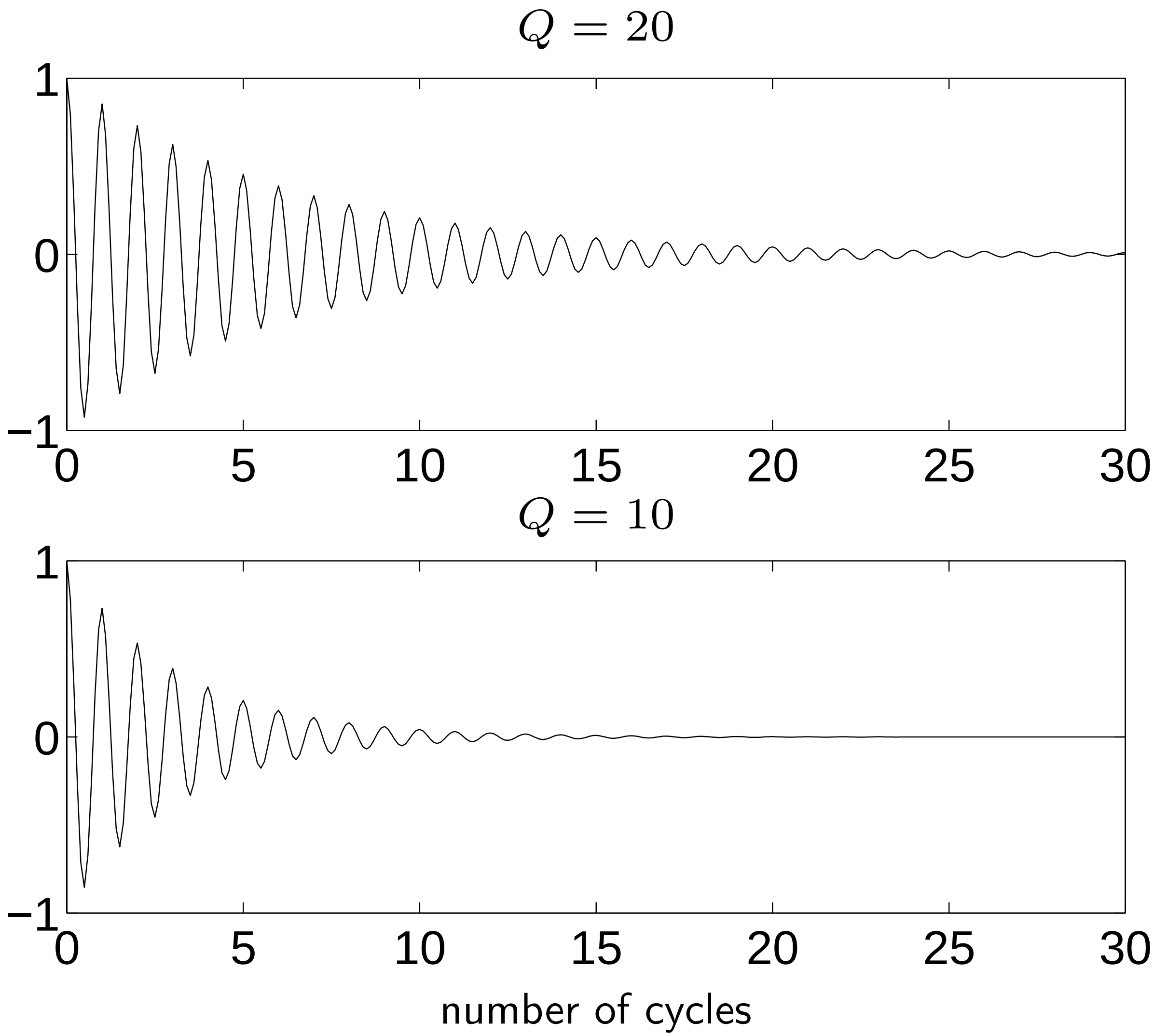
$$N_{1\%} \approx \frac{4.6/|\sigma|}{2\pi/\omega} = 1.46\frac{\omega}{2|\sigma|}$$

rule of thumb (accurate for $Q > 2$ or so):

$$N_{1\%} \approx 1.46Q$$

other rule of thumb: $N_{4\%} \approx Q$

example



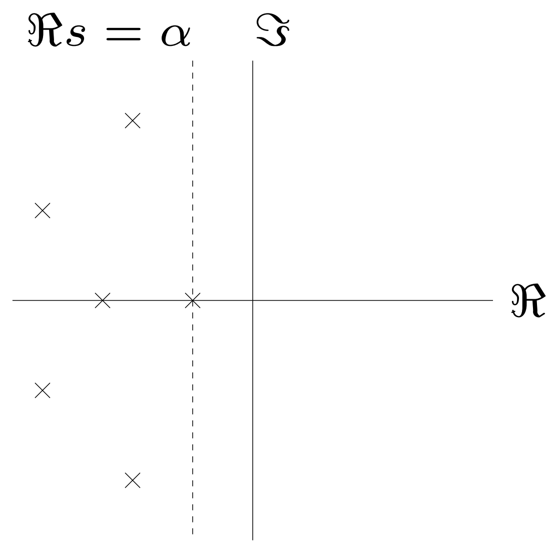
$Q = 20$
1
0
−1
0
5
10
15
20
25
30
$Q = 10$
1
0
−1
0
5
10
15
20
25
30
number of cycles

Dominant poles

suppose the poles of F are $p_1, \ldots, p_n$
the asymptotic growth (or decay if < 0) rate of f is determined by the *maximum real part*:

$$\alpha = \max\{\Re p_1, \ldots, \Re p_n\}$$

- pole (or poles) which achieve this max real part are called **dominant**
- as $t \to \infty$, these terms become larger and larger compared to the other terms, no matter what the residues



example:

$$F(s) = \frac{100}{s+2} + \frac{1}{s+1}, \quad f(t) = 100e^{-2t} + e^{-t}$$

- asymptotic decay rate determined by dominant pole at $s = -1$
- asymptotically, f decays like e^{-t}
- even though residue for nondominant pole is 100 times larger, term associated with dominant pole is larger for $t > 4.6$

Stability of autonomous LCCODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0$$

is *stable* if all solutions converge to zero, regardless of initial condition

take Laplace transform:

$$a_n \left(s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - y^{(n-1)}(0) \right)$$
$$+ a_{n-1} \left(s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{(n-2)}(0) \right) + \cdots + a_0 Y(s) = 0$$

$$Y(s) = \frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{b(s)}{a(s)}$$

where b depends on initial conditions

LCCODE is stable only when all poles of Y have *negative real part*, $i.e.$, roots of a are in left half plane

Initial value theorem

a general property of Laplace transforms (not just for rational F):

$$\lim_{s\to\infty} sF(s) = f(0+)$$

(can take s real in the limit)

makes connection between $f(t)$ for small t, and $F(s)$ for large s

reason: for large (real) s, se^{-st} is bunched up near $t=0$, so

$$sF(s) = \int_0^\infty se^{-st} f(t)\,dt \approx f(0+)\int_0^\infty se^{-st}\,dt = f(0+)$$

examples

- $f(t) = e^{at}$, so $F(s) = 1/(s-a)$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{s-a} = 1 = f(0)$$

- f is unit step at $t = 0$, so $F(s) = 1/s$

$$\lim_{s\to\infty} sF(s) = 1 = f(0+)$$

Final value theorem

makes connection between $f(t)$ for large t and $F(s)$ for small s

$$\lim_{t\to\infty} f(t) = sF(s)|_{s=0}$$

if the limit exists

reason: from relation between Laplace transforms and derivatives,

$$\begin{aligned}
sF(s) - f(0) &= \mathcal{L}(f') = \int_0^\infty f'(t)e^{-st}dt \\
sF(s)|_{s=0} - f(0) &= \int_0^\infty f'(t)dt = \lim_{t\to\infty} f(t) - f(0) \\
sF(s)|_{s=0} &= \lim_{t\to\infty} f(t)
\end{aligned}$$

examples

- $f(t) = 1 - e^{-t}$, so $F(s) = \dfrac{1}{s} - \dfrac{1}{s+1}$, and

$$\lim_{t\to\infty} f(t) = 1 = sF(s)|_{s=0}$$

- $F(s) = \dfrac{s}{s^2+\omega^2}$, so $f(t) = \cos \omega t$ and $\lim_{t\to\infty} f(t)$ does not exist;

  the final value theorem does not apply here