11.4 Sensitivity Specifications

In section 9.3, we considered the sensitivity of the I/O step response at $t = 1$ to plant gain changes, i.e., $\delta F_0^{\text{std}} = \gamma F_0^{\text{std}}$:

$$s_\gamma(1) \triangleq \frac{\partial s(1)}{\partial \gamma} \bigg|_{\gamma=0}.$$

Figure 11.15 shows the subset of $\mathcal{H}_{\text{slice}}$ for which

$$|s_\gamma(1)| \leq 0.75.$$

This specification is equivalent to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1 - T(j\omega))T(j\omega)}{j\omega} e^{j\omega} d\omega \leq 0.75,$$

where

$$T(j\omega) = \alpha H_{13}^{(a)}(j\omega) + \beta H_{13}^{(b)}(j\omega) + (1 - \alpha - \beta)H_{13}^{(c)}(j\omega).$$

As we showed in section 9.3, and as is clear from figure 11.15, the step response sensitivity specification (11.13) is not convex.

![Figure 11.14](image)

Figure 11.14 To first order, the peak change in $|H_{13}(j\omega)|$ for $0 \leq \omega \leq 1$ along the 0.3 contour in figure 11.13, for a 25% gain change in $F_0^{\text{std}}$, will be 0.65dB. The 0.3 contour from figure 11.13 is shown, together with the actual peak change in $|H_{13}(j\omega)|$ for $0 \leq \omega \leq 1$ for several points on the contour.

11.4.2 A Step Response Sensitivity Specification

In section 9.3 we considered the sensitivity of the I/O step response at $t = 1$ to plant gain changes, i.e., $\delta F_0^{\text{std}} = \gamma F_0^{\text{std}}$:

$$s_\gamma(1) \triangleq \frac{\partial s(1)}{\partial \gamma} \bigg|_{\gamma=0}.$$

Figure 11.15 shows the subset of $\mathcal{H}_{\text{slice}}$ for which

$$|s_\gamma(1)| \leq 0.75.$$

This specification is equivalent to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1 - T(j\omega))T(j\omega)}{j\omega} e^{j\omega} d\omega \leq 0.75,$$

where

$$T(j\omega) = \alpha H_{13}^{(a)}(j\omega) + \beta H_{13}^{(b)}(j\omega) + (1 - \alpha - \beta)H_{13}^{(c)}(j\omega).$$

As we showed in section 9.3, and as is clear from figure 11.15, the step response sensitivity specification (11.13) is not convex.
11.5 Robustness Specifications

11.5.1 Gain Margin

We now consider the gain margin specification $D_{+4,-3.5\text{dB}_\text{gm}}$: the system should remain stable for gain changes in $P_\text{std}$ between $+4\text{dB}$ and $-3.5\text{dB}$. In section 10.4.4 we used the small gain theorem to show that

$$
\|T\|_\infty \leq 1.71 \quad (11.14)
$$

$$
\|S\|_\infty \leq 2.02 \quad (11.15)
$$

are (different) inner approximations of the gain margin specification $D_{+4,-3.5\text{dB}_\text{gm}}$.

Figure 11.16 shows the subset of $\mathcal{H}_\text{slice}$ that meets the gain margin specification $D_{+4,-3.5\text{dB}_\text{gm}}$, together with the two inner approximations (11.14–11.15), i.e.,

$$
\left\| \alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta)H_{13}^{(c)} \right\|_\infty \leq 1.71 \quad (11.16)
$$

$$
\left\| 1 - \left( \alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta)H_{13}^{(c)} \right) \right\|_\infty \leq 2.02. \quad (11.17)
$$

In general, the specification $D_{+4,-3.5\text{dB}_\text{gm}}$ is not convex, even though in this case the subset of $\mathcal{H}_\text{slice}$ that satisfies $D_{+4,-3.5\text{dB}_\text{gm}}$ is convex. The two inner approximations (11.16–11.17) are norm-bounds on $H$, and are therefore convex (see sec-
11.5 Robustness Specifications

The two approximations (11.16–11.17) are convex subsets of the exact region that satisfies $D_{+4, -3.5\text{db}_{\text{gm}}}$. 

![Figure 11.16](image)

**Figure 11.16** The boundary of the region where the gain margin specification $D_{+4, -3.5\text{db}_{\text{gm}}}$ is met is shown, together with the boundaries of the two inner approximations (11.16–11.17). In this case the exact region turns out to be convex, but this is not generally so.

The bound on the sensitivity transfer function magnitude given by (10.69) is an inner approximation to each gain margin specification. Figure 11.17 shows the level curves of the peak magnitude of the sensitivity transfer function, i.e.,

$$
\varphi_{\text{max, sens}}(\alpha, \beta) \triangleq \left\| 1 - \left( \alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta)H_{13}^{(c)} \right) \right\|_{\infty}. \quad (11.18)
$$

In section 6.3.2 we showed that a function of the form (11.18) is convex; the level curves in figure 11.17 do indeed bound convex subsets of $H_{\text{gain}}$.

### 11.5.2 Generalized Gain Margin

We now consider a tighter gain margin specification than $D_{+4, -3.5\text{db}_{\text{gm}}}$: for plant gain changes between $+4 \text{dB}$ and $-3.5 \text{dB}$ the stability degree exceeds 0.2, i.e., the closed-loop system poles have real parts less than $-0.2$. In section 10.4.4 we used the small gain theorem to show that

$$
\| S \|_{\infty, 0.2} \leq 2.02 \quad (11.19)
$$

is an inner approximation of this generalized gain margin specification.
Figure 11.17 The level curves of the sensitivity transfer function magnitude, given by (11.18).

Figure 11.18 shows the subset of $\mathcal{H}_{\text{slice}}$ that meets the generalized gain margin specification, together with the inner approximation (11.19), i.e.,

$$\left\| 1 - \left( \alpha H^{(a)}_{13} + \beta H^{(b)}_{13} + (1 - \alpha - \beta) H^{(c)}_{13} \right) \right\|_{\infty,0,2} \leq 2.02.$$  \hfill (11.20)

The generalized gain margin specification is not in general convex, even though in this case the subset of $\mathcal{H}_{\text{slice}}$ that satisfies the generalized gain margin specification is convex. The inner approximation (11.20) is a norm-bound on $H$, and is therefore convex (see section 6.3.2).

### 11.5.3 Robust Stability with Relative Plant Uncertainty

Figure 11.19 shows the subset of $\mathcal{H}_{\text{slice}}$ that meets the specification

$$D_{\text{rob.stab}}(\mathcal{P}),$$ \hfill (11.21)

where $\mathcal{P}$ is the plant perturbation set (10.10), i.e., robust stability with the relative plant uncertainty $W_{\text{rel.err}}$ described in section 10.2.3. The specification (11.21) is equivalent to the convex inner approximation

$$\left\| W_{\text{rel.err}} \left( \alpha H^{(a)}_{13} + \beta H^{(b)}_{13} + (1 - \alpha - \beta) H^{(c)}_{13} \right) \right\|_{\infty} < 1$$ \hfill (11.22)

derived from the small gain theorem in section 10.4.4, i.e., in this case the small gain theorem is not conservative (see the Notes and References in chapter 10).
The boundary of the exact region where the generalized gain margin specification is met is shown, together with the boundary of the inner approximation $\mathcal{D}_{+4, -3.5 \text{dB}_{\text{om}}}$. This specification is tighter than the specification $\mathcal{D}_{+4, -3.5 \text{dB}_{\text{om}}}$ shown in figure 11.16.

The region where the robust stability specification (11.21) is met is shaded. This region is the same as the inner approximation (11.22).
11.5.4 Robust Performance

Figure 11.20 shows the subset of \( \mathcal{H}_{\text{slice}} \) that meets the specification

\[
\mathcal{D}_{\text{rob}}(\mathcal{P}, \| H_{23} \|_\infty \leq 75),
\]

where \( \mathcal{P} \) is the plant perturbation set (10.10), i.e., the plant perturbations described in section 10.2.3 never cause the RMS gain from the reference input to the actuator signal to exceed 75. In section 10.5.2 we showed that an inner approximation of the specification (11.23) is

\[
\left| \alpha H_{13}^{(a)}(j\omega) + \beta H_{13}^{(b)}(j\omega) + (1 - \alpha - \beta) H_{13}^{(c)}(j\omega) \right| < l(\omega) \text{ for } \omega \in \mathbb{R}
\]

(11.24)

(note that \( H_{13} = H_{13}/P_{0}^{\text{std}} \), where

\[
l(\omega) = \frac{1}{\sqrt{2 \left( |W_{\text{rel.err}}(j\omega)|^2 + |1/(75P_{0}^{\text{std}}(j\omega))|^2 \right)}}.
\]

which is also shown in figure 11.20. The exact region is not in general convex, although in this case it happens to be convex (see the Notes and References in chapter 10).

**Figure 11.20** The boundary of the exact region where the robust performance specification (11.23) is met is shown, together with the inner approximation (11.24).

Figure 11.21 shows the exact specification (11.23), together with the convex inner approximation (11.24) and a convex outer approximation (i.e., a specification that
is weaker than (11.23)). The outer approximation is the simultaneous satisfaction of the specifications

\[ D_{\text{rob, stab}}(P) \quad \text{and} \quad \|H_{23}\|_\infty \leq 75, \]  

(11.25)
described in sections 11.5.3 and 11.3.3 respectively. The specifications (11.25) require robust stability, and that the nominal system has an RMS gain from the reference to actuator signal not exceeding 75. The robust performance specification (11.23) therefore implies (11.25), so (11.25) is an outer approximation of the robust performance specification (11.23). The outer approximation in figure 11.21 is the set of \( \alpha, \beta \) for which

\[
\begin{align*}
\| W_{\text{rel.err}} \left( \alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta)H_{13}^{(c)} \right) \|_\infty &< 1, \\
\| \alpha H_{23}^{(a)} + \beta H_{23}^{(b)} + (1 - \alpha - \beta)H_{23}^{(c)} \|_\infty &\leq 75,
\end{align*}
\]

(11.26)

(see figure 10.21).

**Figure 11.21** The boundary of the exact region where the robust performance specification (11.23) is met is shown, together with the inner approximation (11.24) and the outer approximation (11.26). The outer approximation is the intersection of the nominal performance specification and the robust stability specification (see figures 11.12 and 11.19).
11.6 Nonconvex Design Specifications

11.6.1 Controller Stability

Figure 11.22 shows the subset of $\mathcal{H}_{\text{slice}}$ that is achieved by an open-loop stable controller, i.e.,

$$
\left\{ [\alpha \ \beta]^T \left| \alpha H^{(a)} + \beta H^{(b)} + (1 - \alpha - \beta)H^{(c)} \text{ is achieved by a stable controller } K \right. \right\}.
$$

From figure 11.22, we see that (11.27) is a nonconvex subset of $\mathcal{H}_{\text{slice}}$. We conclude that a specification requiring open-loop controller stability,

$$
\mathcal{H}_{k,\text{stab}} = \left\{ H \left| H = P_{zw} + P_{zu} K (I - P_{yu} K)^{-1} P_{yw} \right. \right\},
$$

is not in general convex.

![Figure 11.22](image)

Figure 11.22 Region where the closed-loop transfer matrix $H$ is achieved by a controller that is open-loop stable. It is not convex.

11.7 A Weighted-Max Functional

Consider the functional

$$
\varphi_{\text{wt, max}}(\alpha, \beta) \triangleq \max \{ \varphi_{\text{pk, trk}}(\alpha, \beta), 0.5 \varphi_{\text{max, sens}}(\alpha, \beta), 15 \varphi_{\text{rms, trk}}(\alpha, \beta) \},
$$
where the functions \( \varphi_{\text{pk},k} \), \( \varphi_{\text{max},\text{sens}} \), and \( \varphi_{\text{rms},p} \) are given by (11.2), (11.18), and (11.6). The level curves of the function \( \varphi_{\text{wt},\text{max}}(\alpha, \beta) \) are shown in figure 11.23. The function \( \varphi_{\text{wt},\text{max}} \) will be used for several examples in chapter 14.

\[ \begin{array}{c}
\alpha \\
-1 & -0.5 & 0 & 0.5 & 1 & 1.5 & 2 \\
-1 & -0.5 & 0 & 0.5 & 1 & 1.5 & 2 \\
0 & 0.5 & 1 & 1.5 & 2 & 2.5 & 3 & 3.5 & 4 & 2
\end{array} \]

**Figure 11.23** The level curves of \( \varphi_{\text{wt},\text{max}}(\alpha, \beta) \).
Notes and References

How These Figures Were Computed

Most level curves of convex functions were plotted using a radial bisection method. Consider the problem of plotting the \( \phi(H) = \gamma \) level curve on \( H_{3\times3} \), where \( \phi \) is convex. Assume that we know some point \((\alpha_0, \beta_0)\) inside this level curve, i.e.,

\[
\phi \left( \alpha_0 H^{(a)} + \beta_0 H^{(b)} + (1 - \alpha_0 - \beta_0)H^{(c)} \right) < \gamma.
\]

The value of \( \phi \) along the radial line segment

\[
\alpha = \alpha_0 + \lambda \cos(\theta), \quad \beta = \beta_0 + \lambda \sin(\theta),
\]

where \( \lambda \geq 0 \), is

\[
\varphi_{\lambda}(\lambda) = \phi \left( (\alpha_0 + \lambda \cos(\theta)) \left( H^{(a)} - H^{(c)} \right) + (\beta_0 + \lambda \sin(\theta)) \left( H^{(b)} - H^{(c)} \right) + H^{(c)} \right).
\]

For each \( \theta \), \( \varphi_{\lambda} \) is a convex function from \( \mathbb{R}_+ \) to \( \mathbb{R} \) with \( \varphi_{\lambda}(0) < \gamma \), so there is no more than one \( \lambda \geq 0 \) for which

\[
\varphi_{\lambda}(\lambda) = \gamma.
\]

(11.29) can be solved using a number of standard methods, such as bisection or regula falsi, with only an evaluation of \( \varphi_{\lambda} \) required at each iteration. As \( \theta \) sweeps out the angles \( 0 \leq \theta < 2\pi \), the solution \( \lambda \) to (11.29), together with (11.28), sweeps out the desired level curve. This method was used for figures 11.4, 11.5, 11.12, 11.13, 11.17, 11.19, 11.20, 11.21, and 11.23.

The above method also applies to quasiconvex functionals with the following modification: in place of (11.29) we need to find the largest \( \lambda \) for which \( \varphi_{\lambda}(\lambda) \leq \gamma \).

In certain cases (11.29), or its quasiconvex modification, can be solved directly. For example, consider the quasiconvex settling-time functional \( \Phi_{\text{sett}} \) from section 11.1.1. For a fixed \( \theta \), the step response along the line (11.28) is of the form

\[
s_0(t) + \lambda s_1(t),
\]

where \( s_0 \) is the step response of

\[
\alpha_0 H_{13}^{(a)} + \beta_0 H_{13}^{(b)} + (1 - \alpha_0 - \beta_0)H_{13}^{(c)},
\]

and \( s_1 \) is the step response of

\[
\cos(\theta)H_{13}^{(a)} + \sin(\theta)H_{13}^{(b)} + (-\cos(\theta) - \sin(\theta))H_{13}^{(c)}.
\]

The largest value of \( \lambda \) for which \( \varphi_{\lambda}(\lambda) \leq T_{\text{max}} \) is given by

\[
\lambda^* = \max_{0.95 \leq s_0(t) + \lambda s_1(t) \leq 1.05 \text{ for } t \geq T_{\text{max}}} \lambda.
\]
This is a linear program in the scalar variable $\lambda$ that can be directly solved.

A similar method was used to produce figures 11.8 and 11.10. A similar method could also be used to produce level curves of $H_{12}$ norms; at each frequency $\omega$ a quadratic in $\lambda$ can be solved to find the positive value of $\lambda$ that makes the frequency response magnitude tight at $\omega$. Taking the minimum $\lambda$ over all $\omega$ gives the desired $\lambda^*$.

Figures 11.9 and 11.11 were plotted by directly computing the equations of the level curves using a state-space method. For example, consider $\|H_{12}\|_2^2$, which is one term in the functional $\phi_{\text{vls} - \text{RP}}$. Since

$$\alpha H_{12}^{(a)} + \beta H_{12}^{(b)} + (1 - \alpha - \beta)H_{12}^{(c)} = H^{(c)} + \alpha (H^{(a)} - H^{(c)}) + \beta (H^{(a)} - H^{(c)})$$

is affine in $\alpha$ and $\beta$, it has a state-space realization

$$C(sI - A)^{-1}(B_0 + \alpha B_1 + \beta B_2),$$

where $C$ is a row vector, and $B_0$, $B_1$, and $B_2$ are column vectors. From section 5.6.1, if $W_{\text{obs}}$ is the solution to the Lyapunov equation

$$A^T W_{\text{obs}} + W_{\text{obs}} A + C^T C = 0,$$

we have

$$\left\| \alpha H_{12}^{(a)} + \beta H_{12}^{(b)} + (1 - \alpha - \beta)H_{12}^{(c)} \right\|_2^2 = \begin{bmatrix} 1 & \alpha & \beta \end{bmatrix} E \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix},$$

where

$$E = \begin{bmatrix} B_0^T \\ B_1^T \\ B_2^T \end{bmatrix} W_{\text{obs}} \begin{bmatrix} B_0 & B_1 & B_2 \end{bmatrix}$$

is a positive definite $3 \times 3$ matrix. The level curves of $\|H_{12}\|_2$ on $\mathcal{H}_{\text{slice}}$ are therefore ellipses.

The convex sets shown in figures 11.16 and 11.18 were produced using the standard radial method described above. The exact contour along which the gain margin specification was tight was also found by a radial method. However, since this specification need not be convex, a fine grid search was also used to verify that the entire set had been correctly determined.

The step response sensitivity plot in figure 11.15 was produced by forming an indefinite quadratic in $\alpha$ and $\beta$. The sensitivity, from (11.13), is the step response of

$$\left(1 - \left(\alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta)H_{13}^{(c)}\right)\right)\left(\alpha H_{13}^{(a)} + \beta H_{13}^{(b)} + (1 - \alpha - \beta)H_{13}^{(c)}\right)$$

at $t = 1$. After expansion, the step response of each term, at $t = 1$, gives each of the coefficients in an indefinite quadratic form in $\alpha$ and $\beta$. Figure 11.15 shows the $\alpha$, $\beta$ for
which

\[-0.75 \leq \begin{bmatrix} 1 & \alpha & \beta \\ 0.7149 & -0.0055 & 0.1180 \\ -0.0055 & -0.0074 & 0.0447 \\ 0.1180 & 0.0447 & -0.2493 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} \leq 0.75.\]

The controller stability plot in figure 11.22 was produced by finding points where the transfer function \( S(j\omega) / \omega^2 \) vanished for some frequency \( \omega \). Since \( S = 1 / (1 + P_0^{std} K) \) vanishes wherever \( P_0^{std} \) or \( K \) has a \( j\omega \) axis pole, the \( j\omega \) axis poles of \( K \) are exactly the \( j\omega \) axis zeros of \( S(j\omega) / \omega^2 \); the factor of \( \omega^2 \) cancels the two zeros at \( s = 0 \) that \( S \) inherits from the two \( s = 0 \) poles of \( P_0^{std} \). At each frequency \( \omega \), the linear equations in \( \alpha \) and \( \beta \)

\[
\Re \left( 1 - \left( \alpha H_{13}^{(a)}(j\omega) + \beta H_{13}^{(b)}(j\omega) + (1 - \alpha - \beta)H_{13}^{(c)}(j\omega) \right) \right) / \omega^2 = 0,
\]

\[
\Im \left( 1 - \left( \alpha H_{13}^{(a)}(j\omega) + \beta H_{13}^{(b)}(j\omega) + (1 - \alpha - \beta)H_{13}^{(c)}(j\omega) \right) \right) / \omega^2 = 0
\]

may be dependent, independent, or inconsistent; their solution in the first two cases gives either a line or point in the \((\alpha, \beta)\) plane. When these lines and points are plotted over all frequencies they determine subsets of \( \mathcal{H}_{all} \) over which the controller \( K \) has a constant number of unstable (right half-plane) poles. By checking any one controller inside each subset of \( \mathcal{H}_{all} \) for open-loop stability, each subset of \( \mathcal{H}_{all} \) can be labeled as being achieved by stable or unstable controllers.
Part IV

NUMERICAL METHODS
Chapter 12

Some Analytic Solutions

We describe several families of controller design problems that can be solved rapidly and exactly using standard methods.

12.1 Linear Quadratic Regulator

The linear quadratic regulator (LQR) from optimal control theory can be used to solve a family of regulator design problems in which the state is accessible and regulation and actuator effort are each measured by mean-square deviation. A stochastic formulation of the LQR problem is convenient for us; a more usual formulation is as an optimal control problem (see the Notes and References at the end of this chapter). The system is described by

\[ \dot{x} = Ax + Bu + w, \]

where \( w \) is a zero-mean white noise, i.e., \( w \) has power spectral density matrix \( S_w(\omega) = I \) for all \( \omega \). The state \( x \) is available to the controller, so \( y = x \) in our framework.

The LQR cost function is the sum of the steady-state mean-square weighted state \( x \), and the steady-state mean-square weighted actuator signal \( u \):

\[ J_{\text{LQR}} = \lim_{t \to \infty} \mathbb{E} \left( x(t)^T Q x(t) + u(t)^T R u(t) \right), \]

where \( Q \) and \( R \) are positive semidefinite weight matrices; the first term penalizes deviations of \( x \) from zero, and the second term represents the cost of using the actuator signal. We can express this cost in our framework by forming the regulated output signal

\[ z = \begin{bmatrix} R^{1/2} u \\ Q^{1/2} x \end{bmatrix}, \]
so that
\[ J_{\text{KPC}} = \lim_{t \to -\infty} E z(t)^T z(t), \]
the mean-square deviation of \( z \). Since \( w \) is a white noise, we have (see section 5.2.2)
\[ J_{\text{KPC}} = \|H\|_2^2, \]
the square of the \( H_2 \) norm of the closed-loop transfer matrix.

In our framework, the plant for the LQR regulator problem is given by
\[
A_p = A \\
B_u = B \\
B_w = I \\
C_z = \begin{bmatrix} 0 & Q^{1/2} \end{bmatrix} \\
C_y = I \\
D_{zw} = 0 \\
D_{zu} = \begin{bmatrix} R^{1/2} & 0 \end{bmatrix} \\
D_{yw} = 0 \\
D_{yu} = 0
\]
(the matrices on left-hand side refer to the state-space equations from section 2.5). This is shown in figure 12.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure12_1.png}
\caption{The LQR cost is \( \|H\|_2^2 \).}
\end{figure}

The specifications that we consider are realizability and the functional inequality specification
\[ \|H\|_2 \leq \alpha. \]
(12.1)
Standard assumptions are that $(Q, A)$ is observable, $(A, B)$ is controllable, and $R > 0$, in which case the specification (12.1) is stronger than (i.e., implies) internal stability. (Recall our comment in chapter 7 that internal stability is often a redundant addition to a sensible set of specifications.) With these standard assumptions, there is actually a controller that achieves the smallest achievable LQR cost, and it turns out to be a constant state-feedback,

$$K_{\text{lqr}}(s) = -K_{\text{sfb}},$$

which can be found as follows.

Let $X_{\text{lqr}}$ denote the unique positive definite solution of the algebraic Riccati equation

$$A^T X_{\text{lqr}} + X_{\text{lqr}} A - X_{\text{lqr}} B R^{-1} B^T X_{\text{lqr}} + Q = 0. \tag{12.2}$$

One method of finding this $X_{\text{lqr}}$ is to form the associated Hamiltonian matrix

$$M = \begin{bmatrix} A & -B R^{-1} B^T \\ -Q & -A^T \end{bmatrix}, \tag{12.3}$$

and then compute any matrix $T$ such that

$$T^{-1} M T = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix},$$

where $\tilde{A}_{11}$ is stable. (One good choice is to compute an ordered Schur form of $M$; see the Notes and References in chapter 5.) We then partition $T$ as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

and the solution $X_{\text{lqr}}$ is given by

$$X_{\text{lqr}} = T_{21} T_{11}^{-1}.$$

(We encountered a similar ARE in section 5.6.3; this solution method is analogous to the one described there.)

Once we have found $X_{\text{lqr}}$, we have

$$K_{\text{sfb}} = R^{-1} B^T X_{\text{lqr}},$$

which achieves LQR cost

$$J'_{\text{lqr}} = \text{Tr} X_{\text{lqr}}.$$

In particular, the specification (12.1) (along with realizability) is achievable if and only if $\alpha \geq \sqrt{\text{Tr} X_{\text{lqr}}}$, in which case the LQR-optimal controller $K_{\text{lqr}}$ achieves the specifications.

In practice, this analytic solution is not used to solve the feasibility problem for the one-dimensional family of specifications indexed by $\alpha$; rather it is used to solve multicriterion optimization problems involving actuator effort and state excursion, by solving the LQR problem for various weights $R$ and $Q$. This is explained further in section 12.2.1.
12.2 Linear Quadratic Gaussian Regulator

The linear quadratic Gaussian (LQG) problem is a generalization of the LQR problem to the case in which the state is not sensed directly. For the LQG problem we consider the system given by

\[
\begin{align*}
\dot{x} &= Ax + Bu + w_{\text{proc}} \\
y &= Cx + v_{\text{sensor}},
\end{align*}
\]  

where the process noise \( w_{\text{proc}} \) and measurement noise \( v_{\text{sensor}} \) are independent and have constant power spectral density matrices \( W \) and \( V \), respectively.

The LQG cost function is the sum of the steady-state mean-square weighted state \( x \), and the steady-state mean-square weighted actuator signal \( u \):

\[
J_{\text{lqg}} = \lim_{t \to \infty} \mathbb{E} \left( x(t)^T Q x(t) + u(t)^T R u(t) \right),
\]

where \( Q \) and \( R \) are positive semidefinite weight matrices.

This LQG problem can be cast in our framework as follows. Just as in the LQR problem, we extract the (weighted) plant state \( x \) and actuator signal \( u \) as the regulated output, i.e.,

\[
z = \begin{bmatrix} R^{\frac{1}{2}} u \\ Q^{\frac{1}{2}} x \end{bmatrix}.
\]

The exogenous input consists of the process and measurement noises, which we represent as

\[
\begin{bmatrix} w_{\text{proc}} \\ v_{\text{sensor}} \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} \\ V^{\frac{1}{2}} \end{bmatrix} w,
\]

with \( w \) a white noise signal, i.e., \( S_w(\omega) = I \). The state-space description of the plant for the LQG problem is thus

\[
\begin{align*}
A_P &= A \\
B_w &= \begin{bmatrix} W^{\frac{1}{2}} \\ 0 \end{bmatrix} \\
B_u &= B \\
C_z &= \begin{bmatrix} 0 \\ Q^{\frac{1}{2}} \end{bmatrix} \\
C_y &= C \\
D_{zw} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
D_{zu} &= \begin{bmatrix} R^{\frac{1}{2}} \\ 0 \end{bmatrix} \\
D_{yw} &= \begin{bmatrix} 0 \\ V^{\frac{1}{2}} \end{bmatrix} \\
D_{yu} &= 0.
\end{align*}
\]
This is shown in figure 12.2.

Since \( w \) is a white noise, the LQG cost is simply the variance of \( z \), which is given by

\[
J_{\text{LQG}} = ||H||_2^2.
\]

The specifications for the LQG problem are therefore the same as for the LQR problem: realizability and the \( H_2 \) norm-bound (12.1).

Standard assumptions for the LQG problem are that the plant is controllable from each of \( u \) and \( w \), observable from each of \( z \) and \( y \), a positive weight is used for the actuator signal \((R > 0)\), and the sensor noise satisfies \( V > 0 \). With these standard assumptions in force, there is a unique controller \( K_{\text{LQG}} \) that minimizes the LQG objective. This controller has the form of an estimated-state-feedback controller (see section 7.4); the optimal state-feedback and estimator gains, \( K_{\text{sb}} \) and \( L_{\text{est}} \), can be determined by solving two algebraic Riccati equations as follows. The state-feedback gain is given by

\[
K_{\text{sb}} = R^{-1}B^TX_{\text{LQG}},
\]

where \( X_{\text{LQG}} \) is the unique positive definite solution of the Riccati equation

\[
A^TX_{\text{LQG}} + X_{\text{LQG}}A - X_{\text{LQG}}BR^{-1}B^TX_{\text{LQG}} + Q = 0,
\]

which is the same as (12.2). The estimator gain is given by

\[
L_{\text{est}} = Y_{\text{LQG}}C^TV^{-1},
\]
where $Y_{lqg}$ is the unique positive definite solution of
\[
AY_{lqg} + Y_{lqg}A^T - Y_{lqg}C^TV^{-1}CY_{lqg} + W = 0
\]
(12.19)
(which can be solved using the methods already described in sections 12.1 and 5.6.3). The LQG-optimal controller $K_{lqg}$ is thus
\[
K_{lqg}(s) = -K_{sfb}(sI - A + BK_{sfb} + L_{est}C)^{-1}L_{est},
\]
and the optimal LQG cost is
\[
J_{lqg}^* = \text{Tr}(X_{lqg}W + QY_{lqg} + 2X_{lqg}AY_{lqg}).
\]
(12.21)

The specification $\|H\|_2 \leq \alpha$ (along with realizability) is therefore achievable if and only if $\alpha \geq \sqrt{J_{lqg}^*}$, in which case the LQG-optimal controller $K_{lqg}$ satisfies the specifications.

### 12.2.1 Multicriterion LQG Problem

The LQG objective (12.6) can be interpreted as a weighted-sum objective for a related multicriterion optimization problem. We consider the same system as in the LQG problem, given by (12.4–12.5); the objectives are the variances of the actuator signals,
\[
\|u_1\|_{rms}^2, \ldots, \|u_n\|_{rms}^2,
\]
and some critical variables that are linear combinations of the system state,
\[
\|c_1x\|_{rms}^2, \ldots, \|c_mx\|_{rms}^2,
\]
where the process and measurement noises are the same as for the LQG problem ($c_1, \ldots, c_m$ are row vectors that determine the critical variables).

We describe this multicriterion optimization problem in our framework as follows. We use the same plant as for the LQG problem, substituting
\[
z = \begin{bmatrix}
  u_1 \\
  \vdots \\
  u_n \\
  c_1x \\
  \vdots \\
  c_mx
\end{bmatrix}
\]
for the regulated output used there. The state-space plant equations for the multicriterion LQG problem are therefore given by (12.7–12.15), with
\[
C_z = \begin{bmatrix}
  0 \\
  c_1 \\
  \vdots \\
  c_m
\end{bmatrix}
\]
substituted for (12.10) and
\[ D_{zu} = \begin{bmatrix} I & 0 \end{bmatrix} \]
substituted for (12.13). The objectives are given by the squares of the \( H_2 \) norms of the rows of the closed-loop transfer matrix:
\[ \phi_i(H) \overset{\Delta}{=} \| H^{(i)} \|_2^2, \]
where \( H^{(i)} \) is the \( i \)th row of \( H \) and \( L = n_x = n_u + m \). The hard constraint for this multicriterion optimization problem is realizability.

For \( 1 \leq i \leq n_u \), \( \phi_i(H) \) represents the variance of the \( i \)th actuator signal, and for \( n_u + 1 \leq i \leq L \), \( \phi_i(H) \) represents the variance of the critical variable \( c_{i-n_u}x \). The design specification
\[ \phi_1(H) \leq a_1, \ldots, \phi_L(H) \leq a_L, \]
therefore, limits the RMS values of the actuator signals and critical variables.

Consider the weighted-sum objective associated with this multicriterion optimization problem:
\[ \phi_{\text{wt, sum}}(H) = \lambda_1 \phi_1(H) + \cdots + \lambda_L \phi_L(H), \]
where \( \lambda \geq 0 \). We can express this as
\[ \phi_{\text{wt, sum}}(H) = J_{\text{LQG}} \]
if we choose weight matrices
\[ Q = \lambda_{n_u+1} c_1^T c_1 + \cdots + \lambda_m c_m^T c_m, \]  
\[ R = \text{diag}(\lambda_1, \ldots, \lambda_m), \]
(\( \text{diag} (\cdot) \) is the diagonal matrix with diagonal entries given by the argument list).

Hence by solving an LQG problem, we can find the optimal design for the weighted-sum objective for the multicriterion optimization problem with functionals \( \phi_1, \ldots, \phi_L \). These designs are Pareto optimal for the multicriterion optimization problem; moreover, because the objective functionals and the hard constraint are convex, every Pareto optimal design arises this way for some choice of the weights \( \lambda_1, \ldots, \lambda_L \) (see section 6.5). Roughly speaking, by varying the weights for the LQG problem, we can “search” the whole tradeoff surface.

We note that by solving an LQG problem, we can evaluate the dual function \( \psi \) described in section 3.6.2:
\[ \psi(\lambda) = \min \{ \lambda_1 \phi_1(H) + \cdots + \lambda_L \phi_L(H) \mid H \text{ is realizable} \} \]
\[ = J_{\text{LQG}}^*, \]
given by (12.21), using the weights (12.24–12.25). We will use this fact in section 14.5, where we describe an algorithm for solving the feasibility problem (12.22).
12.3 Minimum Entropy Regulator

The LQG solution method described in section 12.2 was recently modified to find the controller that minimizes the $\gamma$-entropy of $H$, defined in section 5.3.5. Since the $\gamma$-entropy of $H$ is finite if and only if its $H_\infty$ norm is less than $\gamma$, this analytic solution method can be used to solve the feasibility problem with the inequality specification $\|H\|_\infty < \gamma$.

The plant is identical to the one considered for the LQG problem, given by (12.7–12.15); we also make the same standard assumptions for the plant that we made for the LQG case. The design specifications are realizability and the $H_\infty$ norm inequality specification

$$\|H\|_\infty < \gamma$$  \hspace{1cm} (12.26)

(which are stronger than internal stability under the standard assumptions). We will show how to solve the feasibility problem for this one-dimensional family of design specifications.

It turns out that if the design specification (12.26) (along with realizability) is achievable, then it is achievable by a controller that is, except for a scale factor, an estimated-state-feedback controller. This controller can be found as follows. If $\gamma$ is such that the specification (12.26) is feasible, then the two algebraic Riccati equations

$$A^T X_{me} + X_{me} A - X_{me} (BR^{-1} B^T - \gamma^{-2} W) X_{me} + Q = 0$$  \hspace{1cm} (12.27)

(c.f. (12.17)), and

$$A Y_{me} + Y_{me} A^T - Y_{me} (CTV^{-1} C - \gamma^{-2} Q) Y_{me} + W = 0$$  \hspace{1cm} (12.28)

(c.f. (12.19)) have unique positive definite solutions $X_{me}$ and $Y_{me}$, respectively. (The mnemonic subscript “me” stands for “minimum entropy”.) These solutions can be found by the method described in section 12.1, using the associated Hamiltonian matrices

$$\begin{bmatrix} A & -(BR^{-1} B^T - \gamma^{-2} W) \\ -Q & -A^T \end{bmatrix}, \quad \begin{bmatrix} A & -W \\ -(CTV^{-1} C - \gamma^{-2} Q) & -A^T \end{bmatrix}$$

if either of these matrices has imaginary eigenvalues, then the corresponding ARE does not have a positive definite solution, and the specification (12.26) is not feasible.

From $X_{me}$ and $Y_{me}$ we form the matrix

$$X_{me} (I - \gamma^{-2} Y_{me}^{-1} X_{me})^{-1},$$  \hspace{1cm} (12.29)

which can be shown to be symmetric. If this matrix is not positive definite (or the inverse fails to exist), then the specification (12.26) (along with realizability) is not feasible.
If, on the other hand, the positive definite solutions $X_{me}$ and $Y_{me}$ exist, and the matrix (12.29) exists and is positive definite, then the specification (12.26) (along with realizability) is feasible. Let
\[
K_{sfb} = R^{-1}B^TX_{me}(I - \gamma^{-2}Y_{me}X_{me})^{-1}
\]
and
\[
L_{\text{est}} = Y_{me}C^TV^{-1}
\]
(c.f. (12.16) and (12.18)). A controller that achieves the design specifications is given by
\[
K_{me}(s) = -K_{sfb}(sI - A + BK_{sfb} + L_{\text{est}}C - \gamma^{-2}Y_{me}Q)^{-1}L_{\text{est}}
\]
(c.f. the LQG-optimal controller (12.20)).

### 12.4 A Simple Rise Time, Undershoot Example

In this section and the next we show how to find explicit solutions for two specific plants and families of design specifications.

We consider the classical 1-DOF system of section 2.3.2 with
\[
P_0(s) = \frac{s - 1}{(s + 1)^2}.
\]
It is well-known in classical control that since $P_0$ has a real unstable zero at $s = 1$, the step response from the reference input $r$ to the system output $y_p$, $s_1(t)$, must exhibit some undershoot. We will study exactly how much it must undershoot, when we require that a stabilizing controller also meet a minimum rise-time specification.

Our design specifications are internal stability, a limit on undershoot,
\[
\phi_{\text{us}}(H_{13}) \leq U_{\text{max}},
\]
and a limit on rise time,
\[
\phi_{\text{rise}}(H_{13}) \leq T_{\text{max}}.
\]
Thus we have a two-parameter family of design specifications, indexed by $U_{\text{max}}$ and $T_{\text{max}}$.

These design specifications are simple enough that we can readily solve the feasibility problem for each $U_{\text{max}}$ and $T_{\text{max}}$. We will see, however, that these design specifications are not complete enough to guarantee reasonable controller designs; for example, they include no limit on actuator effort. We will return to this point later.
We can express the design specification of internal stability in terms of the interpolation conditions (section 7.2.5) for $T$, the I/O transfer function: $T$ is stable and satisfies

$$T(1) = T(\infty) = 0.$$  \hfill (12.34)

This in turn can be expressed in terms of the step response $s_{13}(t)$: $s_{13}$ is the step response of a stable transfer function and satisfies

$$\int_0^\infty s_{13}(t)e^{-t} dt = 0,$$

$$s_{13}(0) = 0.$$  \hfill (12.35)\hfill (12.36)

Now if (12.33) holds then

$$\int_{T_{\text{max}}}^\infty s_{13}(t)e^{-t} dt \geq 0.8 \int_{T_{\text{max}}}^\infty e^{-t} dt = 0.8e^{-T_{\text{max}}}.$$  \hfill (12.37)

If (12.32) holds then

$$\int_0^{T_{\text{max}}} s_{13}(t)e^{-t} dt \geq -U_{\text{max}} \int_0^{T_{\text{max}}} e^{-t} dt = -U_{\text{max}} (1 - e^{-T_{\text{max}}}).$$  \hfill (12.38)

Adding (12.37) and (12.38) we have

$$0 \geq \int_0^\infty s_{13}(t)e^{-t} dt \geq 0.8e^{-T_{\text{max}}} - U_{\text{max}} (1 - e^{-T_{\text{max}}}).$$

Hence if the design specifications with $U_{\text{max}}$ and $T_{\text{max}}$ are feasible,

$$U_{\text{max}} \geq \frac{0.8e^{-T_{\text{max}}}}{1 - e^{-T_{\text{max}}}}.$$  \hfill (12.39)

This relation is shown in figure 12.3. We have shown that every achievable undershoot, rise-time specification must lie in the shaded region of figure 12.3; in other words, the shaded region in figure 12.3 includes the region of achievable specifications in performance space.

In fact, the specifications with limits $U_{\text{max}}$ and $T_{\text{max}}$ are achievable if and only if

$$U_{\text{max}} > \frac{0.8e^{-T_{\text{max}}}}{1 - e^{-T_{\text{max}}}},$$

so that the shaded region in figure 12.3 is exactly the region of achievable specifications for our family of design specifications.

We will briefly explain why this is true. Suppose that $U_{\text{max}}$ and $T_{\text{max}}$ satisfy (12.39). We can then find a step response $s_{13}(t)$ of a stable rational transfer function, that satisfies the interpolation conditions (12.35–12.36) and the overshoot
12.4 A Simple Rise Time, Undershoot Example

The tradeoff between achievable undershoot and rise-time specifications.

Figure 12.3 The tradeoff between achievable undershoot and rise-time specifications.

and undershoot limits. If $U_{\text{max}}$ and $T_{\text{max}}$ are near the boundary of the region of achievable specifications, this step response will have to “hug” (but not violate) the two constraints. For $U_{\text{max}} = 0.70$ and $T_{\text{max}} = 1.0$ (marked “X" in figure 12.3) a suitable step response is shown in figure 12.4; it is the step response of a 20th order transfer function (and corresponds to a controller $K$ of order 22). (A detailed justification that we can always design such a step response is quite cumbersome; we have tried to give the general idea. See the Notes and References at the end of this chapter for more detail about this particular transfer function.)

The rapid changes near $t = 0$ and $t = 1$ of the step response shown in figure 12.4 suggest very large actuator signals, and this can be verified. It should be clear that for specifications $U_{\text{max}}, T_{\text{max}}$ that are nearly Pareto optimal, such rapid changes in the step response, and hence large actuator signals, will be necessary. So controllers that achieve specifications near the tradeoff curve are probably not reasonable from a practical point of view; but we point out that this “side information” that the actuator signal should be limited was not included in our design specifications. The fact that our specifications do not limit actuator effort, and therefore are probably not sensible, is reflected in the fact that the Pareto optimal specifications, which satisfy

$$U_{\text{max}} = \frac{0.8e^{-T_{\text{max}}}}{1 - e^{-T_{\text{max}}}},$$

are not achievable (see the comments at the end of section 3.5).
Figure 12.4 A step response with an undershoot of 0.70 and a rise time of 
1.0, which achieves the specifications marked “X” in figure 12.3. Undershoot 
as small as 0.466 with a rise time of 1.0 are also achievable.

The tradeoff curve in figure 12.3 is valuable even though the design specifications 
do not limit actuator effort. If we add to our design specifications an appropriate 
limit on actuator effort, the new tradeoff curve will lie above the one we have found. 
Thus, our tradeoff curve identifies design specifications that are not achievable, e.g., 
$U_{\text{max}} = 0.4, T_{\text{max}} = 1.0$, when no limit on actuator effort is made; a fortiori these 
design specifications are not achievable when a limit on actuator effort is included.

We remark that the tradeoff for this example is considerably more general than 
the reader might suspect. (12.39) holds for

- any LTI plant with $R_b(1) = 0$,
- the 2-DOF controller configuration,
- any nonlinear or time-varying controller.

This is because, no matter how the plant input $u$ is generated, the output of $R_b$, $y_p$, must satisfy conditions of the form (12.35–12.36).

### 12.5 A Weighted Peak Tracking Error Example

In this section we present a less trivial example of a plant and family of design 
specifications for which we can explicitly solve the feasibility problem.
12.5 A Weighted Peak Tracking Error Example

We consider the classical 1-DOF system of section 2.3.2 with

\[ P_0(s) = \frac{s - 2}{s^2 - 1}. \]

Designing a controller for this plant is quite demanding, since it has an unstable zero at \( s = 2 \) along with an unstable pole only an octave lower, at \( s = 1 \).

Our design specifications will be internal stability and a limit on a weighted peak gain of the closed-loop tracking error transfer function:

\[ \| WS \|_{pk,\text{gns}} \leq E_{\text{max}}, \tag{12.40} \]

where

\[ W(s) = \frac{1}{1 + sT_{\text{trk}}}, \]

and \(-S\) is the closed-loop transfer function from the reference input \( r \) to the error \( \epsilon = -r + y_p \) (see sections 5.2.5 and 8.1.2). Thus we have a two-parameter family of design specifications, indexed by \( E_{\text{max}} \) and \( T_{\text{trk}} \).

Roughly speaking, \( E_{\text{max}} \) is an approximate limit on the worst case peak mis-tracking that can occur with reference inputs that are bounded by one and have a bandwidth \( 1/T_{\text{trk}} \). Therefore, \( 1/T_{\text{trk}} \) represents a sort of tracking bandwidth for the system. It seems intuitively clear, and turns out to be correct, that small \( E_{\text{max}} \) can only be achieved at the cost of large \( T_{\text{trk}} \).

These design specifications are simple enough that we can explicitly solve the feasibility problem for each \( E_{\text{max}} \) and \( T_{\text{trk}} \). As in the previous section, however, these design specifications are not complete enough to guarantee reasonable controller designs, so the comments made in the previous section hold here as well.

As we did for the previous example, we express internal stability in terms of the interpolation conditions: \( S \) is stable and satisfies

\[ S(1) = 0, \quad S(2) = S(\infty) = 1. \]

Equivalently, \( WS \) is stable, and satisfies

\[ WS(1) = 0, \tag{12.41} \]

\[ WS(2) = (1 + 2T_{\text{trk}})^{-1}, \tag{12.42} \]

\[ \lim_{s \to \infty} sWS(s) = T_{\text{trk}}^{-1}. \tag{12.43} \]

Let \( h \) be the impulse response of \( WS \), so that

\[ \| WS \|_{pk,\text{gns}} = \int_0^\infty |h(t)| \, dt \]
(from (12.43), $h$ does not contain any impulse at $t = 0$). We can express the interpolation conditions in terms of $h$ as

\begin{align}
\int_0^\infty h(t)e^{-t} \, dt &= 0, \\
\int_0^\infty h(t)e^{-2t} \, dt &= (1 + 2T_{trk})^{-1}, \\
h(0) &= T^{-1}_{trk}. 
\end{align}

(12.44) (12.45) (12.46)

We will solve the feasibility problem by solving the optimization problem

\begin{equation}
\min_{\text{subject to (12.44)--(12.46)}} \int_0^\infty |h(t)| \, dt.
\end{equation}

(12.47)

In chapters 13–15 we will describe general numerical methods for solving an infinite-dimensional convex optimization problem such as (12.47); here we will use some specific features to analytically determine the solution. We will first guess a solution, based on some informal reasoning, and then prove, using only simple arguments, that our guess is correct.

We first note that the third constraint, on $h(0)$, should not affect the minimum, since we can always adjust $h$ very near $t = 0$ to satisfy this constraint, without affecting the other two constraints, and only slightly changing the objective. It can be shown that the value of the minimum does not change if we ignore this constraint, so henceforth we will.

Now we consider the two integral constraints. From the second, we see that $h(t)$ will need to be positive over some time interval, and from the first we see that $h(t)$ will also have to be negative over some other time interval. Since the integrand $e^{-2t}$ falls off more rapidly than $e^{-t}$, it seems that the optimal $h(t)$ should first be positive, and later negative, to take advantage of these differing decay rates. Similar reasoning finally leads us to guess that a nearly optimal $h$ should satisfy

\begin{equation}
h(t) = \alpha \delta(t) - \beta \delta(t - T),
\end{equation}

(12.48)

where $\alpha$ and $\beta$ are positive, and $T$ is some appropriate time lapse. The objective is then approximately $\alpha + \beta$.

Given this form, we readily determine that the optimal $\alpha$, $\beta$, and $T$ are given by

\begin{align}
\alpha &= \frac{1}{1 + 2T_{trk}} \left(1 + \frac{1}{2\sqrt{2}}\right), \\
\beta &= \frac{1}{1 + 2T_{trk}} \left(2 + \frac{3}{2\sqrt{2}}\right), \\
T &= \log(1 + \sqrt{2}),
\end{align}

(12.49) (12.50) (12.51)
which corresponds to an objective of
\[ \frac{1}{1 + 2T_{\text{trk}}} \left( 3 + 2\sqrt{2} \right). \] (12.52)

Our guess that the value of (12.47) is given by (12.52) is correct. To verify this, we consider \( \lambda : \mathbb{R} \to \mathbb{R} \) given by
\[ \lambda(t) = -(2 + 2\sqrt{2})e^{-t} + (3 + 2\sqrt{2})e^{-2t}, \] (12.53)
and plotted in figure 12.5. This function has a maximum magnitude of one, i.e., \( \|\lambda\|_{\infty} = 1. \)

![Figure 12.5](image-url) The function \( \lambda(t) \) from (12.53).

Now suppose that \( h \) satisfies the two integral equality constraints in (12.47). Then by linearity we must have
\[ \int_{0}^{\infty} h(t)\lambda(t) \, dt = \frac{1}{1 + 2T_{\text{trk}}} \left( 3 + 2\sqrt{2} \right). \] (12.54)
Since \( |\lambda(t)| \leq 1 \) for all \( t \), we have
\[ \int_{0}^{\infty} h(t)\lambda(t) \, dt \leq \int_{0}^{\infty} |h(t)| \, dt = \|h\|_{1}. \] (12.55)
Combining (12.54) and (12.55), we see that for any \( h \),
\[ \int_{0}^{\infty} h(t)e^{-t} \, dt = 0 \quad \text{and} \quad \int_{0}^{\infty} h(t)e^{-2t} \, dt = (1 + 2T_{\text{trk}})^{-1} \]
\[ \| h \|_1 \geq \frac{1}{1 + 2T_{trk}} \left( 3 + 2\sqrt{2} \right), \]

i.e., any \( h \) that satisfies the constraints in (12.47) has an objective that exceeds the objective of our candidate solution (12.52). This proves that our guess is correct. (The origin of this mysterious \( \lambda \) is explained in the Notes and References.)

From our solution (12.52) of the optimization problem (12.47), we conclude that the specifications corresponding to \( E_{\text{max}} \) and \( T_{trk} \) are achievable if and only if

\[ E_{\text{max}}(1 + 2T_{trk}) > 3 + 2\sqrt{2}. \tag{12.56} \]

(We leave the construction of a controller that meets the specifications for \( E_{\text{max}} \) and \( T_{trk} \) satisfying (12.56) to the reader.) This region of achievable specifications is shown in figure 12.6.

![Figure 12.6 The tradeoff between peak tracking error and tracking bandwidth specifications.](image)

Note that to guarantee that the worst case peak tracking error does not exceed 10%, the weighting filter smoothing time constant must be at least \( T_{trk} \geq 28.64 \), which is much greater than the time constants in the dynamics of \( P_0 \), which are on the order of one second. In classical terminology, the tracking bandwidth is considerably smaller than the open-loop bandwidth. The necessarily poor performance implied by the tradeoff curve (12.56) is a quantitative expression that this plant is “hard to control”.
Notes and References

LQR and LQG-Optimal Controllers

Standard references on LQR and LQG-optimal controllers are the books by Anderson and Moore [AM90], Kwakernaak and Sivan [KS72], Bryson and Ho [BH75], and the special issue edited by Athans [AT71]. Åström and Wittenmark treat minimum variance regulators in [AW90]. The same techniques are readily extended to solve problems that involve an exponentially weighted $H_2$ norm; see, e.g., Anderson and Moore [AM69].

Multicriterion LQG

The articles by Toivonen [Toi84] and Toivonen and Mäkelä [TM89] discuss the multicriterion LQG problem; the latter article has extensive references to other articles on this topic. See also Koussoulas and Leondes [KL86].

Controllers that Satisfy an $H_\infty$ Norm-Bound

In [ZAM81], Zames proposed that the $H_\infty$ norm of some appropriate closed-loop transfer matrix be minimized, although control design specifications that limit the magnitude of closed-loop transfer functions appeared much earlier. The state-space solution of section 12.3 is recent, and is due to Doyle, Glover, Khargonekar, and Francis [DGK89, GD88]. Previous solutions to the feasibility problem with an $H_\infty$ norm-bound on $H$ were considerably more complex.

We noted above that the controller $K_{\text{me}}$ of section 12.3 not only satisfies the specification (12.26); it minimizes the $\gamma$-entropy of $H$. This is discussed in Mustafa and Glover [MG90, GM89]. The minimum entropy controller was developed independently by Whittle [WH90], who calls it the linear exponential quadratic Gaussian (LEQG) optimal controller.

Some Other Analytic Solutions

In [OF85, OF86], O'Young and Francis use Nevanlinna-Pick theory to deduce exact trade-off curves that limit the maximum magnitude of the sensitivity transfer function in two different frequency bands.

Some analytic solutions to discrete-time problems involving the peak gain have been found by Dahleh and Pearson; see [Vid86, DP87b, DP88a, DP87a, DP88a].

About Figure 12.4

The step response shown in figure 12.4 was found as follows. We let

$$T(s) = \sum_{i=1}^{20} x_i \left( \frac{10}{s + 10} \right)^i,$$

where $x \in \mathbb{R}^{20}$ is to be determined. (See chapter 15 for an explanation of this Ritz approximation.) $T(s)$ must satisfy the condition (12.34). The constraint $T(\infty) = 0$ is automatically satisfied; the interpolation condition $T(1) = 0$ yields the equality constraint on $x$;

$$c^T x = 0,$$
where \( c_i = (10/11)i \). The undershoot and rise-time specifications are

\[
\sum_{i=1}^{20} x_i s_i(t) \geq -0.7 \quad \text{for } 0 \leq t \leq 1.0, \tag{12.59}
\]

\[
\sum_{i=1}^{20} x_i s_i(t) \geq 0.8 \quad \text{for } t \geq 1.0, \tag{12.60}
\]

where \( s_i \) is the step response of \((s/10 + 1)^{-i}\). By finely discretizing \( t \), (12.59) and (12.60) yield (many) linear inequality constraints on \( x \), i.e.

\[
a_k^T x \leq b_k, \quad k = 1, \ldots, L. \tag{12.61}
\]

(12.58) and (12.61) can be solved as a feasibility linear program. The particular coefficients that we used, shown in table 12.1, were found by minimizing \(||x||\infty\) subject to (12.58) and (12.61).

### Table 12.1

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<td>9.641</td>
<td>15</td>
<td>9.641</td>
<td>20</td>
<td>-0.343</td>
</tr>
</tbody>
</table>

About the Examples in Sections 12.4 and 12.5

These two examples can be expressed as infinite-dimensional linear programming problems. The references for the next two chapters are relevant; see also Luenberger [Lue69], Rockafellar [Roc74, Roc82], Reiland [Rei80], Anderson and Philpott [AP84], and Anderson and Nash [AN87].

We solved the problem (12.47) (ignoring the third equality constraint) by first solving its dual problem, which is

\[
\max_{\lambda_1, \lambda_2} \lambda_1 (1 + 2T_{kk})^{-1}.
\]

This is a convex optimization problem in \( \mathbb{R}^2 \), which is readily solved. The mysterious \( \lambda(t) \) that we used corresponds exactly to the optimum \( \lambda_1 \) and \( \lambda_2 \) for this dual problem.

This dual problem is sometimes called a *semi-infinite optimization problem* since the constraint involves a "continuum" of inequalities (i.e., \( |\lambda_1 e^{-t} + \lambda_2 e^{-3t}| \leq 1 \) for each \( t \geq 0 \)). Special algorithms have been developed for these problems; see for example the surveys by Polak [Pol83], Polak, Mayne, and Stimler [PMS84], and Hettich [Het78].
Chapter 13

Elements of Convex Analysis

We describe some of the basic tools of convex nondifferentiable analysis: subgradients, directional derivatives, and supporting hyperplanes, emphasizing their geometric interpretations. We show how to compute supporting hyperplanes and subgradients for the various specifications and functionals described in previous chapters.

Many of the specifications and functionals that we have encountered in chapters 8–10 are not smooth—the specifications can have “sharp corners” and the functionals need not be differentiable. Fortunately, for convex sets and functionals, some of the most important analytical tools do not depend on smoothness. In this chapter we study these tools. Perhaps more importantly, there are simple and effective algorithms for convex optimization that do not require smooth constraints or differentiable objectives. We will study some of these algorithms in the next chapter.

13.1 Subgradients

If \( \phi : \mathbb{R}^n \to \mathbb{R} \) is convex and differentiable, we have

\[
\phi(z) \geq \phi(x) + \nabla \phi(x)^T (z - x) \quad \text{for all } z.
\]

This means that the plane tangent to the graph of \( \phi \) at \( x \) always lies below the graph of \( \phi \). If \( \phi : \mathbb{R}^n \to \mathbb{R} \) is convex, but not necessarily differentiable, we will say that \( g \in \mathbb{R}^n \) is a subgradient of \( \phi \) at \( x \) if

\[
\phi(z) \geq \phi(x) + g^T (z - x) \quad \text{for all } z.
\]

From (13.1), the gradient of a differentiable convex function is always a subgradient. A basic result of convex analysis is that every convex function always has at least one subgradient at every point. We will denote the set of all subgradients of \( \phi \) at \( x \) as \( \partial \phi(x) \), the subdifferential of \( \phi \) at \( x \).
We can think of the right-hand side of (13.2) as an affine approximation to \( \phi(z) \), which is exact at \( z = x \). The inequality (13.2) states that the right-hand side is a \textit{global lower bound} on \( \phi \). This is shown in figure 13.1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure131.png}
\caption{A convex function on \( \mathbb{R} \) along with three affine global lower bounds on \( \phi \) derived from subgradients. At \( x_1 \), \( \phi \) is differentiable, and the slope of the tangent line is \( g_1 = \phi'(x_1) \). At \( x_2 \), \( \phi \) is not differentiable; two different tangent lines, corresponding to subgradients \( g_2 \) and \( \tilde{g}_2 \), are shown.}
\end{figure}

We mention two important consequences of \( g \in \partial \phi(x) \). For \( g^T(z-x) > 0 \) we have \( \phi(z) > \phi(x) \), in other words, in the half-space \( \{z \mid g^T(z-x) > 0\} \), the values of \( \phi \) exceed the value of \( \phi \) at \( x \). Thus if we are searching for an \( x^* \) that minimizes \( \phi \), and we know a subgradient \( g \) of \( \phi \) at \( x \), then we can rule out the entire half-space \( g^T(z-x) > 0 \). The hyperplane \( g^T(z-x) = 0 \) is called a \textit{cut} because it cuts off from consideration the half-space \( g^T(z-x) > 0 \) in a search for a minimizer. This is shown in figure 13.2.

An extension of this idea will also be useful. From (13.2), every \( z \) that satisfies \( \phi(z) \leq \alpha \), where \( \alpha < \phi(x) \), must also satisfy \( g^T(z-x) \leq \alpha - \phi(x) \). If we are searching for a \( z \) that satisfies \( \phi(z) \leq \alpha \), we need not consider the half-space \( g^T(z-x) > \alpha - \phi(x) \). The hyperplane \( g^T(z-x) = \alpha - \phi(x) \) is called a \textit{deep-cut} because it rules out a larger set than the simple cut \( g^T(z-x) = 0 \). This is shown in figure 13.3.

\subsection{13.1.1 Subgradients: Infinite-Dimensional Case}

The notion of a subgradient can be generalized to apply to functionals on infinite-dimensional spaces. The books cited in the Notes and References at the end of this chapter contain a detailed and precise treatment of this topic; in this book we will give a simple (but correct) description.
13.1 Subgradients

Figure 13.2 A point $x$ and a subgradient $g$ of $\phi$ at $x$. In the half-space $g^T z > g^T x$, $\phi(z)$ exceeds $\phi(x)$; in particular, any minimizer $x^*$ of $\phi$ must lie in the half-space $g^T z \leq g^T x$.

Figure 13.3 A point $x$ and a subgradient $g$ of $\phi$ at $x$ determine a deep-cut in the search for points that satisfy $\phi(z) \leq \alpha$ (assuming $x$ does not satisfy this inequality). The points in the shaded region need not be considered since they all have $\phi(z) > \alpha$. 
If \( \phi \) is a convex functional on a (possibly infinite-dimensional) vector space \( V \), then we say \( \phi^{\text{reg}} \) is a subgradient for \( \phi \) at \( v \in V \) if \( \phi^{\text{reg}} \) is a linear functional on \( V \), and we have

\[
\phi(z) \geq \phi(v) + \phi^{\text{reg}}(z - v) \quad \text{for all } z \in V.
\] (13.3)

The subdifferential \( \partial \phi(v) \) consists of all subgradients of \( \phi \) at \( v \); note that it is a set of linear functionals on \( V \).

If \( V = \mathbb{R}^n \), then every linear functional on \( V \) has the form \( g^T z \) for some vector \( g \in \mathbb{R}^n \), and our two definitions of subgradient are therefore the same, provided we ignore the distinction between the vector \( g \in \mathbb{R}^n \) and the linear functional on \( \mathbb{R}^n \) given by the inner product with \( g \).

### 13.1.2 Quasigradients

For quasiconvex functions, there is a concept analogous to the subgradient. Suppose \( \phi : \mathbb{R}^n \to \mathbb{R} \) is quasiconvex, which we recall from section 6.2.2 means that

\[
\phi(\lambda x + (1 - \lambda)\bar{x}) \leq \max\{\phi(x), \phi(\bar{x})\} \quad \text{for all } 0 \leq \lambda \leq 1, \ x, \bar{x} \in \mathbb{R}^n.
\]

We say that \( g \) is a quasigradient for \( \phi \) at \( x \) if

\[
\phi(z) \geq \phi(x) \quad \text{whenever } g^T(z - x) \geq 0.
\] (13.4)

This simply means that the hyperplane \( g^T(z - x) = 0 \) forms a simple cut for \( \phi \), exactly as in figure 13.2: if we are searching for a minimizer of \( \phi \), we can rule out the half-space \( g^T(z - x) > 0 \).

If \( \phi \) is differentiable and \( \nabla \phi(x) \neq 0 \), then \( \nabla \phi(x) \) is a quasigradient; if \( \phi \) is convex, then (13.2) shows that any subgradient is also a quasigradient. It can be shown that every quasiconvex function has at least one quasigradient at every point. Note that the length of a quasigradient is irrelevant (for our purposes): all that matters is its direction, or equivalently, the cutting-plane for \( \phi \) that it determines.

Any algorithm for convex optimization that uses only the cutting-planes that are determined by subgradients will also work for quasiconvex functions, if we substitute quasigradients for subgradients. It is not possible to form any deep-cut for a quasiconvex function.

In the infinite-dimensional case, we will say that a linear functional \( \phi^{\text{reg}} \) on \( V \) is a quasigradient for the quasiconvex functional \( \phi \) at \( v \in V \) if

\[
\phi(z) \geq \phi(v) \quad \text{whenever } \phi^{\text{reg}}(z - v) \geq 0.
\]

As discussed above, this agrees with our definition above for \( V = \mathbb{R}^n \), provided we do not distinguish between vectors and the associated inner product linear functionals.
13.1 Subgradients

13.1.3 Subgradients and Directional Derivatives

In this section we briefly discuss the directional derivative, a concept of differential calculus that is more familiar than the subgradient. We will not use this concept in the optimization algorithms we present in the next chapter; we mention it because it is used in descent methods, the most common algorithms for optimization.

We define the directional derivative of $\phi$ at $x$ in the direction $\delta x$ as

$$
\phi'(x; \delta x) \triangleq \lim_{h \to 0} \frac{\phi(x + h\delta x) - \phi(x)}{h}
$$

(the notation $h \searrow 0$ means that $h$ converges to 0 from above). It can be shown that for convex $\phi$ this limit always exists. Of course, if $\phi$ is differentiable at $x$, then

$$
\phi'(x; \delta x) = \nabla \phi(x)^T \delta x.
$$

We say that $\delta x$ is a descent direction for $\phi$ at $x$ if $\phi'(x; \delta x) < 0$.

The directional derivative tells us how $\phi$ changes if $x$ is moved slightly in the direction $\delta x$, since for small $h$,

$$
\phi \left( x + h \frac{\delta x}{\|\delta x\|} \right) \approx \phi(x) + h \frac{\phi'(x; \delta x)}{\|\delta x\|}.
$$

The steepest descent direction of $\phi$ at $x$ is defined as

$$
\delta x_{sd} = \arg\min_{\|\delta x\|=1} \phi'(x; \delta x).
$$

In general the directional derivatives, descent directions, and the steepest descent direction of $\phi$ at $x$ can be described in terms of the subdifferential at $x$ (see the Notes and References at the end of the chapter). In many cases it is considerably more difficult to find a descent direction or the steepest descent direction of $\phi$ at $x$ than a single subgradient of $\phi$ at $x$.

If $\phi$ is differentiable at $x$, and $\nabla \phi(x) \neq 0$, then $-\nabla \phi(x)$ is a descent direction for $\phi$ at $x$. It is not true, however, that the negative of any nonzero subgradient provides a descent direction: we can have $g \in \partial \phi(x)$, $g \neq 0$, but $-g$ not a descent direction for $\phi$ at $x$. As an example, the level curves of a convex function $\phi$ are shown in figure 13.4(a), together with a point $x$ and a nonzero subgradient $g$. Note that $\phi$ increases for any movement along the directions $\pm g$, so, in particular, $-g$ is not a descent direction. Negatives of the subgradients at non-optimal points are, however, descent directions for the distance to a (or any) minimizer, i.e., if $\phi(x^*) = \phi^*$ and $\psi(z) = \|z - x^*\|$, then $\psi'(x; -g) < 0$ for any $g \in \partial \phi(x)$. Thus, moving slightly in the direction $-g$ will decrease the distance to (any) minimizer $x^*$, as shown in figure 13.4(b).

A consequence of these properties is that the optimization algorithms described in the next chapter do not necessarily generate sequences of decreasing functional values (as would a descent method).
Figure 13.4 A point $x$ and a subgradient $g$ of $\phi$ at $x$ is shown in (a), together with three level curves of $\phi$. Note that $-g$ is not a descent direction for $\phi$ at $x$: $\phi$ increases for any movement from $x$ in the directions $\pm g$. However, $-g$ is a descent direction for the distance to any minimizer. In (b) the level curves for the distance $\psi(z) = \|z - x^*\|$ are shown; $-g$ points into the circle through $x$.

13.2 Supporting Hyperplanes

If $C$ is a convex subset of $\mathbb{R}^n$ and $x$ is a point on its boundary, then we say that the hyperplane through $x$ with normal $g$, $\{z \mid g^T(z-x) = 0\}$, is a supporting hyperplane to $C$ at $x$ if $C$ is contained in the half-space $g^T(z-x) \leq 0$. Roughly speaking, if the set $C$ is “smooth” at $x$, then the plane that is tangent to $C$ at $x$ is a supporting hyperplane, and $g$ is its outward normal at $x$, as shown in Figure 13.5. But the notion of supporting hyperplane makes sense even when the set $C$ is not “smooth” at $x$. A basic result of convex analysis is that there is at least one supporting hyperplane at every boundary point of a convex set.

If $C$ has the form of a functional inequality,

$$C = \{z \mid \phi(z) \leq \alpha\},$$

where $\phi$ is convex (or quasiconvex), then a supporting hyperplane to $C$ at a boundary point $x$ is simply $g^T(z-x) = 0$, where $g$ is any subgradient (or quasigradient).

If $C$ is a convex subset of the infinite-dimensional space $V$ and $x$ is a point of its boundary, then we say that the hyperplane

$$\{z \mid \phi^{sh}(z-x) = 0\},$$

where $\phi^{sh}$ is nonzero linear functional on $V$, is a supporting hyperplane for $C$ at point $x$ if

$$\phi^{sh}(z-x) \leq 0 \quad \text{for all } z \in C.$$
Again we note that this general definition agrees with the one above for $V = \mathbb{R}^n$ if we do not distinguish between vectors in $\mathbb{R}^n$ and their associated inner product functionals.

### 13.3 Tools for Computing Subgradients

To use the algorithms that we will describe in the next two chapters we must be able to evaluate convex functionals and find at least one subgradient at any point. In this section, we list some useful tools for subgradient evaluation. Roughly speaking, if one can evaluate a convex functional at a point, then it is usually not much more trouble to determine a subgradient at that point.

These tools come from more general results that describe all subgradients of, for example, the sum or maximum of convex functionals. These results can be found in any of the references mentioned in the Notes and References at the end of this chapter. The more general results, however, are much more than we need, since our purpose is to show how to calculate one subgradient of a convex functional at a point, not all subgradients at a point, a task which in many cases is very difficult, and in any case not necessary for the algorithms we describe in the next two chapters. The more general results have many more technical conditions.

- **Differentiable functional**: If $\phi$ is convex and differentiable at $x$, then its deriv-
tive at $x$ is an element of $\partial \phi(x)$. (In fact, it is the only element of $\partial \phi(x)$.)

- **Scaling**: If $w \geq 0$ and $\phi$ is convex, then a subgradient of $w\phi$ at $x$ is given by $wg$, where $g$ is any subgradient of $\phi$ at $x$.

- **Sum**: If $\phi(x) = \phi_1(x) + \cdots + \phi_m(x)$, where $\phi_1, \ldots, \phi_m$ are convex, then any $g$ of the form $g = g_1 + \cdots + g_m$ is in $\partial \phi(x)$, where $g_i \in \partial \phi_i(x)$.

- **Maximum**: Suppose that
  \[
  \phi(x) = \sup \{ \phi_\alpha(x) \mid \alpha \in \mathcal{A} \},
  \]
where each $\phi_\alpha$ is convex, and $\mathcal{A}$ is any index set. Suppose that $\alpha_{\text{ach}} \in \mathcal{A}$ is such that $\phi_{\alpha_{\text{ach}}}(x) = \phi(x)$ (so that $\phi_{\alpha_{\text{ach}}}(x)$ achieves the maximum). Then if $g \in \partial \phi_{\alpha_{\text{ach}}}(x)$, we have $g \in \partial \phi(x)$. Of course there may be several different indices that achieve the maximum; we need only pick one.

A special case is when $\phi$ is the maximum of the functionals $\phi_1, \ldots, \phi_n$, so that $\mathcal{A} = \{1, \ldots, n\}$. If $\phi(x) = \phi_i(x)$, then any subgradient $g$ of $\phi_i(x)$ is also a subgradient of $\phi(x)$.

From these tools we can derive additional tools for determining a subgradient of a weighted sum or weighted maximum of convex functionals. Their use will become clear in the next section.

For quasiconvex functionals, we have the analogous tools:

- **Differentiable functional**: If $\phi$ is quasiconvex and differentiable at $x$, with nonzero derivative, then its derivative at $x$ is a quasigradient of $\phi$ at $x$.

- **Scaling**: If $w \geq 0$ and $\phi$ is quasiconvex, then any quasigradient of $\phi$ at $x$ is also a quasigradient of $w\phi$ at $x$.

- **Maximum**: Suppose that
  \[
  \phi(x) = \sup \{ \phi_\alpha(x) \mid \alpha \in \mathcal{A} \},
  \]
where each $\phi_\alpha$ is quasiconvex, and $\mathcal{A}$ is any index set. Suppose that $\alpha_{\text{ach}} \in \mathcal{A}$ is such that $\phi_{\alpha_{\text{ach}}}(x) = \phi(x)$. Then if $g$ is a quasigradient of $\phi_{\alpha_{\text{ach}}}$ at $x$, then $g$ is a quasigradient of $\phi$ at $x$.

- **Nested family**: Suppose that $\phi$ is defined in terms of a nested family of convex sets, i.e., $\phi(x) = \inf \{ \alpha \mid x \in C^\alpha \}$, where $C^\alpha \subseteq C^\beta$ whenever $\alpha \leq \beta$ (see section 6.2.2). If $g^\top(z-x) = 0$ defines a supporting hyperplane to $C^{\phi(z)}$ at $x$, then $g$ is a quasigradient of $\phi$ at $x$.

(The sum tool is not applicable because the sum of quasiconvex functionals need not be quasiconvex.)