A Simple Method for Predicting Covariance Matrices of Financial Returns

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Outline

Covariance prediction in finance

Evaluating covariance predictors

Iterated methods

Our method

Empirical study

Extensions and variations
Contributions

- a simple and effective method for predicting covariance matrices of financial returns
- a new method for evaluating a covariance predictor over changing market conditions
- extensive empirical study on several large data sets
- open-source implementation in Python: https://github.com/cvxgrp/cov_pred_finance
Covariance prediction in finance
Financial returns

- $r_t \in \mathbb{R}^n$ is the vector of $n$ financial asset returns over period $t$
- $t = 1, \ldots, T$ are the time periods
- could be days, weeks, months, etc.
- $(r_t)_i$ is the return of asset $i$ over period $t$
- assets could be bonds, stocks, factors, etc.
Gaussian model

**model:** \( r_t \sim \mathcal{N}(0, \Sigma_t) \)

- can demean return data if needed
- for most daily, weekly, or monthly return data

\[
\Sigma_t = \mathbb{E}r_tr_t^T - (\mathbb{E}r_t)(\mathbb{E}r_t)^T \approx \mathbb{E}r_tr_t^T
\]

**objective:** find estimate \( \hat{\Sigma}_t \) of \( \Sigma_t \), based on \( r_1, \ldots, r_{t-1} \)
Rolling window (RW) covariance predictor

\[
\hat{\Sigma}_t = \alpha_t \sum_{\tau=t-M}^{t-1} r_{\tau} r_{\tau}^T, \quad t = 2, 3, \ldots,
\]

- \( \alpha_t = \frac{1}{\min\{t - 1, M\}} \) is the normalizing constant
- \( M \) is the RW memory
Exponentially weighted moving average (EWMA) predictor

\[ \hat{\Sigma}_t = \alpha_t \sum_{\tau=1}^{t-1} \beta^{t-1-\tau} r_\tau r_\tau^T, \quad t = 2, 3, \ldots \]

- \( \alpha_t = \left( \sum_{\tau=1}^{t-1} \beta^{t-1-\tau} \right)^{-1} = \frac{1-\beta}{1-\beta^{t-1}} \) is the normalizing constant
- \( \beta \in (0, 1) \) is the forgetting factor, often expressed in terms of the half-life \( H = -\log 2 / \log \beta \)
Some more complex predictors

- generalized autoregressive conditional heteroskedasticity (GARCH)
  - introduced in the 1980s [Bollerslev, 1986]
  - models univariate volatility
  - Nobel memorial prize awarded for related work [Engle, 1982]

- MGARCH: multivariate extension of GARCH

- currently considered state-of-the-art for volatility and covariance prediction

- MGARCH requires solving non-convex optimization problems, and involves many parameters difficult to estimate reliably
Evaluating covariance predictors
Mean-squared error

- Mean squared error (MSE) of predictions $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_T$

$$
\frac{1}{T} \sum_{t=1}^{T} \| r_t r_t^T - \hat{\Sigma}_t \|_F^2,
$$

(smaller values are better)

- Commonly used in the literature [Patton, 2011]

- MSE best constant predictor is $\Sigma^{\text{emp}} = \frac{1}{T} \sum_{t=1}^{T} r_t r_t^T$
Log-likelihood

- predictions $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_T$ evaluated on average log-likelihood

$$\frac{1}{2T} \sum_{t=1}^{T} \left( -n \log(2\pi) - \log \det \hat{\Sigma}_t - r_t^T \hat{\Sigma}_t^{-1} r_t \right)$$

(larger values are better)

- closely related to (Gaussian) quasi-likelihood (QLIKE) [Patton, 2011; Patton and Sheppard, 2009; Laurent et al., 2013]

- log-likelihood best constant predictor is $\Sigma^{\text{emp}} = \frac{1}{T} \sum_{t=1}^{T} r_t r_t^T$
Log-likelihood regret

- **log-likelihood regret** is the difference between the log-likelihood of the best constant predictor and that of the predictors $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_T$ (smaller values are better).
- Useful when we compute the regret over multiple periods, like months or quarters.
- The regret over multiple periods removes the effect of the log-likelihood of the empirical covariance varying due to changing market conditions.
Portfolio performance

• can evaluate covariance predictor by investment performance
• for example the minimum variance portfolio

\[
\begin{align*}
\text{minimize} & \quad w^T \hat{\Sigma}_t w \\
\text{subject to} & \quad 1^T w = 1, \quad \|w\|_1 \leq L_{\max} \\
& \quad w_{\min} \leq w \leq w_{\max}
\end{align*}
\]

with variable \( w \) (portfolio weight vector)
• other portfolios: risk-parity, max diversification
• performance metrics: realized return, volatility, Sharpe ratio, max drawdown . . .
Volatility control with cash

to more easily compare portfolio performance across different covariance predictors, we mix each portfolio with cash to attain ex-ante volatility target $\sigma^{\text{tar}}$

1. start with portfolio weight $w_t$
2. compute ex-ante volatility $\sigma_t = \sqrt{w_t^T \hat{\Sigma}_t w_t}$
3. add a cash component to attain the new $n+1$ weight vector

$$\begin{bmatrix} \theta w_t \\ (1 - \theta) \end{bmatrix}, \quad \theta = \frac{\sigma^{\text{tar}}}{\sigma_t}$$
Iterated methods
Iterated covariance predictors

1. form initial estimate $\hat{\Sigma}_t^{(1)}$ of $\Sigma_t$
2. form “whitened” returns
   \[ \tilde{r}_t = \left( \hat{\Sigma}_t^{(1)} \right)^{-1/2} r_t, \quad t = 1, \ldots, T \]
3. form estimate $\hat{\Sigma}_t^{(2)}$ of covariance of $\tilde{r}_t$
4. final estimate
   \[ \hat{\Sigma}_t = \left( \hat{\Sigma}_t^{(1)} \right)^{1/2} \hat{\Sigma}_t^{(2)} \left( \hat{\Sigma}_t^{(1)} \right)^{1/2} \]

- variation: let $\hat{\Sigma}_t^{(2)}$ be correlation matrix of $\tilde{r}_t$ [Engle, 2002]
- can iterate [Barratt and Boyd, 2022]
Iterated EWMA (IEWMA) predictor

1. $\Sigma_t^{(1)}$ is diagonal matrix of variances of $r_t$
2. form $\left(\hat{\Sigma}_t^{(1)}\right)_{ii}$ as EWMA of $(r_t)^2_i$ using half-life $H^{vol}$
3. volatility adjusted returns

$$\tilde{r}_t = \left(\hat{\Sigma}_t^{(1)}\right)^{-1/2} r_t, \quad t = 1, \ldots, T$$

4. form $\hat{\Sigma}_t^{(2)}$ as EWMA covariance of $\tilde{r}_t$ using half-life $H^{cor}$

- two parameters: $H^{vol}$ and $H^{cor}$
- proposed in [Engle, 2002]
Our method
Dynamically weighted prediction combiner

1. start with $K$ covariance predictors $\hat{\Sigma}_t^{(k)}$, $k = 1, \ldots, K$
2. Cholesky factorizations of associated precision matrices
   \[
   \left(\hat{\Sigma}_t^{(k)}\right)^{-1} = \hat{L}_t^{(k)}(\hat{L}_t^{(k)})^T, \quad k = 1, \ldots, K
   \]
3. create convex combination
   \[
   \hat{L}_t = \sum_{k=1}^{K} \pi_k \hat{L}_t^{(k)},
   \]
   where $\pi_k \geq 0$ and $\sum_{k=1}^{K} \pi_k = 1$
4. recover covariance predictor as $\hat{\Sigma}_t = \left(\hat{L}_t \hat{L}_t^T\right)^{-1}$
Choosing the weights via convex optimization

• choose weights $\pi$ at time $t$ to maximize log-likelihood over past $N$ time-steps

$$\text{maximize} \quad \sum_{j=1}^{N} \left( \sum_{i=1}^{n} \log \hat{L}_{t-j,ii} - \frac{1}{2} \| \hat{L}_{t-j}^T r_{t-j} \|_2^2 \right)$$

subject to

$$\hat{L}_{\tau} = \sum_{j=1}^{K} \pi_j \hat{L}_{\tau}^{(j)}, \quad \tau = t - 1, \ldots, t - N$$
$$\pi \geq 0, \quad 1^T \pi = 1,$$

• convex problem that can be solved quickly and reliably by many methods
Combined multiple iterated EWMA (CM-IEWMA)

1. choose $K$ half-life pairs $H^\text{vol}_k$ and $H^\text{cor}_k$, $k = 1, \ldots, K$
2. form the $K$ IEWMA predictors $\hat{\Sigma}^{(k)}_t$ for these half-life pairs
3. combine the IEWMAs using the dynamically weighted prediction combiner to get the prediction $\hat{\Sigma}_t = \left(\hat{L}_t\hat{L}_t^T\right)^{-1}$

- parameters: half-life pairs and lookback $N$
Empirical study
Data set and experimental setup

• data: \( n = 49 \) daily industry portfolio returns 1970–2023, \( T = 13,496 \) trading days

• compare six covariance predictors
  – RW with a 500-day window
  – EWMA with 250-day half-life
  – IEWMA with half-lives \( H^{\text{vol}}/H^{\text{cor}} \) of 125/250 (in days)
  – MGARCH with parameters re-estimated annually
  – CM-IEWMA with \( K = 5 \) predictors with half-lives (in days):
    \[
    H^{\text{vol}} \quad 21 \quad 63 \quad 125 \quad 250 \quad 500 \\
    H^{\text{cor}} \quad 63 \quad 125 \quad 250 \quad 500 \quad 1000
    \]

• results on other data sets like stocks and factors are qualitatively similar
### Mean-squared error

<table>
<thead>
<tr>
<th>Predictor</th>
<th>Average/$10^{-4}$</th>
<th>Std. Dev./$10^{-3}$</th>
<th>Max/$10^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RW</td>
<td>7.6</td>
<td>4.0</td>
<td>3.9</td>
</tr>
<tr>
<td>EWMA</td>
<td>7.5</td>
<td>4.0</td>
<td>3.9</td>
</tr>
<tr>
<td>IEWMA</td>
<td>7.4</td>
<td>3.9</td>
<td>3.9</td>
</tr>
<tr>
<td>MGARCH</td>
<td>6.8</td>
<td>3.6</td>
<td>3.8</td>
</tr>
<tr>
<td>CM-IEWMA</td>
<td>6.9</td>
<td>3.6</td>
<td>3.8</td>
</tr>
</tbody>
</table>

- metrics on quarterly MSE, over 212 quarters
- CM-IEWMA and MGARCH perform best
Log-likelihood regret

<table>
<thead>
<tr>
<th>Predictor</th>
<th>Average</th>
<th>Std. dev.</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>RW</td>
<td>20.4</td>
<td>6.9</td>
<td>72.8</td>
</tr>
<tr>
<td>EWMA</td>
<td>19.4</td>
<td>6.2</td>
<td>70.1</td>
</tr>
<tr>
<td>IEWMA</td>
<td>18.2</td>
<td>3.6</td>
<td>41.4</td>
</tr>
<tr>
<td>MGARCH</td>
<td>17.9</td>
<td>3.0</td>
<td>32.8</td>
</tr>
<tr>
<td>CM-IEWMA</td>
<td>16.9</td>
<td>2.4</td>
<td>28.4</td>
</tr>
</tbody>
</table>

- metrics on quarterly regret
- CM-IEWMA performs best
• empirical CDF of quarterly regret (higher is better)
Minimum variance portfolio performance metrics

<table>
<thead>
<tr>
<th>Predictor</th>
<th>Return</th>
<th>Risk</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td>RW</td>
<td>3.1%</td>
<td>5.8%</td>
<td>0.5</td>
</tr>
<tr>
<td>EWMA</td>
<td>3.1%</td>
<td>5.4%</td>
<td>0.6</td>
</tr>
<tr>
<td>IEWMA</td>
<td>3.3%</td>
<td>5.5%</td>
<td>0.6</td>
</tr>
<tr>
<td>MGARCH</td>
<td>4.3%</td>
<td>6.1%</td>
<td>0.7</td>
</tr>
<tr>
<td>CM-IEWMA</td>
<td>3.5%</td>
<td>5.3%</td>
<td>0.7</td>
</tr>
</tbody>
</table>

- minimum variance portfolios cash-adjusted to 5% risk target
- similar performance across predictors
- CM-IEWMA estimates risk better than the other predictors
CM-IEWMA component weights $\pi$

- average weight $\pi_i$, $i = 1, \ldots, 5$ on the five predictors each year
- substantial weight is put on the slower (longer half-life) IEWMAs most years
- during and following volatile periods we see a significant increase in weight on the faster IEWMAs
Extensions and variations
Some practical extensions and variations

• realized covariance
  – uses intraperiod returns
• large universes
  – when $n$ is larger than 100 or so
• smoothing
  – penalize variation in covariance estimate
Realized covariance

- $r_t \in \mathbb{R}^{n \times m}$ return matrix at time $t$, with columns that are $m$ intraperiod return vectors
- $C_t = r_tr_t^T$ realized covariance at time $t$
- realized EWMA (REWMA):

$$\hat{\Sigma}_t = \alpha_t \sum_{\tau=1}^{t-1} \beta^{t-1-\tau} C_\tau, \quad t = 2, 3, \ldots,$$

- CM-REWMA combines REWMAs with different half-lives
Realized covariance empirical results

- $n = 39$ stocks and $m = 77$ intraperiod returns, January 2 2004 to December 30 2016
- CM-IEWMA gives improvement here too
Large universes

- in practice, the number of assets $n$ can be very large
- we describe two closely related methods for large universes
  - traditional factor model
  - fitting a factor model to a (given) covariance matrix
- computational cost of portfolio optimization reduced from $O(n^3)$ to $O(nk^2)$ when using a $k$-factor model [Boyd and Vandenberghe, 2004]
Traditional factor model

- model: \( r_t = F_t f_t + z_t, \quad t = 1, 2, \ldots, \)
  - \( F_t \in \mathbb{R}^{n \times k} \) factor loadings
  - \( f_t \in \mathbb{R}^k \) factor returns
  - \( z_t \in \mathbb{R}^n \) idiosyncratic return
- we end up with covariance of low-rank plus diagonal form
  \[
  \Sigma_t = F_t \Sigma^f_t F_t^T + E_t
  \]
  - \( \Sigma^f_t \) factor return covariance
  - \( E_t \) diagonal matrix of idiosyncratic variances
- never have to store \( n \times n \) covariance
Fitting a factor model to a covariance matrix

• given covariance $\Sigma$

• find one in factor form, $\hat{\Sigma} = FF^T + E$, such that the Kullback-Leibler divergence between $\mathcal{N}(0, \Sigma)$ and $\mathcal{N}(0, \hat{\Sigma})$,

$$K(\Sigma, \hat{\Sigma}) = \frac{1}{2} \left( \log \frac{\det \hat{\Sigma}}{\det \Sigma} - n + \text{Tr} \hat{\Sigma}^{-1} \Sigma \right)$$

is minimized

• equivalent to maximizing the expected log-likelihood of $r \sim \mathcal{N}(0, \Sigma)$ under the model $\mathcal{N}(0, \hat{\Sigma})$

• can be solved via the expectation maximization algorithm (suggested and derived by Emmanuel Candès)
Large universes: empirical setup

- 238 US stocks over 5787 trading days
- traditional factor model
  - create factor model using PCA on two years of data, refitted annually
  - we use $k$ factors and use the CM-IEWMA with half-lives (in days) $H^{vol}/H^{cor}$ of $\lfloor k/2 \rfloor/k$, $k/3k$, and $3k/6k$, to compute the factor covariance
- fitting factor model to covariance
  - use CM-IEWMA directly with half-lives (in days) $H^{vol}/H^{cor}$ of $63/125$, $125/250$, $250/500$, and $500/1000$
  - approximate CM-IEWMA predictor using factor model
Large universes: empirical results

traditional factor model

fitting factor model to covariance
Smooth covariance predictions

- given predictions $\hat{\Sigma}_t$, $t = 1, 2, \ldots$,
- let $\hat{\Sigma}^{sm}_t$ be the EWMA of $\hat{\Sigma}_t$
  - equivalent to minimizing
    $$\left\| \hat{\Sigma}^{sm}_t - \hat{\Sigma}_t \right\|_F^2 + \lambda \left\| \hat{\Sigma}^{sm}_t - \hat{\Sigma}^{sm}_{t-1} \right\|_F^2,$$
    where $\lambda$ is a smoothing parameter
  - yields smooth covariance predictions
- with regularizer $\lambda \left\| \hat{\Sigma}^{sm}_t - \hat{\Sigma}^{sm}_{t-1} \right\|_F$, we obtain piecewise constant predictions
- smoothing can lead to reduced trading and improved portfolio performance
Smooth covariance predictions empirical results

- minimum variance portfolios on five Fama-French factor returns
- portfolio weights for smooth and piecewise constant covariances
Conclusions

- introduced a covariance predictor for financial returns
- relies on solving a small convex optimization problem
- requires little or no tuning or fitting
- interpretable, lightweight, and practically effective
- outperforms popular EWMA and is comparable to MGARCH
from cvx.covariance.combination import from_ewmas
halflife_pairs = [(10, 21), (21, 63), (63, 125)]
combinator = from_ewmas(returns, halflife_pairs)
covariances = {}
for predictor in combinator.solve(window=10):
    covariances[predictor.time] = predictor.covariance

https://github.com/cvxgrp/cov_pred_finance
Thank you!

Questions?