Proceedings of the
American Control Conference
Baltimore, Maryland • June 1994

History of Linear Matrix Inequalities in Control Theory

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Abstract

The purpose of this paper is to give a historical view of Linear Matrix Inequalities in control and system theory. Not surprisingly, it appears that LMIs have been involved in some of the major events of control theory. With the advent of powerful convex optimization techniques, LMIs are now about to become an important practical tool for future control applications.

Definition of an LMI

A linear matrix inequality is a matrix inequality of the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{1}$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T$, $i = 0, \ldots, m$ are given matrices. Thus, a linear matrix inequality is a *constraint* on the variable x. Note that the "nonstrict" version of (1), that is, $F(x) \geq 0$, is also a linear matrix inequality.

Early History

The history of linear matrix inequalities in the analysis of dynamical systems goes back more than 100 years. The story begins in about 1890, when Lyapunov published his seminal work introducing what we now call Lyapunov theory [1]. He showed that the differential equation

$$\frac{d}{dt}x(t) = Ax(t) \tag{2}$$

is stable (*i.e.*, all trajectories converge to zero) if and only if there exists a positive-definite matrix P such that

$$A^T P + PA < 0. \tag{3}$$

The requirement $P > 0$, $A^T P + PA < 0$ is what we now call a Lyapunov inequality on P, which is a special form of a linear matrix inequality. Lyapunov also showed that this first LMI could be explicitly solved. Indeed, we can pick any $Q = Q^T > 0$ and then solve the linear equation $A^T P + PA = -Q$ for the matrix P, which is guaranteed to be positive-definite if the system (2) is stable. In summary, the first LMI used to analyze stability of a dynamical system was the Lyapunov inequality (3), which can be solved analytically (by solving a set of linear equations).

The next major milestone occurs in the 1940's. Lur'e, Postnikov, and others in the Soviet Union applied Lyapunov's methods to some specific practical problems in control engineering, especially, the problem of stability of a control system with a nonlinearity in the actuator [2]. Although they did not explicitly form matrix inequalities, their stability criteria in fact have the form of linear matrix inequalities. These inequalities were reduced to polynomial inequalities which were then checked "by hand" (for, needless to say, small systems). Nevertheless they were justifiably excited by the idea that Lyapunov's theory could be applied to important (and difficult) practical problems in control engineering. From the introduction of Lur'e's 1951 book [3] we find:

> This book represents the first attempt to demonstrate that the ideas expressed 60 years ago by Lyapunov, which even comparatively recently appeared to be remote from practical application, are now about to become a real medium for the exami-

nation of the urgent problems of contemporary engineering.

In summary, Lur'e and others were the first to apply Lyapunov's methods to practical control engineering problems. The LMIs that resulted were solved analytically, by hand. Of course this limited their application to small (second, third order) systems.

The KYP Lemma and Quadratic Optimal Control

The next major breakthrough came in the early 1960's, when Yakubovich, Popov, Kalman, and other researchers succeeded in reducing the solution of the linear matrix inequalities that arose in the problem of Lur'e to simple graphical criteria, using what we now call the Kalman-Yakubovich-Popov (KYP) lemma [4, 5, 6]. This resulted in the celebrated Popov criterion, Circle criterion, Tsypkin criterion [7, 8], and many variations. These criteria could be applied to higher order systems, but did not gracefully or usefully extend to systems containing more than one nonlinearity. From the point of view of our story (LMIs in control theory), the contribution was to show how to solve a certain family of linear matrix inequalities by a graphical method.

The important role of LMIs in control theory was already recognized in the early 1960's, especially by Yakubovich [6, 9, 10]. This is clear simply from the titles of some of his papers from 1962–5, *e.g.*, *The solution of certain matrix inequalities in automatic control theory* (1962), and *The method of matrix inequalities in the stability theory of nonlinear control systems* (1965).

The KYP lemma and extensions were intensively studied in the latter half of the 1960s, and were found to be related to the ideas of passivity, the small-gain criteria introduced by Zames [11, 12] and Sandberg [13, 14, 15], and quadratic optimal control. By 1970, it was known that the LMI appearing in the KYP lemma could be solved not only by graphical means, but also by solving a certain algebraic Riccati equation. The difficulty in solving the LMI directly was noted by Anderson and Vongpanitlerd in their book on Network Synthesis [16, p296]:

> The [quadratic matrix inequality appearing in the KYP lemma] has the general form
>
> $$PAP + PB + B^T P + C \leq 0$$
>
> and is apparently very difficult to solve.

In a 1971 paper on quadratic optimal control [17], J. C. Willems is led to the LMI

$$\begin{bmatrix} A^T P + PA + Q & PB + C^T \\ B^T P + C & R \end{bmatrix} \geq 0, \qquad (4)$$

and points out that it can be solved by studying the symmetric solutions of the Riccati equation

$$A^T P + PA - (PB + C^T)R^{-1}(B^T P + C) + Q = 0,$$

which in turn can be found by an eigendecomposition of a related Hamiltonian matrix. This connection had been observed earlier in the Soviet Union, where the Riccati equation was called the Lur'e resolving equation [18].

So by 1971, researchers knew several methods for solving special types of LMIs: direct (for very small systems), graphical methods, and by solving Lyapunov or Riccati equations. From our point of view, these methods are all "closed-form" or "analytical" solutions that can be used to solve special forms of LMIs. (Most control researchers and engineers consider the Riccati equation to have an "analytical" solution, since the standard algorithms that solve it are very predictable in terms of the effort required, which depends almost entirely on the problem size and not the particular problem data. Of course it cannot be solved exactly in a finite number of arithmetic steps for systems of fifth and higher order.)

In Willems' 1971 paper we find the following striking quote:

> The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example.

Here Willems refers to the specific LMI (4), and not the more general form (1). Still, Willems' suggestion that linear matrix inequalities might have some advantages in computational algorithms (as compared to the corresponding Riccati equations) foreshadows the next chapter in the story.

LMIs and Convexity

The next major advance (in our view) was the simple observation that:

> The LMIs that arise in control and systems theory can be formulated as *convex optimization problems* that are amenable to computer solution.

Although this is a simple observation, it has some important consequences, the most important of which is that we can reliably solve many LMIs for which no "analytical solution" has been found (or is likely to be found).

This observation was made explicitly by several researchers. Pyatnitskii and Skorodinskii [19, 20] were perhaps the first researchers to make this point, clearly and completely. They reduced the original problem of Lur'e (extended to the case of multiple nonlinearities) to a convex optimization problem involving linear matrix inequalities, which they then solved using the ellipsoid algorithm. (This problem had been studied before, but the "solutions" involved an arbitrary scaling matrix.) Pyatnitskii and Skorodinskii were the first, as far as we know, to formulate the search for a Lyapunov function as a convex optimization problem, and then apply an algorithm guaranteed to solve the optimization problem.

We should also mention several precursors. In a 1976 paper, Horisberger and Bélanger [21] had remarked that the existence of a quadratic Lyapunov function that simultaneously proves stability of a collection of linear systems is a convex problem involving LMIs. And of course, the idea of having a computer search for a Lyapunov function was not new—it appears, for example, in a 1965 paper by Schultz et al. [22].

Recent Advances in Convex Optimization

The final chapter in our story is quite recent and of great practical importance: the development of powerful and efficient interior-point methods to solve the LMIs that arise in control and systems theory. In 1984, N. Karmarkar [23] introduced a new linear programming algorithm that solved linear programs in polynomial-time, like the ellipsoid method, but in contrast to the ellipsoid method, was also very efficient in practice. Karmarkar's work produced a sensation and spurred an enormous amount of work in the area of interior-point methods for linear programming (including the rediscovery of efficient methods that were developed in the 1960s but ignored). Essentially all of this research activity concentrated on algorithms for linear programs. Then in 1988, Nesterov and Nemirovskii [24] developed interior-point methods that apply directly to convex problems involving matrix inequalities, and in particular, to the problems encountered in control theory. Although there remains much to be done in this area, several interior-point algorithms for LMI problems have been implemented and tested on specific families of LMIs that arise in control theory, and found to be extremely efficient.

Summary

A summary of key events in the history of LMIs in Control theory is then:

- **1890:** First LMI appears; analytical solution of the Lyapunov LMI via Lyapunov equation.
- **1940's:** Application of Lyapunov's methods to real control engineering problems. Small LMIs solved "by hand".
- **Early 1960's:** KYP lemma gives graphical techniques for solving another family of LMIs.
- **Late 1960's:** Observation that the same family of LMIs can be solved by solving an ARE.
- **Early 1980's:** Recognition that many LMIs can be solved by computer via convex programming.
- **Late 1980's:** Development of interior-point algorithms for LMIs.

It is fair to say that Yakubovich is the father of the field, and Lyapunov the grandfather of the field. The new development is the ability to directly *solve* (general) LMIs.

References

[1] A. M. Lyapunov. *Problème général de la stabilité du mouvement*, volume 17 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 1947.

[2] A. I. Lur'e and V. N. Postnikov. On the theory of stability of control systems. *Applied mathematics and mechanics*, 8(3), 1944. In Russian.

[3] A. I. Lur'e. *Some Nonlinear Problems in the Theory of Automatic Control.* H. M. Stationery Off., London, 1957. In Russian, 1951.

[4] R. E. Kalman. Lyapunov functions for the problem of Lur'e in automatic control. *Proc. Nat. Acad. Sci., USA*, 49:201–205, 1963.

[5] V. M. Popov. Absolute stability of nonlinear systems of automatic control. *Automation and Remote Control*, 22:857–875, 1962.

[6] V. A. Yakubovich. The solution of certain matrix inequalities in automatic control theory. *Soviet Math. Dokl.*, 3:620–623, 1962. In Russian, 1961.

[7] Ya. A. Tsypkin. Frequency criteria for the absolute stability of nonlinear sampled data systems. *Aut. Remote Control*, 25:261–267, 1964.

[8] Ya. Z. Tsypkin. A criterion for absolute stability of automatic pulse systems with monotonic characteristics of the nonlinear element. *Sov. Phys. Doklady*, 9:263–266, 1964.

[9] V. A. Yakubovich. Solution of certain matrix inequalities encountered in nonlinear control theory. *Soviet Math. Dokl.*, 5:652–656, 1964.

[10] V. A. Yakubovich. The method of matrix inequalities in the stability theory of nonlinear control systems, *I*, *II*, *III*. *Automation and Remote Control*, 25-26(4):905–917, 577–592, 753–763, April 1967.

[11] G. Zames. On the input-output stability of time-varying nonlinear feedback systems—Part I: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Trans. Aut. Control*, AC-11:228–238, April 1966.

[12] G. Zames. On the input-output stability of time-varying nonlinear feedback systems—Part II: Conditions involving circles in the frequency plane and sector nonlinearities. *IEEE Trans. Aut. Control*, AC-11:465–476, July 1966.

[13] I. W. Sandberg. A frequency-domain condition for the stability of feedback systems containing a single time-varying nonlinear element. *Bell Syst. Tech. J.*, 43(3):1601–1608, July 1964.

[14] I. W. Sandberg. On the boundedness of solutions of non-linear integral equations. *Bell Syst. Tech. J.*, 44:439–453, 1965.

[15] I. W. Sandberg. Some results in the theory of physical systems governed by nonlinear functional equations. *Bell Syst. Tech. J.*, 44:871–898, 1965.

[16] B. D. O. Anderson and S. Vongpanitlerd. *Network analysis and synthesis: a modern systems theory approach.* Prentice-Hall, 1973.

[17] J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Aut. Control*, AC-16(6):621–634, December 1971.

[18] V. A. Yakubovich. Dichotomy and absolute stability of nonlinear systems with periodically nonstationary linear part. *Syst. Control Letters*, 1988.

[19] E. S. Pyatnitskii and V. I. Skorodinskii. Numerical methods of Lyapunov function construction and their application to the absolute stability problem. *Syst. Control Letters*, 2(2):130–135, August 1982.

[20] E. S. Pyatnitskii and V. I. Skorodinskii. Numerical method of construction of Lyapunov functions and absolute stability criteria in the form of numerical procedures. *Automation and Remote Control*, 44(11):1427–1437, 1983.

[21] H. P. Horisberger and P. R. Bélanger. Regulators for linear, time invariant plants with uncertain parameters. *IEEE Trans. Aut. Control*, AC-21:705–708, 1976.

[22] D. G. Schultz, F. T. Smith, H. C. Hsieh, and C. D. Johnson. The generation of Liapunov functions. In C. T. Leondes, editor, *Advances in Control Systems*, volume 2, pages 1–64. Academic Press, New York, New York, 1965.

[23] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984.

[24] Yu. Nesterov and A. Nemirovsky. *Interior point polynomial methods in convex programming: Theory and applications.* SIAM, 1993.