9.1 Bode’s Log Sensitivities

When the phase factor \((10 - s)/(10 + s) \) in \(P_0^{ad}\) is replaced by \((5 - s)/(5 + s)\), the magnitude of the I/O transfer function changes from \(|T|\) to \(|T^{pert}|\). \(|T^{pert}|\) is a first order approximation of \(|T^{pert}|\) computed from (9.16).

![Diagram](image)

**Figure 9.3** When the phase factor \((10 - s)/(10 + s) \) in \(P_0^{ad}\) is replaced by \((5 - s)/(5 + s)\), the magnitude of the I/O transfer function changes from \(|T|\) to \(|T^{pert}|\). \(|T^{pert}|\) is a first order approximation of \(|T^{pert}|\) computed from (9.16).

**Table 9.1** The logarithmic sensitivity of some important closed-loop transfer functions are also closed-loop transfer functions. In the general case, however, the logarithmic sensitivity of a closed-loop transfer function need not be another closed-loop transfer function.
is equivalent to the closed-loop convex specification

\[
\frac{-P_0(j\omega)K(j\omega)}{1 + P_0(j\omega)K(j\omega)} \leq 2 \quad \text{for } \omega \leq \omega_{bw}.
\] (9.18)

In (9.17) we might interpret $-S$ as the command input to tracking error transfer function, so that the specification (9.17), and hence (9.18), limits the logarithmic sensitivity of the command input to tracking error transfer function with respect to changes in $P_0$.

### 9.2 MAMS Log Sensitivity

It is possible to generalize Bode’s results to the MAMS case. We consider the MAMS 2-DOF control system (see section 2.3.4), with I/O transfer matrix

\[ T = (I + P_0 K_y)^{-1} P_0 K_r. \]

If the plant is perturbed so that $P_0$ becomes $P_0 + \delta P_0$, we have

\[ T + \delta T = (I + (P_0 + \delta P_0) K_y)^{-1} (P_0 + \delta P_0) K_r. \]

Retaining only first order terms in $\delta P_0$ we have

\[
\begin{align*}
\delta T & \simeq (I + P_0 K_y)^{-1} \delta P_0 K_r - (I + P_0 K_y)^{-1} \delta P_0 K_y (I + P_0 K_y)^{-1} P_0 K_r \\
& = (I + P_0 K_y)^{-1} \delta P_0 (I + K_y P_0)^{-1} K_r \\
& = S \delta P_0 (I + K_y P_0)^{-1} K_r,
\end{align*}
\] (9.19)

where $S = (I + P_0 K_y)^{-1}$ is the (output-referred) sensitivity matrix of the MAMS 2-DOF control system.

Now suppose that we can express the change in $P_0$ as

\[ \delta P_0 = \delta P_0^{\text{frac}} P_0. \]

We can interpret $\delta P_0^{\text{frac}}$ as an output-referred fractional perturbation of $P_0$, as shown in figure 9.4. Then from (9.19) we have

\[ \delta T \simeq S \delta P_0^{\text{frac}} P_0 (I + K_y P_0)^{-1} K_r = S \delta P_0^{\text{frac}} T, \]

so that

\[ \delta T \simeq \delta T^{\text{frac}} T, \]

where

\[ \delta T^{\text{frac}} \simeq S \delta P_0^{\text{frac}}. \] (9.20)

This is analogous to (9.2): it states that the output-referred fractional change in the I/O transfer matrix $T$ is, to first order, the sensitivity matrix $S$ times the output-referred fractional change in $P_0$. 
The design specification,

\[ \sigma_{\text{max}}(\delta T^{\text{frac}}(j\omega)) \leq 0.01 \quad \text{for} \quad \omega \leq \omega_{\text{bw}}, \quad \sigma_{\text{max}}(\delta P_0^{\text{frac}}(j\omega)) \leq 0.20, \quad (9.21) \]

which limits the first order fractional change in \( T \) to 1% over the bandwidth \( \omega_{\text{bw}} \), despite variations in \( P_0 \) of 20%, is therefore equivalent to the closed-loop convex specification

\[ \sigma_{\text{max}}(S(j\omega)) \leq 0.05 \quad \text{for} \quad \omega \leq \omega_{\text{bw}}. \quad (9.22) \]

We remind the reader that the inequality in (9.21) holds only to first order in \( \delta P_0^{\text{frac}} \); its precise meaning is

\[ \lim_{\delta P_0 \to 0} \frac{\sigma_{\text{max}}(\delta T^{\text{frac}}(j\omega))}{\sigma_{\text{max}}(\delta P_0^{\text{frac}}(j\omega))} \leq \frac{0.01}{0.20} \quad \text{for} \quad \omega \leq \omega_{\text{bw}}. \]

### 9.2.1 Cruz and Perkins' Comparison Sensitivity

Cruz and Perkins gave another generalization of Bode's log sensitivity to the MAMS 2-DOF control system using the concept of *comparison sensitivity*, in which the perturbed closed-loop system is compared to an equivalent open-loop system.

The open-loop equivalent system consists of \( P_0 + \delta P_0 \) driven by the unperturbed actuator signal, as shown at the top of figure 9.5. For \( \delta P_0 = 0 \), the open-loop equivalent system is identical to the closed-loop system shown at the bottom of figure 9.5. For \( \delta P_0 \) nonzero, however, the two systems differ. By comparing the first order changes in these two systems, we can directly see the effect of the feedback on the perturbation \( \delta P_0 \).

The transfer matrix of the open-loop equivalent system is

\[ T^{\text{ole}} = (P_0 + \delta P_0)(I + K_{\delta}P_0)^{-1}K_r, \]

so that

\[ \delta T^{\text{ole}} = \delta P_0(I + K_{\delta}P_0)^{-1}K_r. \]
Figure 9.5 In the open-loop equivalent system, shown at top, the actuator signal $u$ drives $P_0 + \delta P_0$, so there is no feedback around the perturbation $\delta P_0$. The benefit of feedback can be seen by comparing the first order changes in the transfer matrices from $r$ to $y_p^\text{ole}$ and $y_p^\text{pert}$, respectively (see (9.23)).

Comparing this to the first order change in the I/O transfer matrix, given in (9.19), we have

$$\delta T \simeq \delta T^\text{ole}.$$  \hfill (9.23)

This simple equation shows that the first order variation in the I/O transfer matrix is equal to the sensitivity transfer matrix times the first order variation in the open-loop equivalent system. It follows that the specification (9.22) can be interpreted as limiting the sensitivity of the I/O transfer matrix to be no more than 5% of the sensitivity of I/O transfer matrix of the open-loop equivalent system.

### 9.3 General Differential Sensitivity

The general expression for the first order change in the closed-loop transfer matrix $H$ due to a change in the plant transfer matrix is

$$\delta H \simeq \delta P_{zw} + \delta P_{zu} K (I - P_{yu} K)^{-1} P_{yw} + P_{zu} K (I - P_{yu} K)^{-1} \delta P_{yw} + P_{zw} K (I - P_{yu} K)^{-1} \delta P_{yu} K (I - P_{yw} K)^{-1} P_{yw},$$  \hfill (9.24)
The last term shows that \( \frac{\partial H}{\partial P_{yu}} \) (which is a complicated object with four indices) has components that are given by the product of two closed-loop transfer functions. It is usually the case that design specifications that limit the size of \( \frac{\partial H}{\partial P_{yu}} \) are not closed-loop convex, since, roughly speaking, a product can be made small by making either of its terms small.

### 9.3.1 An Example

Using the standard example plant and controller \( K^{(a)} \), described in section 2.4, we consider the sensitivity of the I/O step response with respect to gain variations in \( P^{\text{std}}_0 \). Since \( \delta P^{\text{std}}_0 = \alpha P^{\text{std}}_0 \), the sensitivity

\[
s_\alpha(t) \triangleq \left. \frac{\partial s(t)}{\partial \alpha} \right|_{\alpha=0}
\]

is simply the unit step response of the transfer function

\[
\frac{P^{\text{std}}_0 K^{(a)}}{(1 + P^{\text{std}}_0 K^{(a)})^2} = ST. \tag{9.25}
\]

This transfer function is the product of two closed-loop transfer functions, which is consistent with our general comments above.

In figure 9.6 the actual effect of a 20% gain reduction in \( P^{\text{std}}_0 \) on the step response is compared to the step response predicted by the first order perturbational analysis,

\[
s^{\text{approx}}(t) = s(t) - 0.2s_\alpha(t),
\]

with the controller \( K^{(a)} \). The step response sensitivity with this controller is shown in figure 9.7. For plant gain changes between ±20%, the first order approximation to the step response falls in the shaded envelope \( s(t) ± 0.2s_\alpha(t) \).

We now consider the specification

\[
|s_\alpha(1)| \leq 0.75, \tag{9.26}
\]

which limits the sensitivity of the step response at time \( t = 1 \) to gain variations in \( P^{\text{std}}_0 \). We will show that this specification is not convex.

The controller \( K^{(a)} \) yields a closed-loop transfer matrix \( H^{(a)} \) with \( s_\alpha^{(a)}(1) = 0.697 \), so \( H^{(a)} \) satisfies the specification (9.26). The controller \( K^{(b)} \) yields a closed-loop transfer matrix \( H^{(b)} \) with \( s_\alpha^{(b)}(1) = 0.702 \), so \( H^{(b)} \) also satisfies the specification (9.26). However, the average of these two transfer matrices, \( (H^{(a)} + H^{(b)})/2 \), has a step response sensitivity at \( t = 1 \) of 0.786, so \( (H^{(a)} + H^{(b)})/2 \) does not satisfy the specification (9.26). Therefore the specification (9.26) is not convex.
When $P_{\text{std}}^0$ is replaced by $0.8P_{\text{std}}^0$, the step response changes from $s$ to $s^\text{pert}$. The first order approximation of $s^\text{pert}$ is given by $s^\text{approx}(t) = s(t) - 0.2s_{\alpha}(t)$.

The sensitivity of the step response to plant gain changes is shown for the controller $K^{[s]}$. The first order approximation of the step response falls in the shaded envelope when $P_{\text{std}}^0$ is replaced by $\alpha P_{\text{std}}^0$, for $0.8 \leq \alpha \leq 1.2$. 

Figure 9.6

Figure 9.7
9.3.2 Some Convex Approximations

In many cases there are useful convex approximations to specifications that limit general differential sensitivities of the closed-loop system.

Consider the specification

\[ |s_\alpha(t)| \leq 0.75 \quad \text{for } t \geq 0, \]  

(9.27)

which limits the sensitivity of the step response to gain variations in \( P_0 \). This specification is equivalent to

\[ \|ST\|_{pk, step} \leq 0.75, \]

which is not closed-loop convex. We will describe two convex approximations for the nonconvex specification (9.27).

Suppose

\[ s_{\min}(t) \leq s(t) \leq s_{\max}(t) \quad \text{for } t \geq 0 \]  

(9.28)

is a design specification (see figure 8.5). A weak approximation of the sensitivity specification (9.26) (along with the step response specification (9.28)) is that a typical (and therefore fixed) step response satisfies the specification:

\[ \|ST_{typ}\|_{pk, step} \leq 0.75, \]

where \( T_{typ} \) is the transfer function that has unit step response

\[ s_{typ}(t) = \frac{s_{\min}(t) + s_{\max}(t)}{2}. \]

A stronger approximation of (9.27) (along with the step response specification (9.28)) requires that the sensitivity specification be met for every step response that satisfies (9.28):

\[ \max \{ \|Sv\|_\infty \mid s_{\min}(t) \leq v(t) \leq s_{\max}(t) \text{ for } t \geq 0 \} \leq 0.75. \]

This is an inner approximation of (9.27), meaning that it is tighter than (9.27).
Notes and References

Feedback and Sensitivity

The ability of feedback to make a system less sensitive to changes in the plant is discussed in essentially every book on feedback and control; see Mayr [May70] for a history of this idea. An early discussion (in the context of feedback amplifiers) can be found in Black [BLA34], in which we find:

\[ \text{... by building an amplifier whose gain is deliberately made, say 40dB higher than necessary, and then feeding the output back on the input in such a way as to throw away the excess gain, it has been found possible to effect extraordinary improvement in constancy of amplification ...} \]

By employing this feedback principle, amplifiers have been built and used whose gain varied less than 0.01dB with a change in plate voltage from 24V to 26V [whereas for an amplifier of conventional design and comparable size this change would have been 0.7dB].

For a later discussion see Horowitz [Hor63, ch3]. A concise discussion appears in chapter 1, On the Advantages of Feedback, of Callier and Desoer [CD82A].

Differential Sensitivity

Bode [BOD45] was the first to systematically study the effect of small (differential) changes in closed-loop transfer functions due to small (differential) changes in the plant. On page 33 of [BOD45] we find (with our corresponding notation substituted),

\[ \text{The variation in the final gain characteristic [T] in dB, per dB change in the gain of [P], is reduced in the ratio [S].} \]

A recent exposition of differential sensitivity can be found in chapter 3 of Lunze [LUN89].

Comparison Sensitivity

The notion of comparison sensitivity was introduced by Cruz and Perkins in [CP64]; see also the book edited by Cruz [Cru73]. The idea of an open-loop equivalent system, however, is older. In [NGK57, §1.7], it is called the equivalent cascade configuration of the control system. Recent discussions of comparison sensitivity can be found in Callier and Desoer [CD82A, ch1] and Anderson and Moore [AM90, §5.3].

Sensitivity Specifications that Limit Control System Performance

The idea that sensitivity or robustness specifications can limit the achievable control system performance is explicitly expressed in, e.g., Newton, Gould, and Kaiser [NGK57, p23]:

\[ \text{Control systems often employ mechanical, hydraulic, or pneumatic elements which have less reproducible behavior than high quality electric circuit elements. This practical problem often causes the control designer to stop short of an optimum design because he knows full well that the parameters of the physical system may deviate considerably from the data on which he bases his design.} \]

A more recent paper that raised this issue, in the context of regulators designed by state-space methods, is Doyle and Stein [DS81].
Chapter 10

Robustness Specifications via Gain Bounds

In this chapter we consider robustness specifications, which limit the worst case variation in the closed-loop system that can be caused by a specific set of plant variations. We describe a powerful method for formulating inner approximations of robustness specifications as norm-bounds on the nominal closed-loop transfer matrix. These specifications are closed-loop convex.

In the previous chapter we studied the differential sensitivity of the closed-loop system to variations in the plant. Differential sensitivity analysis often gives a good prediction of the changes that occur in the closed-loop system when the plant changes by a moderate (non-vanishing) amount, and hence, designs that satisfy differential sensitivity specifications are often robust to moderate changes in the plant. But differential sensitivity specifications cannot guarantee that the closed-loop system does not change dramatically (e.g., become unstable) when the plant changes by a non-vanishing amount.

In this chapter we describe robustness specifications, which, like differential sensitivity specifications, limit the variation in the closed-loop system that can be caused by a change or perturbation in the plant. In this approach, however,

- the sizes of plant variations are explicitly described, e.g., a particular gain varies \( \pm 1 \) dB,

- robustness specifications limit the worst case change in the closed-loop system that can be caused by one of the possible plant perturbations.

By contrast, in the differential sensitivity approach,

- the sizes of plant variations are not explicitly described; they are vaguely described as “small”,
differential sensitivity specifications limit the first order changes in the closed-loop system that can be caused by the plant perturbations.

Robustness specifications give guaranteed bounds on the performance deterioration, even for "large" plant variations, for which extrapolations from differential sensitivity specifications are dubious. Offsetting this advantage are some possible disadvantages of robustness specifications over differential sensitivity specifications:

- It may not be possible to model the actual variations in the plant in the precise way required by robustness specifications. For example, we may not know whether to expect a ±1dB or a ±0.5dB variation in a particular gain.

- It may not be desirable to limit the worst case variation in the closed-loop system, which results in a conservative design. A specification that limits the typical variations in the closed-loop system (however we may define typical) may better capture the designer's intention.

Robustness specifications are often not closed-loop convex, just as the most general specifications that limit differential sensitivity are not closed-loop convex. We will describe a general small gain method for formulating convex inner approximations of robustness specifications; the Notes and References for this chapter describe some of the attempts that have been made to make approximations of robustness specifications that are less conservative, but not convex. Since we will be describing convex approximations of robustness specifications, we should add the following item to the list of possible disadvantages:

- The small gain based convex inner approximations of robustness specifications can be poor approximations. Thus, designs based on these approximations can be conservative.

This topic is addressed in some of the references at the end of this chapter.

In the next section we give a precise and general definition of a robustness specification, which may appear abstract on first reading. In the remainder of this chapter we describe the framework for small gain methods, and then the small gain methods themselves. The framework and methods are demonstrated on some simple, specific examples that are based on our standard example SASS 1-DOF control system described in section 2.4. These examples continue throughout the chapter.

10.1 Robustness Specifications

10.1.1 Some Definitions

In this section we give a careful definition of a robustness specification; we defer until the next section examples of common robustness specifications. Roughly speaking,
10.1 ROBUSTNESS SPECIFICATIONS

A robustness specification requires that some design specification \( \mathcal{D} \) must hold, even if the plant \( P \) is replaced by any \( P^{pert} \) from a specified set \( \mathcal{P} \) of possible perturbed plants.

Let us be more precise. Suppose that \( \mathcal{P} \) is any set of \((n_w + n_u) \times (n_z + n_y)\) transfer matrices. We will refer to \( \mathcal{P} \) as the perturbed plant set, and its elements as perturbed plants. Let \( \mathcal{D} \) denote some design specification, i.e., a boolean function on \( n_z \times n_w \) transfer matrices, and let \( K \) denote any \( n_u \times n_y \) transfer matrix.

**Definition 10.1:** We say \( \mathcal{D} \) holds robustly for \( K \) and \( \mathcal{P} \) if for each \( P^{pert} \in \mathcal{P} \), \( \mathcal{D} \) holds for the transfer matrix \( P^{pert}_{zw} + P^{pert}_{zu}K(I - P^{pert}_{yu}K)^{-1}P^{pert}_{yw} \).

In words, the design specification \( \mathcal{D} \) holds robustly for \( K \) and \( \mathcal{P} \) if the controller \( K \) connected to any of the perturbed plants \( P^{pert} \in \mathcal{P} \) yields a closed-loop system that satisfies \( \mathcal{D} \). Definition 10.1 is not, by itself, a design specification: it is a property of a controller and a set of transfer matrices. Note also that definition 10.1 makes no mention of the plant \( P \).

Once we have the concept of a design specification holding robustly for a given controller and perturbed plant set, we can define the notion of a robustness specification, which will involve the plant \( P \).

**Definition 10.2:** The robustness specification \( \mathcal{D}_{rob} \) formed from \( \mathcal{D} \), \( \mathcal{P} \), and \( P \) is given by:

\[
\mathcal{D}_{rob} : \quad \mathcal{D} \text{ holds robustly for } K \text{ and } \mathcal{P},
\]

for every \( K \) that satisfies

\[
H = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}.
\] (10.1)

Thus a robustness specification is formed from a design specification \( \mathcal{D} \), a perturbation plant set \( \mathcal{P} \), and the plant \( P \). The reader should note that \( \mathcal{D}_{rob} \) is indeed a design specification: it is a boolean function of \( H \). We can interpret \( \mathcal{D}_{rob} \) as follows: if \( H \) satisfies \( \mathcal{D}_{rob} \) and \( K \) is any controller that yields the closed-loop transfer matrix \( H \) (when connected to \( P \)), the closed-loop transfer matrix that results from connecting \( K \) to any \( P^{pert} \in \mathcal{P} \) will all satisfy \( \mathcal{D} \).

A sensible formulation of the plant will including signals such as sensor and actuator noises and the sensor and actuator signals (recall chapter 7). In this case the controller \( K \) is uniquely determined by a closed-loop transfer matrix \( H \) that is realizable, since \((I - P_{yu}K)^{-1}\) will appear as a submatrix of \( H \), and we can determine \( K \) from this transfer matrix. In these cases we may substitute “the \( K \)” for “every \( K \)” in the definition 10.2.

In many cases, \( P \in \mathcal{P} \), and \( \mathcal{P} \) consists of transfer matrices that are “close” to \( P \). In this context \( P \) is sometimes called the nominal plant. In this case the robustness specification \( \mathcal{D}_{rob} \) requires that even with the worst perturbed plant substituted for the nominal plant, the design specification \( \mathcal{D} \) will continue to hold.
If the design specification $D$ is $D_{\text{stable}}$, i.e., closed-loop stability (see chapter 7), we call $D_{\text{rob}}$ the robust stability design specification associated with $P$ and $P$.

Throughout this chapter, $P$ is understood, so the robustness specification will be written

$$D_{\text{rob}}(P, D).$$

The robust stability specification associated with the perturbed plant set $P$ will be denoted

$$D_{\text{rob, stab}}(P) \triangleq D_{\text{rob}}(P, D_{\text{stable}}).$$

### 10.1.2 Time-Varying and Nonlinear Perturbations

It is possible to extend the perturbed plant set to include time-varying or nonlinear systems, although this requires some care since many of our basic concepts and notation depend on our assumption 2.2 that the plant is LTI. Such an extension is useful for designing a controller $K$ for a nonlinear or time-varying plant $P_{\text{nonlin}}$. The controller $K$ is often designed for a “nominal” LTI plant $P$ that is in some sense “close” to $P_{\text{nonlin}}$; $P_{\text{nonlin}}$ is then considered to be a perturbation of $P$.

In this section we briefly and informally describe how we may modify our framework to include such nonlinear or time-varying perturbations. In this case the perturbed plant is a nonlinear or time-varying system with $n_w + n_u$ inputs and $n_z + n_y$ outputs. The perturbed closed-loop system, obtained by connecting the controller between the signals $y$ and $u$ of the perturbed plant, is now also nonlinear or time-varying, so the perturbed closed-loop system cannot be described by an $n_z \times n_w$ transfer matrix, as in (10.1). Instead, the closed-loop system is described by the nonlinear or time-varying closed-loop operator that maps $w$ into the resulting $z$.

A design specification will simply be a predicate of the closed-loop system. The only predicate that we will consider is closed-loop stability, which, roughly speaking, means that $z$ is bounded whenever $w$ is bounded (the definition of closed-loop stability given in chapter 7 does not apply here, since it refers to the transfer matrix $H$). The reader can consult the references given at the end of this chapter for precise and detailed definitions of closed-loop stability of nonlinear or time-varying systems.

The robust stability specification $D_{\text{rob, stab}}$ will mean that when the (LTI) controller $K$, which is designed on the basis of the (LTI) plant $P$, is connected to any of the nonlinear or time-varying perturbed plants in $P$, the resulting (nonlinear or time-varying) closed-loop system is stable.

### 10.2 Examples of Robustness Specifications

In this section we consider some examples of robustness specifications, organized by their associated plant perturbation sets. Most of these robustness specifications
are not convex, but later in this chapter we describe a general method of forming convex inner approximations of these specifications.

10.2.1 Finite Plant Perturbation Sets

A simple but important case occurs when $\mathcal{P}$ is a finite set:

$$\mathcal{P} = \{P_1, \ldots, P_N\}.$$ (10.2)

Neglected Dynamics

Recall from chapter 1 that $P$ may be a simple (but not very accurate) model of the system to be controlled. Our perturbed plant set might then be $\mathcal{P} = \{P^{\text{comp}}\}$, where $P^{\text{comp}}$ is a complex, detailed, and accurate model of the system to be controlled. In this case, the robustness specification $\mathcal{D}_{\text{rob}}$ guarantees that the controller we design using the simple model $P$, will, when connected to $P^{\text{comp}}$, satisfy the design specification $\mathcal{D}$.

As a specific example, suppose that our plant is our standard numerical example, the 1-DOF controller described in section 2.4, with plant

$$P = \left[ \begin{array}{c|c} P_{zw} & P_{zy} \\ \hline P_{yw} & P_{yu} \end{array} \right] = \left[ \begin{array}{c|c|c} P^0_{\text{std}} & 0 & 0 \\ 0 & 0 & 0 \\ -P^0_{\text{std}} & -1 & 1 \\ -P^0_{\text{std}} & 0 & 1 \end{array} \right].$$

The more detailed model of the system to be controlled might take into account a high frequency resonance and roll-off in the system dynamics and some fast sensor dynamics, neither of which is included in the plant model $P$:

$$P^{\text{comp}} = \left[ \begin{array}{c|c|c} P^0_{\text{comp}} & 0 & 0 \\ 0 & 0 & 0 \\ -P^0_{\text{comp}} H_{\text{sens}} & -H_{\text{sens}} & 1 \\ -P^0_{\text{comp}} H_{\text{sens}} & 0 & 1 \end{array} \right],$$

where

$$P^0_{\text{comp}}(s) = \frac{P^0_{\text{std}}(s)}{1 + 1.25(s/100) + (s/100)^2}, \quad H_{\text{sens}}(s) = \frac{1}{1 + s/80}.$$

This is shown in figure 10.1 below (c.f. figure 2.11).

For this example, the perturbed plant set is

$$\mathcal{P} = \{P^{\text{comp}}\}.$$ (10.3)

The robust stability specification $\mathcal{D}_{\text{rob,stab}}$ that corresponds to (10.3) requires that the controller designed on the basis of the nominal plant $P$ will also stabilize the complex model $P^{\text{comp}}$ of the system to be controlled. Roughly speaking, $\mathcal{D}_{\text{rob,stab}}$ requires that the system cannot be made unstable by the high frequency resonance and roll-off in the system dynamics and the dynamics of the sensor, which are ignored in the model $P$. 
Failure Modes

The perturbed plants in (10.2) may represent different failure modes of the system to be controlled. For example, $P_1$ might be a model of the system to be controlled after an actuator has failed (i.e., $P_1$ is $P$, but with the column associated with the failed actuator set to zero). In this case the specification of robust stability guarantees that the closed-loop system will remain stable, despite the failures modeled by $P_1,\ldots,P_N$.

10.2.2 Parametrized Plant Perturbations

In some cases the perturbed plant set $\mathcal{P}$ can be described by some parameters that vary over ranges:

$$\mathcal{P} = \{ P_{\text{pert}}(\alpha) \mid L_1 \leq \alpha_1 \leq U_1, \ldots, L_k \leq \alpha_k \leq U_k \}.$$  

In this case we often have $P \in \mathcal{P}$; the corresponding parameter is called the nominal parameter:

$$P = P_{\text{pert}}(\alpha^{\text{nom}}).$$

Parametrized plant perturbation sets can be used to model several different types of plant variation:

- **Component tolerances.** A single controller $K$ is to be designed for many plants, for example, a manufacturing run of the system to be controlled. The controller is designed on the basis of a nominal plant; the parameter variations represent the (slight, one hopes) differences in the individual manufactured systems. Designing a controller that robustly achieves the design specifications avoids the need and cost of tuning each manufactured control system (see section 1.1.5).
10.2 Examples of Robustness Specifications

- **Component drift or aging.** A controller is designed for a system that is well modeled by \( P \), but it is desired that the system should continue to work if or when the system to be controlled changes, due to aging or drift in its components. Designing a controller that robustly achieves the design specifications avoids the need and cost of periodically re-tuning the control system.

- **Externally induced changes.** The system to be controlled may be well modeled as an LTI system that depends on an external operating condition, which changes slowly compared to the system dynamics. Examples include temperature induced variations in a system, and the effects of varying aerodynamic pressure on aircraft dynamics. Designing a controller that robustly achieves the design specifications can avoid the need for a gain-scheduled or adaptive controller. (See the Notes and References in chapter 2.)

- **Model parameter uncertainty.** A parametrized perturbed plant set can model uncertainty in modeling the system to be controlled (see section 1.1.2). In a model developed from physical principles, the \( \alpha_i \) may represent physical parameters such as lengths, masses, and heat conduction coefficients, and the bounds \( L_i \) and \( U_i \) are then minimum and maximum values that could be expected to occur. In a black box model derived from an identification procedure, the \( \alpha_i \) could represent transfer function coefficients, and the \( L_i \) and \( U_i \) might represent the 90% confidence bands for the identified model.

**Example: Gain Margins**

Gain margin specifications are examples of classical robustness specifications that are associated with a parametrized plant perturbation set. We consider the classical 1-DOF controller, with perturbed plant set described informally by

\[
P_0^{\text{pert}}(s) = \alpha P_0(s), \quad L \leq \alpha \leq U.
\]

More precisely, we have the perturbed plant set

\[
P = \left\{ \begin{bmatrix} \alpha P_0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha P_0 & -1 & 1 \end{bmatrix} \left| \begin{array}{c} L \leq \alpha \leq U \end{array} \right. \right\}, \quad \alpha^{\text{nom}} = 1, \quad (10.4)
\]

where \( 0 < L \leq 1 \leq U \).

The specification of robust stability with the perturbed plant set (10.4) is described in classical terms as a positive gain margin of \( 20 \log_{10} U \text{dB} \) and a negative gain margin of \( 20 \log_{10} L \text{dB} \).

As an example we will use later, the robustness specification that requires gain margins of +4dB and -3.5dB is given by

\[
P_{+4,-3.5\text{dB gm}} : \quad \mathcal{D}_{\text{rob,stab}}(P) \quad \text{with} \quad L = 0.668, \quad U = 1.585, \quad (10.5)
\]

with the plant perturbation set (10.4).
Example: Pole Variation

The parameter vector $\alpha$ may determine pole or zero locations in the plant. As a specific example, consider the perturbed plant set for the standard 1-DOF example of section 2.4, described informally by

$$P^\text{pert}_0(s) = \frac{1}{\frac{\alpha - s}{s \alpha + s}}, \quad 5 \leq \alpha \leq 15, \quad \alpha^\text{nom} = 10. \quad (10.6)$$

Some of these phase variations in $P^\text{std}_0$ are shown in figure 10.2.

![Figure 10.2](image)

**Figure 10.2** The perturbed plants described by (10.6) consist of a phase shift in $P^\text{std}_0$: shown here are the phase shifts for $\alpha - \alpha^\text{nom} = \pm 1, \pm 3, \pm 5$. (cf. figure 9.2, which shows a particular phase shift.)

The robust stability specification $D^\text{rob,stab}_\alpha$ is a strengthening of the stability specification $D^\text{stable}_\alpha$: $D^\text{rob,stab}_\alpha$ requires that the controller stabilize not only the plant $P$, but also the perturbed plants (10.6). $D^\text{rob,stab}_\alpha$ can be thought of as a type of phase margin specification.

### 10.2.3 Unknown-but-Bounded Transfer Function Perturbations

It is often useful to model the uncertainty in the plant (as a model of the system to be controlled) as frequency-dependent errors in the frequency responses of its entries. Such plant perturbation sets can be used to account for:

- **Model uncertainty.** The plant transfer functions may inaccurately model the system to be controlled because of measurement or identification errors. For
10.2 Examples of Robustness Specifications

Examples of robustness specifications can be illustrated by considering the transfer functions of the system to be controlled. These may have been measured at each frequency to an accuracy of 1%, or these measurements might be repeatable only to 1%.

- **High frequency parasitic dynamics.** A model of the system to be controlled may become less accurate at high frequencies because of unknown or unmodeled parasitic dynamics. Moreover, these parasitic dynamics may change with time or other physical parameters, and so cannot be confidently modeled. In electrical systems, for example, we may have small stray capacitances and (self and mutual) inductances between conductors; these parasitic dynamics can change significantly when the electrical or magnetic environment of the system is changed.

In state-space plant descriptions, the addition of high frequency parasitic dynamics is called a *singular perturbation*, because the perturbed plant has more states than the plant.

Since these plant perturbation sets cannot be described by the variation of a small number of real parameters, they are sometimes called *nonparametric plant perturbations*.

There are some subtle distinctions between intentionally neglected system dynamics that could in principle be modeled, and parasitic dynamics that cannot be confidently modeled. For example, a model of a mechanical system may be developed on the assumption that a drive train is rigid, an assumption that is good at low frequencies, but poor at high frequencies. If the high frequency dynamics of this drive train could be accurately modeled or consistently measured, then we could develop a more accurate (and more complex) model of the system to be controlled, as in section 10.2.1. However, it may be the case that these high frequency dynamics are very sensitive to minor physical variations in the system, such as might be induced by temperature changes, bearing wear, and so on. In this case the drive train dynamics could reasonably be modeled as an unknown transfer function that is close to one at low frequencies, and less close at high frequencies.

**Example: Relative Uncertainty in** $P_0$

We consider again our standard SASS 1-DOF control system example. Suppose we believe that the relative or fractional error in the transfer function $P_0$ (as a model of the system to be controlled) is about 20% at low frequencies (say, $\omega \leq 5$), and much larger at high frequencies (say, up to 400% for $\omega \geq 500$). We define the relative error as

$$\dot{P}_0^\text{rel.err} \triangleq \frac{P_0^\text{pert} - P_0}{P_0},$$

so that

$$P_0^\text{pert} = (1 + \dot{P}_0^\text{rel.err})P_0. \tag{10.7}$$
Our plant model uncertainty can be described by a frequency-dependent limit on the magnitude of $P_0^{\text{rel.err}}$, e.g.,
\[
\|P_0^{\text{rel.err}}/W_{\text{rel.err}}\|_\infty \leq 1,
\]
where
\[
W_{\text{rel.err}}(s) = \frac{0.2 + s/10}{1 + s/200}. \tag{10.9}
\]
We interpret $|W_{\text{rel.err}}(j\omega)|$, which is plotted in figure 10.3, as the maximum relative error in $P_0(j\omega)$. We say that $P_0^{\text{rel.err}}$ is an unknown-but-bounded transfer function. One interpretation is shown in figure 10.4.

![Figure 10.3](image)

**Figure 10.3** From (10.8), $|W_{\text{rel.err}}(j\omega)|$ is the maximum relative error in $P_0(j\omega)$. This represents a relative error of 20% at low frequencies and up to 400% at high frequencies.

The perturbed plant set for this example is thus
\[
\mathcal{P} = \left\{ \begin{bmatrix} P_0^{\text{pert}} & 0 & 0 \\ 0 & 0 & 0 \\ -P_0^{\text{pert}} & -1 & 1 \end{bmatrix} | \| P_0^{\text{rel.err}}/W_{\text{rel.err}} \|_\infty \leq 1 \right\}, \tag{10.10}
\]
where $P_0^{\text{pert}}$ is given by (10.7).

We note for future reference two robustness specifications using the perturbed plant set (10.10). The first is robust stability, $\mathcal{D}_{\text{rob.stab}}$, and the second is the
Figure 10.4 The complex transfer function $P_0^{pert}(j\omega)$ is shown versus frequency $\omega$. Circles that are centered at the nominal plant transfer function, $P_0(j\omega)$, with radius $|W_{red,\omega}(j\omega)P_0(j\omega)|$ are also shown. (10.7) and (10.8) require that the perturbed plant transfer function, $P_0^{pert}$, must lie within the region enclosed by these circles.

A stronger specification that these plant perturbations never cause the RMS gain from the reference input to the actuator signal to exceed $75$:

$$D_{rob}(P, \|T/P_0\|_\infty \leq 75)$$ (10.11)

($T/P_0$ is the transfer function from the reference input to the actuator signal).

### 10.2.4 Neglected Nonlinearities

**Example: Actuator Saturation**

A common nonlinearity encountered in systems to be controlled is saturation of the actuator signals, shown in figure 10.5. This system is described by

$$\begin{bmatrix} z \\ y \end{bmatrix} = P_{lin} \begin{bmatrix} w \\ q_{sat} \end{bmatrix}, \quad (10.12)$$

$$u_{sat}(t) = S_i Sat(u_i(t)/S_i), \quad i = 1, \ldots, n_u, \quad (10.13)$$
where $P^\text{lin}$ is an LTI system and the unit saturation function $\text{Sat}: \mathbb{R} \to \mathbb{R}$ is defined by

$$\text{Sat}(a) \triangleq \begin{cases} 
    a & |a| \leq 1 \\
    1 & a > 1 \\
    -1 & a < -1.
\end{cases}$$  \hfill (10.14)

$S_i$ is called the saturation threshold of the $i$th actuator: if the magnitude of each actuator signal is always below its saturation threshold, then the system is LTI, with transfer matrix $P^\text{lin}$.

One approach to designing a controller for this nonlinear system is to define the LTI plant $P = P^\text{lin}$, and consider the saturation as a nonlinear perturbation of $P$. Thus we consider the perturbed plant set consisting of the single nonlinear system given by (10.12–10.13):

$$P = \{P^\text{nonlin}\}.$$  \hfill (10.15)

By designing an LTI controller for $P$ that yields robust stability for (10.15), we are guaranteed that when the controller is connected to the nonlinear system to be controlled, the resulting nonlinear closed-loop system will at least be stable. If the actuator signals only occasionally exceed their thresholds, then the closed-loop transfer matrix $H$ can give a good approximation of the behavior of the nonlinear closed-loop system. (Another useful approach, also based on the idea of considering the saturators as a perturbation of an LTI plant, is described in the Notes and References.)

We should also mention the describing function method, a heuristic approach that is often successful in practice. Roughly speaking, the describing function method approximates the effect of the saturators as a gain reduction in the actuator channels: such perturbations are handled via gain margin specifications. Of course, gain margin specifications do not guarantee that the nonlinear closed-loop system is stable.
10.3 Perturbation Feedback Form

In many cases the perturbed plant set $\mathcal{P}$ can be represented as the nominal plant with an internal feedback, as shown in figure 10.6. When the internal feedback $\Delta$ is zero, we recover the nominal plant $P$; each perturbed plant in $\mathcal{P}$ corresponds to a particular feedback $\Delta \in \Delta$, where $\Delta$ is a set of transfer matrices of the appropriate size.

![Figure 10.6](image)

Each perturbed plant is equivalent to the nominal plant modified by the internal feedback $\Delta$.

We will call $\Delta$ the feedback perturbation. The perturbed plant that results from the feedback perturbation $\Delta$ will be denoted $P_{\text{pert}}(\Delta)$, and $\Delta$ will be called the feedback perturbation set that corresponds to $\mathcal{P}$:

$$\mathcal{P} = \{ P_{\text{pert}}(\Delta) \mid \Delta \in \Delta \}. \quad (10.16)$$

The symbol $\Delta$ emphasizes its role in “changing” the plant $P$ into the perturbed plant $P_{\text{pert}}$.

The input signal to the perturbation feedback, denoted $q$, can be considered an output signal of the plant $P$. Similarly, the output signal from the perturbation feedback, denoted $p$, can be considered an input signal to the plant $P$. Throughout this chapter we will assume that the exogenous input signal $w$ and the regulated output signal $z$ are augmented to contain $p$ and $q$, respectively:

$$w = \begin{bmatrix} \tilde{w} \\ p \end{bmatrix}, \quad z = \begin{bmatrix} \tilde{z} \\ q \end{bmatrix},$$

where $\tilde{w}$ and $\tilde{z}$ denote the original signals from figure 10.6. This is shown in figure 10.7.

To call $p$ an exogenous input signal can be misleading, since this signal does not originate “outside” the plant, like command inputs or disturbance signals, as
the term exogenous implies. We can think of the signal \( p \) as originating outside the nominal plant, as in figure 10.6.

To describe a perturbation feedback form of a perturbed plant set \( \mathcal{P} \), we give the (augmented) plant transfer matrix

\[
P = \begin{bmatrix}
P_{\tilde{z}\tilde{w}} & P_{\tilde{z}p} & P_{\tilde{z}u} \\
P_{q\tilde{w}} & P_{qp} & P_{qu} \\
P_{y\tilde{w}} & P_{yp} & P_{yu}
\end{bmatrix},
\]

along with the set \( \Delta \) of perturbation feedbacks. Our original perturbed plant can be expressed as

\[
P_{\text{pert}}(\Delta) = \begin{bmatrix} P_{\tilde{z}\tilde{w}} & P_{\tilde{z}u} \\
P_{q\tilde{w}} & P_{qu} \end{bmatrix} + \begin{bmatrix} P_{q\tilde{w}} & P_{q\bar{u}} \end{bmatrix} \Delta(I - P_{qp} \Delta)^{-1} \begin{bmatrix} P_{q\tilde{w}} & P_{qu} \end{bmatrix}. \tag{10.17}
\]

The perturbation feedback form, i.e., the transfer matrix \( P \) in (10.17) and the set \( \Delta \), is not uniquely determined by the perturbed plant set \( \mathcal{P} \). This fact will be important later.

When \( \mathcal{P} \) contains nonlinear or time-varying systems, the perturbation feedback form consists of an LTI \( P \) and a set \( \Delta \) of nonlinear or time-varying systems. Roughly speaking, the feedback perturbation \( \Delta \) represents the extracted nonlinear or time-varying part of the system. We will see an example of this later.

### 10.3.1 Perturbation Feedback Form: Closed-Loop

Suppose now that the controller \( K \) is connected to the perturbed plant \( P_{\text{pert}}(\Delta) \), as shown in figures 10.8 and 10.9.
When the perturbed plant set is expressed in the perturbation feedback form shown in figure 10.6, the perturbed closed-loop system can be represented as the nominal plant $P$, with the controller $K$ connected between $y$ and $u$ as usual, and the perturbation feedback $\Delta$ connected between $q$ and $p$.

The perturbed closed-loop system can be represented as the nominal closed-loop system with feedback $\Delta$ connected from $q$ (a part of $z$) to $p$ (a part of $w$). Note the similarity to figure 2.2.
By substituting (10.17) into (2.7) we find that the transfer matrix of the perturbed closed-loop system is

\[ H_{\text{pert}}(\Delta) = H_{z\tilde{u}} + H_{z\tilde{p}} \Delta(I - H_{qp}\Delta)^{-1}H_{q\tilde{w}}, \]

where

\[
\begin{align*}
H_{z\tilde{u}} &= P_{z\tilde{u}} + P_{zu}K(I - P_{yu}K)^{-1}P_{y\tilde{u}} \\
H_{z\tilde{p}} &= P_{z\tilde{p}} + P_{zu}K(I - P_{yu}K)^{-1}P_{yp} \\
H_{q\tilde{w}} &= P_{q\tilde{w}} + P_{qu}K(I - P_{yu}K)^{-1}P_{y\tilde{w}} \\
H_{qp} &= P_{qp} + P_{qu}K(I - P_{yu}K)^{-1}P_{yp}.
\end{align*}
\]

Note the similarities between figures 10.9 and 2.2, and the corresponding equations (10.18) and (2.7). Figure 2.2 and equation (2.7) show the effect of connecting the controller to the nominal plant to form the nominal closed-loop system; figure 10.9 and equation (10.18) show the effect of connecting the feedback perturbation \( \Delta \) to the nominal closed-loop system to form the perturbed closed-loop system.

We may interpret

\[ H_{\text{pert}}(\Delta) - H_{z\tilde{u}} = H_{z\tilde{p}} \Delta(I - H_{qp}\Delta)^{-1}H_{q\tilde{w}} \]

as the change in the closed-loop transfer matrix that is caused by the feedback perturbation \( \Delta \). We have the following interpretations:

- \( H_{z\tilde{u}} \) is the closed-loop transfer matrix of the nominal system, before its exogenous input and regulated output were augmented with the signals \( p \) and \( q \).

- \( H_{q\tilde{w}} \) is the closed-loop transfer matrix from the original exogenous input signal \( \tilde{w} \) to \( q \). If \( H_{q\tilde{w}} \) is “large”, then so will be the signal \( q \) that drives or excites the feedback perturbation \( \Delta \).

- \( H_{z\tilde{p}} \) is the closed-loop transfer matrix from \( p \) to the original regulated output signal \( \tilde{z} \). If \( H_{z\tilde{p}} \) is “large”, then so will be the effect on \( \tilde{z} \) of the signal \( p \), which is generated by the feedback perturbation \( \Delta \).

- \( H_{qp} \) is the closed-loop transfer matrix from \( p \) to \( q \). We can interpret \( H_{qp} \) as the feedback seen by \( \Delta \), looking into the nominal closed-loop system.

Thus, if the three closed-loop transfer matrices \( H_{z\tilde{p}} \), \( H_{q\tilde{w}} \), and \( H_{qp} \) are all “small”, then our design will be “robust” to the perturbations, i.e., the change in the closed-loop transfer matrix, which is given in (10.23), will also be “small”. This vague idea will be made more precise later in this chapter.
10.3.2 Examples of Perturbation Feedback Form

In this section $\ast$ will denote a transfer function that we have already given elsewhere. In this way we emphasize the transfer functions that are directly relevant to the perturbation feedback form.

Neglected Dynamics

Figure 10.10 shows one way to represent the perturbed plant set $\mathcal{P} = \{ \mathcal{P}^{\text{complex}} \}$ described in section 10.2.1 in perturbation feedback form. In this block diagram, the perturbation feedback $\Delta$ acts as a switch: $\Delta = 0$ yields the nominal plant; $\Delta = I$ turns on the perturbation, to yield the perturbed plant $\mathcal{P}^{\text{complex}}$.

This perturbation feedback form is described by the augmented plant

$$
\begin{bmatrix}
P_{z\bar{z}} & P_{z p} & P_{z u} \\
P_{q\bar{q}} & P_{q p} & P_{q u} \\
P_{y\bar{y}} & P_{y p} & P_{y u}
\end{bmatrix}
= 
\begin{bmatrix}
\ast & \ast & P_{\text{err}}^{(1)} & 0 & 0 & \ast \\
\ast & \ast & 0 & 0 & P_{\text{err}}^{(1)} & 0 \\
\ast & \ast & -P_{\text{err}}^{(1)} & -P_{\text{err}}^{(2)} & \ast
\end{bmatrix},
$$

(10.24)

where

$$
P_{\text{err}}^{(1)}(s) = \frac{-1.25(s/100) - (s/100)^2}{1 + 1.25(s/100) + (s/100)^2}, \quad P_{\text{err}}^{(2)}(s) = \frac{-s/80}{1 + s/80},
$$

and the feedback perturbation set

$$\Delta = \{ I \}.
$$

(10.25)

When the controller is connected, we have

$$
H_{q\bar{q}} = \begin{bmatrix}
P_{0}^{\text{std}} S & -T & T \\
-\frac{P_{0}^{\text{std}} S}{P_{0}^{\text{std}}} & S & T
\end{bmatrix},
$$

(10.26)

$$
H_{z\bar{q}} = \begin{bmatrix}
P_{\text{err}}^{(1)} S & -P_{\text{err}}^{(2)} T \\
-\frac{P_{\text{err}}^{(1)} S}{P_{\text{err}}^{(2)}} & -\frac{P_{\text{err}}^{(2)} T}{P_{0}^{\text{std}}}
\end{bmatrix},
$$

(10.27)

$$
H_{q\bar{p}} = \begin{bmatrix}
-P_{\text{err}}^{(1)} T & -T P_{\text{err}}^{(2)} \\
-P_{\text{err}}^{(1)} S & -T P_{\text{err}}^{(2)}
\end{bmatrix}.
$$

(10.28)

Gain Margin: Perturbation Feedback Form 1

The perturbed plant set for the classical gain margin specification, given by $\mathcal{P}$ in (10.4), can be expressed in the perturbation feedback form shown in figure 10.11.
This perturbation feedback form is described by the augmented plant

\[
\begin{bmatrix}
P_{zw} & P_{zp} & P_{zu} \\
\dot{P}_{qw} & \dot{P}_{qp} & \dot{P}_{qu} \\
\dot{P}_{yw} & \dot{P}_{yp} & \dot{P}_{yu}
\end{bmatrix} = \begin{bmatrix}
* & * & * & P_0 & * \\
* & * & * & 0 & * \\
0 & 0 & 0 & 0 & 1 \\
* & * & * & -P_0 & *
\end{bmatrix},
\]  \hspace{1cm} (10.29)

and the perturbation feedback set

\[\Delta = [L - 1, U - 1],\] \hspace{1cm} (10.30)

which is an interval. Thus, the feedback perturbations are real constants, or gains. Informally, the perturbation \(\Delta\)causes \(P_0\) to become \((1 + \Delta)P_0\).

For this perturbation feedback form, the transfer matrices \(H_{qw}, H_{zp},\) and \(H_{qp}\) are given by

\[
H_{qw} = \begin{bmatrix}
-T & -T/P_0 & T/P_0
\end{bmatrix},
\] \hspace{1cm} (10.31)

\[
H_{zp} = \begin{bmatrix}
P_0S \\
-T
\end{bmatrix},
\] \hspace{1cm} (10.32)

\[
H_{qp} = -T.
\] \hspace{1cm} (10.33)
10.3 Perturbation Feedback Form

Figure 10.11 One possible perturbation feedback form for the classical gain margin specification.

Gain Margin: Perturbation Feedback Form 2

The same perturbed plant set, (10.4), can be described in the different perturbation feedback form shown in figure 10.12, for which

\[
\begin{bmatrix}
P_{zz} & P_{zp} & P_{zu} \\
P_{qw} & P_{wp} & P_{wu} \\
P_{yq} & P_{yp} & P_{yu}
\end{bmatrix} =
\begin{bmatrix}
* & * & * & P_0 & * \\
* & * & * & 0 & * \\
0 & 0 & 0 & 1 & 1 \\
* & * & * & -P_0 & *
\end{bmatrix}
\] (10.34)

(only one entry differs from (10.29)), and

\[\Delta = [1 - 1/L, 1 - 1/U],\] (10.35)

which is a different interval than (10.30). For this perturbation feedback form, \(\Delta\) represents a constant that causes \(P_0\) to become \(P_0/(1 - \Delta)\).

Figure 10.12 Another perturbation feedback form for the classical gain margin specification.

For this second perturbation feedback form, the transfer matrices \(H_{qw}, H_{zp},\) and \(H_{qp}\) are given by

\[H_{qw} = \begin{bmatrix}
-T & -T/P_0 & T/P_0
\end{bmatrix},\] (10.36)
\[ H_{zp} = \begin{bmatrix} P_0 s \\ -T \end{bmatrix}, \quad (10.37) \]

\[ H_{qp} = S. \quad (10.38) \]

Only \( H_{qp} \) differs from its corresponding expression for the previous perturbation feedback form.

**Pole Variation**

In some cases, a plant pole or zero that depends on a parameter can be expressed in perturbation feedback form. As a specific example, figure 10.13 shows one way to express the specific example described in section 10.2.2 in perturbation feedback form.

![Diagram](image)

**Figure 10.13** The variation in the phase shift of \( P_0 \), described by \((10.6)\), can be represented as the effect of a varying feedback gain \( \Delta \) inside the plant.

Here we have

\[
\begin{bmatrix}
P_{z\phi} & P_{z\theta} & P_{zu} \\
P_{q\phi} & P_{q\theta} & P_{qu} \\
P_{y\phi} & P_{y\theta} & P_{yu}
\end{bmatrix} = 
\begin{bmatrix}
* & * & * \\
* & * & * \\
2 & 0 & 0 \\
\frac{s}{s + 10} & 0 & \frac{-1}{s + 10} \\
\frac{-s}{s + 10} & \frac{2}{s^2(s + 10)} & *
\end{bmatrix}
\quad (10.39)
\]

with feedback perturbation set

\[ \Delta = [-5, 5]. \quad (10.40) \]
The closed-loop transfer matrices are given by

\[
H_{\tilde{q}w} = \begin{bmatrix} \frac{2}{s^2(s + 10)}S & \frac{2}{s - 10}T & \frac{-2}{s - 10}T \end{bmatrix},
\]

(10.41)

\[
H_{zp} = \begin{bmatrix} \frac{s}{s + 10}S \\ \frac{s^3}{s - 10}T \end{bmatrix},
\]

(10.42)

\[
H_{qp} = \frac{2s}{s^2 - 100}T - \frac{1}{s + 10}.
\]

(10.43)

(The reader worried that these transfer matrices may be unstable should recall that the interpolation conditions of section 7.2.5 require that \(T(10) = 0\); similar conditions guarantee that these transfer matrices are proper, and have no pole at \(s = 0\).)

Relative Uncertainty in \(P_0\)

The plant perturbation set (10.10) can be expressed in the perturbation feedback form shown in figure 10.14, for which

\[
\begin{bmatrix} P_{\tilde{z}w} & P_{\tilde{z}p} & P_{\tilde{z}u} \\ P_{\tilde{q}w} & P_{\tilde{q}p} & P_{\tilde{q}u} \\ P_{y\tilde{w}} & P_{y\tilde{p}} & P_{yu} \end{bmatrix} = \begin{bmatrix} * & * & W_{relerr}P_0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * \\ 0 \\ -W_{relerr}P_0 \end{bmatrix},
\]

(10.44)

and

\[
\Delta = \{ \Delta \mid \|\Delta\|_\infty \leq 1 \}.
\]

(10.45)

In this case the feedback perturbations are normalized unknown-but-bounded transfer functions, that cause the transfer function \(P_0\) to become \((1 + W_{relerr}\Delta)P_0\).

For this perturbation feedback form, the transfer matrices \(H_{\tilde{q}w}, H_{zp},\) and \(H_{qp}\) are given by

\[
H_{\tilde{q}w} = \begin{bmatrix} -T & -T/P_0 & T/P_0 \end{bmatrix},
\]

(10.46)

\[
H_{zp} = \begin{bmatrix} W_{relerr}P_0S \\ -W_{relerr}P_0S \\ -W_{relerr}T \end{bmatrix},
\]

(10.47)

\[
H_{qp} = -W_{relerr}T.
\]

(10.48)

Saturation Actuators

We consider the perturbed plant set (10.15) which consists of the single nonlinear system \(\{P_{nonlin}\}\). This can be expressed in perturbation feedback form by express-
ing each saturator as a straight signal path perturbed by a dead-zone nonlinearity,
\[ Dz(a) = a - Sat(a), \]
as shown in figure 10.15.

This perturbation feedback form is described by the augmented plant

\[
\begin{bmatrix}
P_{z\bar{w}} & P_{z\bar{p}} & P_{z\bar{u}} \\
P_{q\bar{w}} & P_{q\bar{p}} & P_{q\bar{u}} \\
\end{bmatrix}
= \begin{bmatrix}
P_{\text{lin}} & -P_{\text{lin}} & P_{\text{lin}} \\
0 & 0 & I \\
0 & P_{\text{lin}} & -P_{\text{lin}} \\
\end{bmatrix},
\]
(10.49)
and $\Delta = \{ \Delta^{\text{nonlin}} \}$, where $p = \Delta^{\text{nonlin}}(q)$ is defined by

$$p_i(t) = S_i D z(q_i(t)/S_i), \quad i = 1, \ldots, n_u. \quad (10.50)$$

In this case the feedback perturbation $\Delta^{\text{nonlin}}$ is a memoryless nonlinearity.

For this perturbation feedback form, the transfer matrices $H_{q \hat{w}}$, $H_{z_p}$, and $H_{q p}$ are given by

$$H_{q \hat{w}} = K(I - P_{yu}^{\text{lin}} K)^{-1} P_{yu}^{\text{lin}}, \quad (10.51)$$

$$H_{z_p} = -P_{zu}^{\text{lin}}(I - K K_{yu}^{\text{lin}})^{-1}, \quad (10.52)$$

$$H_{qp} = -K P_{yu}^{\text{lin}}(I - K K_{yu}^{\text{lin}})^{-1}. \quad (10.53)$$

### 10.4 Small Gain Method for Robust Stability

#### 10.4.1 A Convex Inner Approximation

We consider a perturbed plant set $\mathcal{P}$ that is given by a perturbation feedback form. Suppose that the norm $\| \cdot \|_{\text{gn}}$ is a gain (see chapter 5). Let $M$ denote the maximum gain of the possible feedback perturbations, i.e.,

$$M = \sup_{\Delta \in \Delta} \| \Delta \|_{\text{gn}}. \quad (10.54)$$

$M$ is thus a measure of how “big” the feedback perturbations can be.

Then from the small gain theorem described in section 5.4.2 (equations (5.29–5.31) with $H_1 = \Delta$ and $H_2 = H_{qp}$) we know that if

$$\| H_{qp} \|_{\text{gn}} M < 1 \quad (10.55)$$

then we have for all $\Delta \in \Delta$,

$$\| \Delta(I - H_{qp} \Delta)^{-1} \|_{\text{gn}} \leq \frac{M}{1 - M\| H_{qp} \|_{\text{gn}}}.$$

From (10.23) we therefore have

$$\| H^{\text{pert}}(\Delta) - H_{z \hat{w}} \|_{\text{gn}} \leq \frac{M\| H_{z_p} \|_{\text{gn}}\| H_{q \hat{w}} \|_{\text{gn}}}{1 - M\| H_{qp} \|_{\text{gn}}} \quad \text{for all } \Delta \in \Delta. \quad (10.56)$$

We will refer to the closed-loop convex specification (10.55) as the small gain condition (for the perturbation feedback form and gain used). (10.55) and (10.56) are a precise statement of the idea expressed in section 10.3.1: the closed-loop system will be robust if the three closed-loop transfer matrices $H_{z_p}$, $H_{q \hat{w}}$, and $H_{qp}$ are “small enough”.

It follows that the closed-loop convex specification on $H$ given by
\begin{align}
\|H_{qp}\|_{gn} &< 1/M, \quad (10.57) \\
\|H_{zp}\|_{gn} &< \infty, \quad (10.58) \\
\|H_{q\hat{w}}\|_{gn} &< \infty, \quad (10.59) \\
\|H_{\hat{z}w}\|_{gn} &< \infty, \quad (10.60)
\end{align}
implies that
\[
\|H_{\text{pert}}\|_{gn} < \infty \quad \text{for all} \quad \Delta \in \Delta,
\]
i.e., the robustness specification formed from the perturbed plant set $\mathcal{P}$ and the specification $\|H\|_{gn} < \infty$ holds. If the gain $\|\cdot\|_{gn}$ is finite only for stable transfer matrices, then the specification (10.57–10.60) implies that $H_{\text{pert}}$ is stable, and thus the specification (10.57–10.60) is stronger than the specification of robust stability. In this case, we may think of the specification (10.57–10.60) as a closed-loop convex specification that guarantees robust stability.

As a more specific example, the RMS gain ($H_{\infty}$ norm) $\|\cdot\|_{\infty}$ is finite only for stable transfer matrices, so the specification $\|H_{qp}\|_{\infty} < 1/M$, along with stability of $H_{zp}$, $H_{q\hat{w}}$, and $H_{\hat{z}w}$ (which is usually implied by internal stability), guarantees that the robust stability specification $\mathcal{D}_{\text{robust}}$ holds for $H$.

The specification (10.57–10.60) can be used to form a convex inner approximation of a robust generalized stability specification, for various generalizations of internal stability (see section 7.5), by using other gains. As an example, consider the $\alpha$-shifted $H_{\infty}$ norm, which is finite if and only if the poles of its argument have real parts less than $-\alpha$. If the specification (10.57–10.60) holds for this norm, then we may conclude that the feedback perturbations cannot cause the poles of the closed-loop system to have real parts equal to or exceeding $-\alpha$. (We comment that changing the gain used will generally change $M$, and will give a different specification.)

Since the small gain condition (10.55) depends only on $M$, the largest gain of the feedback perturbations, it follows that the conclusion (10.56) actually holds for a set of feedback perturbations that may be larger than $\Delta$:
\[
\Delta_{\text{set}} = \{ \Delta \mid \|\Delta\|_{gn} \leq M \} \supseteq \Delta.
\]

By using different perturbation feedback forms of a perturbed plant set, and different gains that are finite only for stable transfer matrices, the small gain condition (10.55) can be used to form different convex inner approximations of the robust stability specification.

### 10.4.2 An Extrapolated First Order Bound

It is interesting to compare (10.56) to a corresponding bound that is based on a first order differential analysis. Since
\[
\Delta(I - H_{qp} \Delta)^{-1} \simeq \Delta
\]
(recall that \( \simeq \) means equals, to first order in \( \Delta \)), the first order variation in the closed-loop transfer matrix is given by

\[
H_{\text{pert}}(\Delta) - H_{z\Delta} \simeq H_{z\Delta} \Delta H_{\hat{z}w} \tag{10.62}
\]

(c.f. the exact expression given in (10.23)). If we use this first order analysis to extrapolate the effects of any \( \Delta \in \Delta \), we have the approximate bound

\[
\|H_{\text{pert}}(\Delta) - H_{z\Delta}\|_{\text{gn}} \lesssim M\|H_{z\Delta}\|_{\text{gn}}\|H_{\hat{z}w}\|_{\text{gn}} \tag{10.63}
\]

(c.f. the small gain bound given in (10.56); \( \lesssim \) means that the inequality holds to first order in \( \Delta \)).

The small gain bound (10.56) can be interpreted as the extrapolated first order differential bound (10.63), aggravated (increased) by a term that represents the “margin” in the small gain condition, i.e.

\[
1 \quad \frac{1}{1 - M\|H_{qp}\|_{\text{gn}}}.
\]

(10.64)

Of course, the small gain bound (10.56) is correct, whereas extrapolations from the first order bound (10.63) need not hold.

Continuing this comparison, we can interpret the term (10.64) as representing the higher order effects of the feedback perturbation \( \Delta \), since

\[
\frac{1}{1 - M\|H_{qp}\|_{\text{gn}}} = 1 + (M\|H_{qp}\|_{\text{gn}}) + (M\|H_{qp}\|_{\text{gn}})^2 + (M\|H_{qp}\|_{\text{gn}})^3 + \cdots.
\]

The bound (10.63), which keeps just the first term in this series, only accounts for the first order effects.

It is interesting that the transfer matrix \( H_{qp} \), which is important in the small gain based approximation of robust stability, has no effect whatever on the first order change in \( H \), given by (10.62). Thus \( H_{qp} \), the feedback “seen” by the feedback perturbation, has no first order effects, but is key to robust stability.

Conversely, the transfer matrices \( H_{z\Delta} \) and \( H_{\hat{z}w} \), which by (10.62) determine the first order variation of \( H \) with respect to changes in \( \Delta \), have little to do with robust stability. Thus, differential sensitivity of \( H \), which depends only on \( H_{z\Delta} \) and \( H_{\hat{z}w} \), and robust stability, which depends only on \( H_{qp} \), measure different aspects of the robustness of the closed-loop system.

### 10.4.3 Nonlinear or Time-Varying Perturbations

A variation of the small gain method can be used to form an inner approximation of the robust stability specification, when the feedback perturbations are nonlinear or time-varying. In this case we define \( M \) by

\[
M = \sup \left\{ \frac{\|\Delta w\|_{\text{sig}}}{\|w\|_{\text{sig}}} \mid \Delta \in \Delta, \|w\|_{\text{sig}} > 0 \right\} \tag{10.65}
\]
where $\| \cdot \|_{\text{sig}}$ denotes the norm on signals that determines the gain $\| \cdot \|_{\text{gn}}$.

With this definition of $M$ (which coincides with (10.54) when each $\Delta \in \Delta$ is LTI), the specification (10.57–10.60) implies robust stability. In fact, a close analog of (10.56) holds: for all $\Delta \in \Delta$ and exogenous inputs $\tilde{w}$, we have

$$\| \tilde{z}_{\text{pert}} - \tilde{z} \|_{\text{sig}} \leq \alpha \| \tilde{w} \|_{\text{sig}},$$

where

$$\alpha = \frac{M \| H_{z\tilde{w}} \|_{\text{gn}} \| H_{q\tilde{u}} \|_{\text{gn}}}{1 - M \| H_{q\tilde{p}} \|_{\text{gn}}},$$

Thus, we have a bound on the perturbations in the regulated variables that can be induced by the nonlinear or time-varying perturbations.

### 10.4.4 Examples

In this section we apply the small gain method to form inner approximations of some of the robust stability specifications that we have considered so far. In a few cases we will derive different convex inner approximations of the same robust stability specification, either by using different perturbation feedback forms for the same perturbed plant set, or by using different gains in the small gain theorem.

#### Neglected Dynamics: RMS Gain

We now consider the specification of internal stability along with the robust stability specification for the perturbation plant set (10.3), i.e., $D_{\text{rob, stab}} \cap D_{\text{stable}}$. We will use the perturbation feedback form given by (10.24–10.25). The specification that $H_{z\tilde{w}}$, $H_{q\tilde{u}}$, and $H_{z\tilde{p}}$ are stable is weaker than $D_{\text{stable}}$, so we will concentrate on the small gain condition (10.57). No matter which gain we use, $M$ is one, since $\Delta$ contains only one transfer matrix, the $2 \times 2$ identity matrix.

We first use the RMS gain, i.e., the $H_{\infty}$ norm $\| \cdot \|_{\infty}$, which is finite only for stable transfer matrices. The small gain condition is then

$$\| H_{q\tilde{p}} \|_{\infty} = \left\| \begin{bmatrix} -P_{\text{err}}^{(1)} T & -TP_{\text{err}}^{(2)} \\ P_{\text{err}}^{(1)} S & -TP_{\text{err}}^{(2)} \end{bmatrix} \right\|_{\infty} < 1. \quad (10.66)$$

The closed-loop specification (10.66) (together with internal stability) is stronger than the robust stability specification $D_{\text{rob, stab}}$ with the plant perturbation set (10.25) (together with internal stability): if (10.66) is satisfied, then the corresponding controller also stabilizes $P_{\text{exmp}}^{\text{pix}}$.

We can interpret the specification (10.66) as limiting the bandwidth of the closed-loop system. The specification (10.66) can be crudely considered a frequency-dependent limit on the size of $T$; since $P_{\text{err}}^{(1)}$ and $P_{\text{err}}^{(2)}$ are each highpass filters, this limit is large for low frequencies, but less than one at high frequencies where $P_{\text{err}}^{(1)}$ and
$P_C^{(2)}$ have magnitudes nearly one, e.g., $\omega \geq 100$. Thus, (10.66) requires that $|T(j\omega)|$ is less than one above about $\omega = 100$; in classical terms, the control bandwidth is less than $100\text{rad/sec}$. Figure 10.16(a) shows the actual region of the complex plane that the specification (10.66) requires $T(20j)$ to lie in; figure 10.16(b) shows the same region for $T(200j)$.

Figure 10.16 The specification (10.66) requires $T(j\omega)$ to lie in the shaded region (a) for $\omega = 20$, and (b) for $\omega = 200$.

Neglected Dynamics: Scaled RMS Gain

We now apply the small gain method to the same example, using the same perturbation feedback form, substituting a scaled $H_\infty$ norm for the $H_\infty$ norm used above (see section 5.3.6). We use the scaling

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

and the associated scaled gain,

$$\|DH_{qp}D^{-1}\|_\infty = \left\| \begin{bmatrix} P_C^{(1)} & -TP_C^{(2)} \\ P_C^{(1)}S/2 & TP_C^{(2)} \end{bmatrix} \right\|_\infty.$$

This norm is finite only for stable transfer matrices, so the small gain condition

$$\left\| \begin{bmatrix} P_C^{(1)} & -TP_C^{(2)} \\ P_C^{(1)}S/2 & TP_C^{(2)} \end{bmatrix} \right\|_\infty < 1,$$

(10.67)

together with internal stability, is stronger than the robust stability specification $D_{\text{rob,stab}}$ with the plant perturbation set (10.25) (together with internal stability). Like the specification (10.66), the specification (10.67) can be thought of as limiting the bandwidth of the closed-loop system.
Figure 10.17(a) shows the actual region of the complex plane that the specification (10.67) requires $T(20j)$ to lie in; figure 10.17(b) shows the same region for $T(200j)$; comparing these figures to figures 10.16(a) and 10.16(b), we see that the two inner approximations of robust stability given by (10.66) and (10.67) are different: neither is stronger than the other.

![Figure 10.17](image)

**Figure 10.17** The specification (10.67) requires $T(j\omega)$ to lie in the shaded region (a) for $\omega = 20$, and (b) for $\omega = 200$. The boundaries of the regions from figure 10.16, for the specification (10.66), are shown with a dashed line.

Gain Margin: Perturbation Feedback Form 1

We now consider the gain margin specification, using the RMS gain in the small gain theorem, with the perturbation feedback form given by (10.29–10.30). The maximum of the RMS gains of the perturbations is

$$M = \max\{\|L - 1\|_\infty, \|U - 1\|_\infty\} = \max\{1 - L, U - 1\}.$$ 

For this perturbation feedback form, $H_{qp} = -T$, so the small gain condition is

$$\|T\|_\infty < 1/M = \min\left\{\frac{1}{1-L}, \frac{1}{U-1}\right\}.$$ 

For the specific gain margins of +4dB and -3.5dB, i.e., the robustness specification (10.5), the convex inner approximation is

$$\|T\|_\infty < 1.71.$$  \hspace{1cm} (10.68)

The closed-loop convex specification (10.68) is stronger than the gain margin specification (10.5).
Gain Margin: Perturbation Feedback Form 2

We now use the second perturbation feedback form for the gain margin problem, given by (10.34–10.35). For this perturbation feedback form, we have

\[ M = \max \{1/L - 1, 1 - 1/U\}, \]

and \( H_{qp} = S \), so the small gain condition is

\[ \|S\|_\infty < 1/M = \min \left\{ \frac{L}{1-L}, \frac{U}{U-1} \right\}. \]

For the specific gain margins of +4dB, −3.5dB, we have

\[ \|S\|_\infty < 2.02. \]

The closed-loop convex specification (10.70) is a different inner approximation of the gain margin specification (10.5) than (10.68).

Thus we have

\[ \|T\|_\infty \leq 1.71 \implies \mathcal{D}_{+4,-3.5\text{dB}_\text{gm}}, \quad \|S\|_\infty \leq 2.02 \implies \mathcal{D}_{+4,-3.5\text{dB}_\text{gm}}, \]

but neither of the convex specifications on the left-hand sides is stronger than the other. We can contrast the two specifications by expressing (10.70) as \( \|1 - T\|_\infty < 2.02 \); see figure 10.18.

\[ \begin{array}{c}
|1 - T(j\omega)| \leq 2.02 \\
|T(j\omega)| \leq 1.71 \\
\end{array} \]

\[ \begin{array}{c}
\text{ST}(j\omega) \\
\text{RT}(j\omega) \\
\end{array} \]

Figure 10.18 The specifications (10.68) and (10.70) require the complex number \( T(j\omega) \) to lie in the indicated circles at each frequency \( \omega \).
A Generalized Gain Margin

We consider again the perturbed plant set for the gain margin specification, but tighten our robustness specification to require that the perturbed closed-loop system should have a stability degree that exceeds $a > 0$, i.e., the poles of $H_{pbrt}^\text{pert}$ should have real parts less than $-a$. To form a convex inner approximation of this robustness specification, we apply the small gain method to the perturbation feedback form given by (10.34–10.35) and the $a$-shifted $H_\infty$ norm. For this norm (indeed, for any gain) we find that

$$M = \max \{1/L - 1, 1 - 1/U\},$$

just as for the unshifted $H_\infty$ norm.

The convex inner approximation is then: $H_{pbrt}^\text{pert}$, $H_{z^p}$, and $H_{q^\text{pert}}^\text{pert}$ have stability degrees exceeding $a$ (i.e., finite $\|\cdot\|_{\infty,a}$ norms) and

$$\|S\|_{\infty,a} < \min \left\{ \frac{L}{1-L}, \frac{U}{U-1} \right\}.$$  

For the specific generalized gain margin of $+4\text{dB}$, $-3.5\text{dB}$, with a minimum stability degree of 0.2, we have the convex inner approximation

$$\|S\|_{\infty,0.2} < 2.02.$$

(10.71)

Pole Variation

We now consider the perturbed plant set (10.6) from section 10.2.2. We will form the small gain condition using the perturbation feedback form (10.39–10.40), and the RMS gain. The maximum RMS gain of the feedback perturbations is 5, so we have the approximation

$$\left\| \frac{2s}{s^2 - 100} T - \frac{1}{s + 10} \right\|_\infty < 1/5.$$  

(10.72)

We can interpret this closed-loop convex specification as follows. The weighting on $T$ is a bandpass filter, whose peak magnitude is 0.1, at $\omega = 10$. The specification (10.72), roughly speaking, constrains $T$ for frequencies near $\omega = 10$.

Relative Uncertainty in $P_0$

We will use the perturbation feedback form (10.44) described above, with the RMS gain. In this case we have $M = 1$, so the small gain condition becomes the weighted $H_\infty$ norm specification

$$\|W_{\text{rel.err}} T\|_\infty < 1.$$  

(10.73)

This requires that the magnitude of $T$ lie below the frequency-dependent limit $1/|W_{\text{rel.err}}(j\omega)|$, as shown in figure 10.19.
10.5 Small Gain Method for Robust Performance

10.5.1 A Convex Inner Approximation

A variation of the small gain method can be used to form convex inner approximations of robustness specifications that involve a gain bound such as

\[ \|H_{\tilde{w}}\|_{\infty} \leq \alpha, \]

Saturating Actuators

We consider the perturbed plant set (10.15), with the perturbation feedback form given by (10.49) and (10.50). We will use the RMS gain or $H_\infty$ norm. From formula (10.65) we find that $M = 1$, since the RMS value of the dead-zone output is less than the RMS value of its input, and for large constant signals, the two RMS values are close. The small gain condition for robust stability is thus

\[ \|K P_{y u}^{\text{lin}} (I - K P_{y u}^{\text{lin}})^{-1}\|_{\infty} < 1. \] (10.74)

Thus, if the closed-loop convex specification (10.74) is satisfied, then an LTI controller designed on the basis of the linear model $P_{y u}^{\text{lin}}$ will at least stabilize the nonlinear system that includes the actuator saturation.
where $H_{\tilde{z} \tilde{w}}$ is some entry or submatrix of $H$ (c.f. robust stability, which involves the gain bound $\|H_{\tilde{z} \tilde{w}}\|_\infty < \infty$).

Throughout this section, we will consider the robustness specification that is formed from the perturbed plant set $\mathcal{P}$ and the RMS gain bound specification

$$\|H_{\tilde{z} \tilde{w}}\|_\infty \leq 1. \quad (10.75)$$

We will refer to this robust performance specification as $D_{\text{rob perf}}$. We will also assume that the perturbed plant set $\mathcal{P}$ is described by a perturbation feedback form for which the maximum RMS gain of the feedback perturbations is one, i.e., $M = 1$ in (10.54).

The inner approximation of $D_{\text{rob perf}}$ is

$$\left\| \begin{bmatrix} H_{\tilde{z} \tilde{w}} & H_{\tilde{z} p} \\ H_{q \tilde{w}} & H_{q p} \end{bmatrix} \right\|_\infty < 1. \quad (10.76)$$

Like the inner approximation (10.57–10.60) of the robust stability specification $D_{\text{rob stab}}$, we can interpret (10.76) as limiting the size of $H_{\tilde{z} p}, H_{q \tilde{w}},$ and $H_{q p}$.

Let us show that (10.76) implies that the specification (10.75) holds robustly, i.e.,

$$\|H_{\tilde{z} \tilde{w}} + H_{\tilde{z} p} (I - H_{q p} \Delta)^{-1} H_{q \tilde{w}}\|_\infty \leq 1 \text{ for all } \Delta \in \Delta. \quad (10.77)$$

Assume that (10.76) holds, so that for any signals $\tilde{w}$ and $p$ we have

$$\left\| \begin{bmatrix} \tilde{z} \\ q \end{bmatrix} \right\|_{\text{rms}} < \left\| \begin{bmatrix} \tilde{w} \\ p \end{bmatrix} \right\|_{\text{rms}}, \quad (10.78)$$

where

$$\begin{bmatrix} \tilde{z} \\ q \end{bmatrix} = \begin{bmatrix} H_{\tilde{z} \tilde{w}} & H_{\tilde{z} p} \\ H_{q \tilde{w}} & H_{q p} \end{bmatrix} \begin{bmatrix} \tilde{w} \\ p \end{bmatrix}.$$ 

The inequality (10.78) can be rewritten

$$\|\tilde{z}\|_{\text{rms}}^2 + \|q\|_{\text{rms}}^2 < \|\tilde{w}\|_{\text{rms}}^2 + \|p\|_{\text{rms}}^2. \quad (10.79)$$

Now assume that $p = \Delta q$, where $\Delta \in \Delta$, so that these signals correspond to closed-loop behavior of the perturbed system, i.e.,

$$\tilde{z} = \left(H_{\tilde{z} \tilde{w}} + H_{\tilde{z} p} (I - H_{q p} \Delta)^{-1} H_{q \tilde{w}}\right) \tilde{w}. \quad (10.80)$$

Since $\|\Delta\|_\infty \leq 1$, we have

$$\|p\|_{\text{rms}} \leq \|q\|_{\text{rms}}. \quad (10.81)$$

From (10.79–10.81) we conclude that

$$\|\tilde{z}\|_{\text{rms}} = \left\| \left(H_{\tilde{z} \tilde{w}} + H_{\tilde{z} p} (I - H_{q p} \Delta)^{-1} H_{q \tilde{w}}\right) \tilde{w} \right\|_{\text{rms}} \leq \|\tilde{w}\|_{\text{rms}}.$$
Since this holds for any \( \tilde{w} \), (10.77) follows.

Doyle has interpreted the specification (10.76) as a small gain based robust stability condition (10.55) for a perturbed plant set that includes an unknown-but-bounded transfer matrix connected from \( \tilde{z} \) to \( \tilde{w} \). This “performance loop” is shown in figure 10.20.

If the condition (10.76) holds, then the closed-loop system in figure 10.20 is robustly stable for all \( \Delta \) with \( \|\Delta\|_\infty \leq 1 \). In particular, the closed-loop system in figure 10.20 will be robustly stable for all \( \Delta \) of the form

\[
\tilde{\Delta} = \begin{bmatrix}
\Delta & 0 \\
0 & \Delta_{\text{perf}}
\end{bmatrix},
\]

with \( \|\Delta\|_\infty \leq 1 \) and \( \|\Delta_{\text{perf}}\|_\infty \leq 1 \). This is equivalent to the specification (10.75) holding robustly for all \( \Delta \) with \( \|\Delta\|_\infty \leq 1 \).

**Figure 10.20** Doyle’s performance loop connects a feedback \( \Delta_{\text{perf}} \) from the critical regulated variable \( \tilde{z} \) back to the critical exogenous input \( \tilde{w} \). If the resulting system is robustly stable (with \( \|\Delta_{\text{perf}}\|_\infty \leq 1 \) and \( \|\Delta\|_\infty \leq 1 \)), then the original system robustly satisfies the performance specification \( \|H_{zc}\|_\infty \leq 1 \).

### 10.5.2 An Example

We consider the plant perturbation set (10.10), i.e., frequency-dependent relative uncertainty in the transfer function \( P_0 \), and the robustness specification (10.11) which limits the RMS gain from the reference input to the actuator signal to be less than or equal to 75, for all possible perturbations.
Applying the method above yields the convex specification

\[ \left\| \begin{bmatrix} T/(75P_0) & -W_{rel\text{-err}} T \\ T/(75P_0) & -W_{rel\text{-err}} T \end{bmatrix} \right\|_{\infty} < 1, \]  

which guarantees that the robust performance specification holds.

The inner approximation (10.82) can be simplified by factoring \( T \) out of the matrix, and assuming \( T \) is stable, in which case it is equivalent to

\[ |T(j\omega)| \sigma \left( \begin{bmatrix} 1/(75P_0(j\omega)) & -W_{rel\text{-err}}(j\omega) \\ 1/(75P_0(j\omega)) & -W_{rel\text{-err}}(j\omega) \end{bmatrix} \right) \]

\[ = |T(j\omega)| \sqrt{2(|W_{rel\text{-err}}(j\omega)|^2 + |1/(75P_0(j\omega))|^2)} < 1 \quad \text{for} \; \omega \in \mathbb{R}. \]

Hence, (10.82) is equivalent to the specification that \( T \) be stable and satisfy the frequency-dependent magnitude limit

\[ |T(j\omega)| < \frac{1}{\sqrt{2(|W_{rel\text{-err}}(j\omega)|^2 + |1/(75P_0(j\omega))|^2)}} \quad \text{for} \; \omega \in \mathbb{R}, \]

which is plotted in figure 10.21. The reader should compare the specification (10.83) to (10.73), which guarantees robust stability, and is also plotted in figure 10.21.
If the closed-loop transfer function $T$ is stable and its magnitude lies below the limit shown, then the relative uncertainty in $P_0$ cannot cause the perturbed closed-loop transfer function from reference input to actuator signal to have RMS gain exceeding 75. The dotted limit shows the previously determined bound that guarantees closed-loop stability despite the relative uncertainty in $P_0$ (see figure 10.19). The dashed line is the bound imposed by the performance specification $\|T/P_0\|_\infty \leq 75$. For this example, the specification of robust stability and nominal performance is not much looser than the small gain based inner approximation of robust performance.
Notes and References

Comparison with Differential Sensitivity

In [LUN89, P48–49], Lunze states:

Sensitivity is a local property that describes how strongly the system performance is affected by very small perturbations around a given nominal point $\alpha$. No information about the amount of perturbations is used. However, as many properties depend continuously on the parameter vector $\alpha$, extrapolations can be made from very small to finite deviations. Therefore, sensitivity analysis yields guidelines for the attenuation of severe parameter perturbation and, hence, the achievement of robustness. But sensitivity analysis alone cannot ensure this robustness because the range of validity of the results is not known.

(See also table 3.1 in [LUN89, p49].)

The distinction between parametrized and other perturbed plant sets is not as clear as it might seem. In [BOY86], it is observed that robust stability specifications with unknown-but-bounded transfer function perturbations can be recast as robust stability specifications with parametrized plant perturbation sets. (There does not appear to be any advantage in doing so.)

Relation to Classical Control Ideas

Many of the small gain based approximations of robustness specifications that we have seen can be interpreted as limiting the magnitude of $T$ or $S$. The idea that a closed-loop system with “large” $T$ or $S$ can be very sensitive to changes in $P_0$ is well-known in classical control; the specification $\|T\|_\infty \leq M$ is called an $M$-circle constraint. Horowitz [HOR83, P148] states

$\ldots$ it is not necessarily a useful practical system, if the locus [of $L$] passes very close to the $-1$ point. In the first place, a slight change of gain or time constant may sufficiently shift the locus so as to lead to an unstable system. In the second place, the closed-loop system response has $1 + L$ for its denominator. At those frequencies for which $L$ is close to $-1$, $1 + L$ is close to zero, leading to large peaking in the system frequency response.

A classical interpretation of the small gain specification $\|S\|_\infty \leq \alpha$ is that the Nyquist plot maintains a distance of at least $1/\alpha$ from the critical point $-1$ (see [BBN90]).

For a discussion of singular perturbations of control systems, see Kokotovic, Khalil, and O’Reilly [KKO86].

Small Gain Methods

Small gain methods are discussed in the books by Desoer and Vidyasagar [DV75] and Vidyasagar [Vid78]. In most discussions the method is used to establish stability despite nonlinear and time-varying perturbations.

The small gain theorem for linear $\Delta$ is a basic result of Functional Analysis, often attributed to Banach; see, e.g., Kantorovich and Akilov [KA82, p154]. Its use in the analysis of feedback systems was introduced by Zames [ZAM66A].
Many papers that discuss robustness specifications and small gain approaches are reprinted in the volume edited by Dorato [Dor87]; see also the recent books by Lunze [LUN89, CH.8] and Maciejowski [MAC89, §3.10].

Conservatism of Small Gain Methods

Since the small gain method yields inner approximations of robustness specifications, it is natural to ask how "conservative" these approximations can be. Doyle, Wall, Stein, Chen, and Desoer [DWS82, DS81, CD82b] observed that for the special case when

$$\Delta = \{ \Delta \mid \|\Delta\|_{\infty} \leq M \},$$

the small gain condition for robust stability, i.e.,

$$\|H_{\infty}\|_{\infty} < 1/M,$$

is exactly equivalent to the robust stability specification, and not just an approximation. In such cases, therefore, the specification of robust stability is closed-loop convex. Thus in our example of robust stability despite relative uncertainty in $R_0$, the small gain condition (10.73) is the same as the robust stability specification.

In other cases the approximation can be arbitrarily conservative; see for example the papers by Doyle [DOY82, DOY78] and Safonov and Doyle [SD84]. These papers suggest various ways this conservatism can be reduced, for example, by choosing an optimally-scaled norm for the small gain theorem. (We saw in our unmodeled plant dynamics example that scaling can affect the inner approximation produced by the small gain method.) But limits on the optimally-scaled gain are not closed-loop convex.

In these papers, Doyle introduces the structured singular value; if the structured singular value is substituted for the norm-bound in the small gain theorem, then there is no conservatism for robust stability problems with certain types of feedback perturbation sets. Specifications that limit the structured singular value are, however, not closed-loop convex.

Small Gain Theorem and Lyapunov Stability

Many of the specifications that we have encountered in this chapter have the form of a (possibly weighted) $H_{\infty}$ norm-bound on $H_{\infty}$. If such a specification is satisfied, then we can compute the positive definite solution $X$ of the ARE (5.42) as we described in section 5.6.3. This matrix provides a Lyapunov function, $V(z) = z^TXz$, that proves that robust stability holds. See [BBR89] and [WIL73].

Circle Theorem

Several of the small gain robustness specifications that we encountered can be interpreted as instances of the circle criterion, developed by Zames [ZAM66], Sandberg [SAN64], and Narendra and Goldman [NG64]. Multivariable versions were developed by Safonov and Athans [SAF80, SA81] and others. Part III of the collection edited by MacFarlane [MAC79] contains reprints of many of these original articles.

Boyd, Barratt, and Norman [BBN90] show that a general circle criterion specification is closed-loop convex.
About the Examples

The gain margin specification has been completely analyzed by Tannenbaum in [TAN80] and [TAN82]; his analysis shows that it is not closed-loop convex; indeed, the set of transfer matrices that satisfy a gain margin specification need not be connected.

The convex inner approximation (10.71) of the generalized gain margin, which limits the shifted $H_\infty$ norm of $S$, is a special form of a generalized circle theorem (Moore [MOO68]).

The robust performance specification example turns out to be closed-loop convex, since it can be shown to be equivalent to

$$|W_{rel, err}(j\omega)T'(j\omega)| + |(1/75)T(j\omega)/P_0(j\omega)| < 1 \quad \text{for all } \omega.$$  

In fact, for the 1-DOF control system, most robust performance specifications that are expressed in terms of weighted $H_\infty$ norm-bounds also turn out to be closed-loop convex.

Another example is described in Francis [FRA88], and discussed in Boyd, Barratt and Norman [BBN90].

Robust Performance Method

This method was introduced by Doyle in [DOY82, DOY78], and is discussed in Maciejowski [MAC89, §3.12].

Describing Function Method

The describing function method is described in [GV68] and [VID78, ch4]. Several modifications can make the describing function method nonheuristic; see, e.g., Mees and Bergen [MB75].

An Extension of the Saturating Actuators Example

We saw that it may be possible to design an LTI controller for a plant that is linear except for saturating actuators, in such a way that we can guarantee that the resulting nonlinear closed-loop system is stable, by requiring the specification (10.74) to be met. In this section we briefly describe an extension of this idea that has been very useful in practice, and is interesting because the method effectively synthesizes a nonlinear controller.

The control system architecture is shown in figure 10.22. The nonlinear controller $K_{nonlin}$ consists of a two-input, one-output LTI system, with its output saturated and fed back to its first input; the output is identical to the signal that drives $P_{lin}$.

We redraw this control system as shown in figure 10.23, and consider the dead-zone nonlinearity as a perturbation. We augment our design specifications with the small gain condition $\|H_{sp}\|_\infty < 1$; any resulting design is guaranteed to yield a stable nonlinear closed-loop system. (Indeed, by our comments above, we can produce a Lyapunov function that proves stability of the closed-loop system.)

This scheme is discussed in, for example, Åström and Wittenmark [AW90], and Morari and Zafirou [MZ89, §3.2.3].
The nonlinear controller $K_{\text{nonlin}}$ consists of a two-input, one-output LTI system, with its output saturated and fed back to its first input (this is the signal that drives $P_{\text{lin}}$).

The saturators in figure 10.22 are treated as a dead-zone nonlinearity perturbation to a linear system.
Chapter 11

A Pictorial Example

The sets of transfer matrices that satisfy specifications are generally infinite-dimensional. In this chapter we consider our standard example described in section 2.4 with an additional two-dimensional affine specification. This allows us to visualize a two-dimensional “slice” through the various specifications we have encountered. The reader can directly see, for this example, that specifications we have claimed are convex are indeed convex.

Recall from section 2.4 that $H^{(a)}$, $H^{(b)}$, and $H^{(c)}$ are the closed-loop transfer matrices resulting from the three controllers $K^{(a)}$, $K^{(b)}$, and $K^{(c)}$ given there. The closed-loop affine specification

$$
\mathcal{H}_{\text{slice}} = \{ H \mid H = \alpha H^{(a)} + \beta H^{(b)} + (1 - \alpha - \beta)H^{(c)} \text{ for some } \alpha, \beta \in \mathbb{R} \}
$$

requires $H$ to lie on the plane passing through these three transfer matrices. The specification $\mathcal{H}_{\text{slice}}$ has no practical use, but we will use it throughout this chapter to allow us to plot two-dimensional “slices” through other (useful) specifications.

Figure 11.1 shows the subset $-1 \leq \alpha \leq 2, -1 \leq \beta \leq 2$ of $\mathcal{H}_{\text{slice}}$. Most plots that we will see in this chapter use this range. Each point in figure 11.1 corresponds to a closed-loop transfer matrix; for example, $H^{(a)}$ corresponds to the point $\alpha = 1$, $\beta = 0$, $H^{(b)}$ corresponds to the point $\alpha = 0$, $\beta = 1$, and $H^{(c)}$ corresponds to the point $\alpha = 0$, $\beta = 0$. Also shown in figure 11.1 are the points

$$
0.6H^{(a)} + 0.3H^{(b)} + 0.1H^{(c)} \quad \text{and} \quad -0.2H^{(a)} - 0.6H^{(b)} + 1.8H^{(c)}.
$$

Each point in figure 11.1 also corresponds to a particular controller, although we will not usually be concerned with the controller itself. The controller that realizes the closed-loop transfer matrix

$$
\alpha H^{(a)} + \beta H^{(b)} + (1 - \alpha - \beta)H^{(c)}
$$

can be computed by two applications of equation (7.10) from chapter 7.
11.1 I/O Specifications

11.1.1 A Settling Time Limit

The step responses from the reference input $r$ to $y_p$ for the three closed-loop systems are shown in figure 11.2. Figure 11.3 shows the level curves of the reference $r$ to $y_p$ settling-time functional $\phi_{\text{settle}}$, i.e.,

$$\phi_{\text{settle}} \left( \alpha H_{13} + \beta H_{13} + (1 - \alpha - \beta)H_{13} \right).$$  \hspace{1cm} (11.1)

Recall from section 8.1.1 that $\phi_{\text{settle}}$, and therefore (11.1), are quasiconvex. From figure 11.3 it can be seen that the level curves bound convex subsets.
11.1 I/O Specifications

Figure 11.2 The step responses from the reference input, $r$, to plant output, $y_p$, for the closed-loop transfer matrices $H^{(a)}$, $H^{(b)}$, and $H^{(c)}$.

Figure 11.3 Level curves of the step response settling time, from the reference $r$ to $y_p$, given by (11.1).
11.1.2 Some Worst Case Tracking Error Specifications

We now consider the tracking error. For our example, the tracking error transfer function is given by

\[ H_{13} - 1 \]

(we have not set up our example with the tracking error explicitly included in the regulated variables). Figure 11.4 shows the level curves of the weighted peak tracking error, i.e.,

\[ \varphi_{pk, \text{trk}}(\alpha, \beta) \equiv \| W \left( \alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta) H_{13}^{(c)} - 1 \right) \|_{pk, \text{trk}}, \quad (11.2) \]

where the weight is

\[ W(s) = \frac{1}{2s + 1}. \]

(We use the symbol \( \varphi \) to denote the restriction of a functional to a finite-dimensional domain.) Recall from section 8.1.2 that the weighted peak tracking error is a convex functional of \( H \), and therefore \( \varphi_{pk, \text{trk}} \) is a convex function of \( \alpha \) and \( \beta \). From figure 11.4 it can be seen that the level curves bound convex subsets.

Figure 11.5 shows the level curves of the peak tracking error, for reference inputs bounded and slew-rate limited by 1, i.e.,

\[ \left\| \alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta) H_{13}^{(c)} - 1 \right\|_{wc}. \quad (11.3) \]

In section 6.3.2 we showed that a function of the form (11.3) is convex; as expected, the level curves in figure 11.5 bound convex subsets of \( \mathcal{H}_{\text{slice}} \).

In section 5.5.2 we showed that, for any transfer function \( H \),

\[ ||HW||_{pk, \text{trk}} \leq ||H||_{wc} \leq 3||HW||_{pk, \text{trk}}. \]

The reader should compare the level curves in figures 11.4 and 11.5 with this relation in mind.
Figure 11.4 Level curves of the weighted peak tracking error, given by (11.2).

Figure 11.5 Level curves of the peak tracking error, for reference inputs bounded and slew-rate limited by 1, given by (11.3).
11.2 Regulation

11.2.1 Asymptotic Rejection of Constant Disturbances

We first consider the effect on $y_p$ of a constant disturbance applied to $n_{proc}$. Figure 11.6 shows the subset of $H_{slice}$ where such a disturbance asymptotically has no effect on $y_p$, i.e., where the affine function

$$\alpha H_{11}^{(a)}(0) + \beta H_{11}^{(b)}(0) + (1 - \alpha - \beta) H_{11}^{(c)}(0)$$

vanishes.

![Figure 11.6 Asymptotic rejection of constant actuator-referred disturbances on $y_p$.](image)

11.2.2 Rejection of a Particular Disturbance

Suppose now that an actuator-referred disturbance is the waveform $d_{part}(t)$ shown in figure 11.7. Figure 11.8 shows the level curves of the peak output $y_p$ due to the actuator-referred disturbance $d_{part}$, i.e.,

$$\left\| \left( \alpha H_{11}^{(a)} + \beta H_{11}^{(b)} + (1 - \alpha - \beta) H_{11}^{(c)} \right) d_{part} \right\|_\infty,$$

which is a convex function on $\mathbb{R}^2$. 
11.2 Regulation

Figure 11.7 A particular actuator-referred process disturbance signal, $d_{\text{part}}(t)$.

Figure 11.8 Level curves of the peak of $y_p$ due to the particular actuator-referred disturbance $d_{\text{part}}(t)$ shown in figure 11.7, given by (11.5).
11.2.3 RMS Regulation

Suppose that $n_{\text{proc}}$ and $n_{\text{sensor}}$ are independent, zero-mean stochastic processes with power spectral densities

$$S_{\text{proc}}(\omega) = W_{\text{proc}}^2,$$
$$S_{\text{sensor}}(\omega) = W_{\text{sensor}}^2,$$

where

$$W_{\text{proc}} = 0.04,$$
$$W_{\text{sensor}} = 0.01$$

(i.e., scaled white noises). Figure 11.9 shows the level curves of the RMS value of $y_p$ due to these noises, i.e., the level curves of the function

$$\varphi_{\text{rms,y_p}}(\alpha, \beta) \triangleq \phi_{\text{rms,y_p}} \left( \alpha H^{(a)} + \beta H^{(b)} + (1 - \alpha - \beta)H^{(c)} \right),$$

(11.6)

where

$$\phi_{\text{rms,y_p}}(H) \triangleq \left( \|H_{11}W_{\text{proc}}\|_2^2 + \|H_{12}W_{\text{sensor}}\|_2^2 \right)^{1/2}.$$ 

(11.7)

Recall from section 8.2.2 that the RMS response to independent stochastic inputs with known power spectral densities is a convex functional of $H$; therefore $\phi_{\text{rms,y_p}}$ is a convex function of $H$, and $\varphi_{\text{rms,y_p}}$ is a convex function of $\alpha$ and $\beta$.

11.3 Actuator Effort

11.3.1 A Particular Disturbance

We consider again the particular actuator-referred disturbance $d_{\text{part}}(t)$ shown in figure 11.7. Figure 11.10 shows the peak actuator signal $u$ due to the actuator-referred disturbance $d_{\text{part}}$, i.e.,

$$\left\| \left( \alpha H^{(a)}_{21} + \beta H^{(b)}_{21} + (1 - \alpha - \beta)H^{(c)}_{21} \right) d_{\text{part}} \right\|_\infty,$$

(11.8)

which is a convex function on $\mathbb{R}^2$.

11.3.2 RMS Limit

Figure 11.11 shows the level curves of the RMS value of $u$ due to the noises described in section 11.2.3, i.e., the level curves of the function

$$\phi_{\text{rms,u}} \left( \alpha H^{(a)} + \beta H^{(b)} + (1 - \alpha - \beta)H^{(c)} \right),$$

(11.9)
Figure 11.9 Level curves of the RMS value of \( y_p \), with sensor and actuator noises, given by (11.6).

Figure 11.10 Level curves of the peak actuator signal \( u \), due to the particular actuator-referred disturbance \( d_{\text{part}}(t) \) shown in figure 11.7, given by (11.8).
where
\[
\phi_{\text{rms, actuator}}(H) \overset{\Delta}{=} \left( \|H_{21} W_{\text{proc}}\|_2^2 + \|H_{22} W_{\text{sensor}}\|_2^2 \right)^{1/2}. \tag{11.10}
\]

Recall from section 8.2.2 that the RMS response to independent stochastic inputs with known power spectral densities is a convex functional of \(H\); therefore \(\phi_{\text{rms, actuator}}\) is a convex function of \(H\), and (11.9) is a convex function of \(\alpha\) and \(\beta\).

![Graph](image)

**Figure 11.11** Level curves of the RMS value of the actuator signal \(u\), with sensor and actuator noises, given by (11.9).

### 11.3.3 RMS Gain Limit

Figure 11.12 shows the level curves of the worst case RMS actuator signal \(u\) for any reference input \(r\) with RMS value bounded by 1, i.e.,
\[
\left\| \alpha H_{23}^{(a)} + \beta H_{23}^{(b)} + (1 - \alpha - \beta) H_{23}^{(c)} \right\|_\infty.
\tag{11.11}
\]
Figure 11.12 Level curves of the worst case RMS actuator signal $u$ for any reference input $r$ with RMS value bounded by 1, given by (11.11).
11.4 Sensitivity Specifications

11.4.1 A Log Sensitivity Specification

We consider the plant perturbation

\[ \delta P^{st}_{0}(s) = \gamma P^{st}_{0}(s), \]

i.e., a gain variation in \( P^{st}_{0} \) (see section 9.1.3). Figure 11.13 shows the level curves of the maximum logarithmic sensitivity of the magnitude of the I/O transfer function \( H_{13} \), over the frequency range \( 0 \leq \omega \leq 1 \), to these gain changes, i.e.,

\[
\sup_{0 \leq \omega \leq 1} \left| \frac{\partial}{\partial \gamma} \log |H_{13}(j\omega)| \right| = \max_{0 \leq \omega \leq 1} |\Re S(j\omega)|, \tag{11.12}
\]

where

\[
S(j\omega) = 1 - \left( \alpha H_{13}^{(a)}(j\omega) + \beta H_{13}^{(b)}(j\omega) + (1 - \alpha - \beta)H_{13}^{(c)}(j\omega) \right). \]

As expected, the level curves in figure 11.13 bound convex subsets of \( \mathcal{H}_{\text{slice}} \).

![Figure 11.13](image-url) Level curves of the logarithmic sensitivity of the magnitude of the I/O transfer function \( H_{13} \), over the frequency range \( 0 \leq \omega \leq 1 \), to gain changes in the plant \( P^{st}_{0} \), given by (11.12).

When the function (11.12) takes on the value 0.3, the maximum first order change in \( |H_{13}(j\omega)| \), over \( 0 \leq \omega \leq 1 \), with a 25% plant gain change is \( \exp(0.075) \), or 0.65dB. In figure 11.14 the actual maximum change in \( |H_{13}(j\omega)| \) is shown for points on the 0.3 contour of the function (11.12).