Markowitz Portfolio Construction at Seventy

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Abstract

More than seventy years ago Harry Markowitz formulated portfolio construction as an optimization problem that trades off expected return and risk, defined as the standard deviation of the portfolio returns. Since then the method has been extended to include many practical constraints and objective terms, such as transaction cost or leverage limits. Despite several criticisms of Markowitz's method, for example its sensitivity to poor forecasts of the return statistics, it has become the dominant quantitative method for portfolio construction in practice. In this article we describe an extension of Markowitz's method that addresses many practical effects and gracefully handles the uncertainty inherent in return statistics forecasting. Like Markowitz's original formulation, the extension is also a convex optimization problem, which can be solved with high reliability and speed.

*Alphabetical order.

Contents

1 Introduction

Harry Markowitz's 1952 paper *Portfolio Selection* [Mar52] was a true breakthrough in our understanding of and approach to investing. Before Markowitz there was (almost) no mathematical approach to investing. As a 25-year-old graduate student, Markowitz founded modern portfolio theory, and methods inspired by him would become the most widely used portfolio construction practices over the next 70 years (and counting).

Before Markowitz, diversification and risk were fuzzy concepts. Investors loosely connected risk to the probability of loss, but with no analytical rigor around that connection. Ben Graham, who along with David Dodd wrote *Security Analysis* [GD09], once commented that investors should own "a minimum of ten different issues and a maximum of about thirty" [Gra73].

There were a few precursors, such as an article by de Finetti, that contained some similar ideas before Markowitz; see [dF40, Rub06] for a discussion and more of the history of mathematical formulation of portfolio construction. Another notable precursor is John Burr Williams' 1938 *Theory of Investment Value* [Wil38]. He argued that the value of a company was the present value of future dividends. His book is full of mathematics, and Williams predicted that "mathematical analysis is a new tool of great power, whose use promises to lead to notable advances in investment analysis". That prediction came true with Markowitz's work. Indeed, Markowitz considered Williams' book as part of his inspiration. According to Markowitz, "the basic concepts of portfolio theory came to me one afternoon in the library while reading John Burr Williams' *Theory of Investment Value*".

For many years, the lack of data and accessible computational power [Mar19] rendered Markowitz's ideas impractical, despite his pragmatic approach. In 1963, William Sharpe published his market model [Sha63], designed to speed up the Markowitz calculations. This model was a one-factor risk model (the factor was the market return), with the assumption that all residual returns are uncorrelated. His paper stated that solving a 100-asset problem on an IBM 7090 computer required 33 minutes, but his simplified risk model reduced it to 30 seconds. He also commented that computers could only handle 249 assets at most with a full covariance matrix, but 2000 assets with the simplified risk model. Today such a problem can be solved in microseconds; we can routinely solve problems with tens of thousands of assets and substantially more factors in well under one second.

Markowitz portfolio construction has thrived for many years in spite of claims of various alleged deficiencies. These have included the method's sensitivity to data errors and estimation uncertainty, its single-period nature to handle what is fundamentally a multi-period problem, its symmetric definition of risk, and its neglect of higher moments like skewness and kurtosis. We will address these alleged criticisms and show that standard techniques in modern approaches to optimization effectively deal with them without altering Markowitz's vision for portfolio selection.

In 1990 Markowitz was awarded the Nobel Memorial Prize in Economics for his work on portfolio theory, shared with Merton Miller and William Sharpe. For more light on the fascinating historic details we recommend an interview with Markowitz [Mar19], his acceptance speech for the Nobel Prize [Mar23], and his remarks in the introduction to the

Handbook of Portfolio Construction [Gue10].

1.1 The original Markowitz idea

Markowitz identified two steps in the portfolio selection process. In a first step, the investor forms beliefs about the expected returns of the assets, expressed as a vector μ, and their covariances, expressed as a covariance matrix Σ, which gives the volatilities of asset returns and the correlations among them. These beliefs are the core inputs for the second step, which is the optimization of the portfolio based on these quantities.

He introduced the expected returns–variance of returns (E–V) rule, which states that an investor desires to achieve the maximum expected return for a portfolio while keeping its variance or risk below a given threshold. Convex programming was not a well developed field at that time, and Markowitz used a geometric interpretation in the space of portfolio weights [Mar52] to solve the problem we would now express as

$$\begin{array}{ll} \text{maximize} & \mu^T w \\ \text{subject to} & w^T \Sigma w \leq (\sigma^{\text{tar}})^2, \\ & \mathbf{1}^T w = 1, \end{array} \tag{1}$$

with variable $w \in \mathbf{R}^n$, the set of portfolio weights, where $\mathbf{1}$ is the vector with all entries one. The data in the problem are $\mu \in \mathbf{R}^n$, the vector of expected asset returns, and Σ, the $n \times n$ covariance matrix of asset returns. The positive parameter σ^{tar} is the target portfolio return standard deviation or volatility. (We define the weights and describe the problem more carefully in §2.)

There are many other ways to formulate the trade-off of expected return and risk as an optimization problem [BV04, ApS23]. One very popular method maximizes the *risk-adjusted return*, which is the expected portfolio return minus its variance, scaled by a positive *risk-aversion parameter*. This leads to the optimization problem [GK00]

$$\begin{array}{ll} \text{maximize} & \mu^T w - \gamma w^T \Sigma w \\ \text{subject to} & \mathbf{1}^T w = 1, \end{array} \tag{2}$$

where γ is the risk-aversion parameter that controls the trade-off between risk and return. Both problems (1) and (2) give the full trade-off curve of Pareto optimal weights, as σ^{tar} or γ vary from 0 to ∞ (although (1) can be infeasible when σ^{tar} is too small). One advantage of the first formulation (1) is that the parameter σ^{tar} that controls the volatility is interpretable as, simply, the target risk level. The risk-aversion parameter γ appearing in (2) is less interpretable. We will have more to say about the parameters that control trade-offs in portfolio construction in §4.

Both problems (1) and (2) have analytical solutions. For example the solution of (2) is given by

$$w^\star = \frac{1}{2\gamma} \Sigma^{-1} (\mu + \nu^\star \mathbf{1}), \qquad \nu^\star = \frac{2\gamma - \mathbf{1}^T \Sigma^{-1} \mu}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}.$$

(The scalar $\nu^\star$ is the optimal dual variable [BV04, Chap. 5].) We note here the appearance of the inverse covariance matrix. To compute $w^\star$ we would not compute the inverse, but rather solve two sets of equations to find $\Sigma^{-1}\mu$ and $\Sigma^{-1}\mathbf{1}$ [BV18]. Still, the appearance of Σ^{-1} in the expressions for the solutions give us a hint that the method can be sensitive to the input data when the covariance matrix Σ is nearly singular. These analytical formulas can also be used to back out so-called implied returns, *i.e.*, the mean μ for which a given portfolio is optimal. For example the *market implied return* μ^{mkt} is the return for which the optimal weights are the market weights, *i.e.*, proportional to asset capitalization.

Both formulations (1) and (2) are referred to as the basic Markowitz problem, or *mean-variance optimization*, since they both trade off the mean and variance of the portfolio return. In his original paper Markowitz also noted that additional constraints can be added to the problem, specifically the constraint that $w \geq 0$ (elementwise), which means the portfolio is long-only, *i.e.*, it does not contain any short positions. With this added constraint, the two problems above do not have simple analytical solutions. But the formulation (2), with the additional constraint $w \geq 0$, is a quadratic program (QP), a type of convex optimization problem for which numerical solvers were developed already in the late 1950s [Wol59]. In that early paper on QP, solving the Markowitz problem (2) with the long-only constraint $w \geq 0$ was listed as a prime application. Today we can solve either formulation reliably, with essentially any set of convex portfolio constraints.

Since the 1950s we have seen a truly stunning increase in computer power, as well as the development of convex optimization methods that are fast and reliable, and high-level languages that allow users to express complex convex optimization problems in a few lines of clear code. These advances allow us to extend Markowitz's formulation to include a large number of practical constraints and additional terms, such as transaction cost or the cost incurred when holding short positions. In addition to directly handling a number of practical issues, these generalizations of the basic Markowitz method also address the issue of sensitivity to the input data μ and Σ. This paper describes one such generalization of the basic Markowitz problem, that works well in practice.

Out of respect for Markowitz, and because the more generalized formulation we present here is nothing more than an extension of his original idea, we will refer to these more complex portfolio construction methods also as Markowitz methods. When we need to distinguish the extension of Markowitz's portfolio construction that we recommend from the basic Markowitz method, we refer to it as Markowitz++. (In computer science, the post-script ++ denotes the successor.)

1.2 Alleged deficiencies

The frequent criticism of Markowitz's work is a testament to its importance. These criticisms usually fall into one or more of the following (related) categories.

It's sensitive to data errors and estimation uncertainty. The sensitivity of Markowitz portfolio construction to input data is well documented [Mul93, MM08, SH13, Bra10, CY16], and already hinted at by the inverse covariance that appears in the analytical solutions of the basic Markowitz method. This sensitivity, coupled with the challenge of estimating the mean and covariance of the return, leads to portfolios that exacerbate errors or deficiencies in the input data to find unrealistic and poorly performing portfolios. Some authors argue that choosing a portfolio by optimization, as Markowitz's method does, is essentially an estimation-error maximization method. This is still a research topic that draws much attention. In the recent papers [GPS22, Shk23] the authors quantify how the (basic) Markowitz portfolio is affected by estimation errors in the covariance matrix.

This criticism is justified, on the surface. Markowitz portfolio construction can perform poorly when it is naïvely implemented, for example by using empirical estimates of mean and covariance on a trailing window of past returns. But the critical practical issues of taming sensitivity and gracefully handling estimation errors are readily addressed using techniques such as regularization and robust optimization, described in more detail in §1.3.

It implicitly assumes risk symmetry. Markowitz portfolio construction uses variance of the portfolio return as its risk measure. With this risk measure a portfolio return well above the mean is just as bad as one that is well below the mean, whereas the former is clearly a good event, not a bad one. This observation should at least make one suspicious of the formulation, and has motivated a host of proposed alternatives, such as defining the risk taking into account only the downside [Mar59, Chap. IX]. This criticism is also valid, on the surface. But when the parameters are chosen appropriately, and the data are reasonable, portfolios constructed from mean-variance optimization do not suffer from this alleged deficiency.

We should maximize expected utility. A more academic version of the previous criticism is that portfolios should be constructed by maximizing the expected value of a concave increasing utility function of the portfolio return [VNM47]. The utility in mean-variance optimization (with risk-adjusted return objective) is $U(R) = R - \gamma R^2$, where R is the portfolio return. This utility function is concave, but only increasing for $R < 1/(2\gamma)$; above that value of return, it decreases, putting us in the awkward position of seeming to prefer smaller returns over larger ones.

This criticism is also valid, taken at face value; the quadratic utility above is indeed not increasing. Markowitz himself addressed the issue in a 1979 paper with H. Levy that argued that while mean-variance optimization does not appear to be the same as maximizing an expected utility, it is a very good approximation; see [LM79] and [MB14, Chap. 2]. But in fact it turns out that Markowitz portfolio construction *does* maximize the expected value of a concave increasing utility function. Specifically if we model the returns as Gaussian, and use the exponential utility $U(R) = 1 - \exp(-\gamma R)$, then the expected utility is the risk-adjusted return, up to an additive constant [LB23]. In other words, Markowitz portfolio construction *does* maximize expected utility of portfolio return, for a specific concave increasing utility function and a specific asset return distribution.

It considers only the first and second moments of the return. Mean-variance optimization naturally only considers the first two moments of the distribution. It would seem that taking higher moments like skewness and kurtosis into account might better describe investor preferences [Caj22, ZP21]. This, coupled with the fact that the tails of asset returns are not well modeled by a Gaussian distribution [Fam65], suggests that portfolio construction should consider higher moments than the first and second.

While it is possible to construct small academic examples where mean-variance optimization does poorly due to its neglect of higher moments, simple mean-variance optimization does very well on practical problems. In [LB23] the authors extend Markowitz by maximizing exponential utility, but with a more complex Gaussian mixture model of asset returns. Such a distribution is general, in that it can approximate any distribution. Their method evidently handles higher moments, but empirically gives no boost in performance on practical problems.

Markowitz himself addressed the common misconception that he labeled the "Great Confusion" [Mar19, MB14, Mar99, Mar09], stating that Gaussian returns are merely a sufficient but not a necessary condition on the return distribution for mean-variance optimization to work well and that mean and variance are good approximations for expected utility.

It's a greedy method. Portfolios are generally not just set up and then held for one investment period; they are rebalanced, and sometimes often. Problems in which a sequence of decisions are made, based on newly available information, are more accurately modeled not as simple optimization problems, but instead as stochastic control problems, also known as sequential decision making under uncertainty [Koc15, KWW22, Bel66, Ber12]. In the context of stochastic control, methods that take into account only the current decision and not future ones are called *greedy*, and in some cases can perform very poorly. This criticism is also, on its face, valid. Using Markowitz portfolio construction repeatedly, as is always done in practice, is a greedy method.

We can readily counter this criticism. First, in the special case with risk-adjusted return and quadratic transaction costs, and no additional constraints, the stochastic optimal policy can be worked out, and coincides with a single-period Markowitz portfolio [GK20, BB21]. This suggests that when other constraints are present, and the transaction cost is not quadratic, the (greedy) Markowitz method should not be too far from stochastic optimal.

Second, there are extensions of Markowitz portfolio construction, called *multi-period* methods, that plan a sequence of trades over a horizon, and then execute only the first trade; see, *e.g.*, [BBD+17, LUM22]. These multi-period methods can work better than so-called single-period methods, for example when a portfolio is transitioning between two managers, or being set up or liquidated over multiple periods. But in almost all other cases, single-period methods work just as well as multi-period ones.

The third response to this criticism more directly addresses the question. In the paper *Performance Bounds and Suboptimal Policies for Multi-Period Investment* [BMOW13], the authors develop bounds on how well a full stochastic control trading policy can do, and show empirically that single-period Markowitz trading essentially does as well as a full stochastic control policy (which is impractical if there are more than a handful of assets). So while there are applications where greedy policies do much more poorly than a true stochastic control policy, it seems that multi-period trading is not one of them.

1.3 Robust optimization and regularization

Here we directly address the question of sensitivity of Markowitz portfolio construction to the input data μ and Σ. As mentioned above, the basic methods are indeed sensitive to these parameters. But this sensitivity can be mitigated and tamed using techniques that are widely used in other applications and fields, robust optimization and regularization.

Robust optimization. Modifying an optimization-based method to make it more robust to data uncertainty is done in many fields, using techniques that have differing names. When optimization is used in almost any application, some of the data are not known exactly, and solving the optimization problem without recognizing this uncertainty, for example by using some kind of mean or typical values of the parameters, can lead to very poor practical performance. *Robust optimization* is a subfield of optimization that develops methods to handle or mitigate the adverse effects of parameter uncertainty; see, *e.g.*, [BTEGN09, TK04, GMT14, BTN02, BBC11, Lob00]. These methods tend to fall in one of two approaches: statistical or worst-case deterministic. In a statistical model, the uncertain parameters are modeled as random variables and the goal is to optimize the expected value of the objective under this distribution, leading to a *stochastic optimization problem* [SDR21], [BV04, Chap. 6.4.1]. A worst-case deterministic uncertainty model posits a set of possible values for parameters, and the goal is to optimize the worst-case value of the objective over the possible parameter values [BS07], [BV04, Chap. 6.4.2]. Another name for worst-case robust optimization is *adversarial optimization*, since we can model the problem as our choosing values for the variables to obtain the best objective, after which an adversary chooses the values of the parameters so as to achieve the worst possible objective. Worst-case robust optimization has many variations and goes by many names. For example when the set of possible parameter values is finite, they are called *scenarios* or *regimes*, and optimizing for the worst-case scenario is called *worst-case scenario optimization*. While these general approaches sound quite different, they often lead to very similar solutions, and both can work well in applications. Robust optimization methods work by modifying the objective or constraints to model the possible variation in the data.

One very successful application of robust optimization is in *robust control*, where a control system is designed so that the control performance is not too sensitive to changes in the system dynamics [ZD98, KDG96]. So-called linear quadratic optimal control was developed around 1960, and used in many applications. Its occasional sensitivity to the data (in this case, the dynamic model of the system being controlled) was noted then; by the early 1990s robust control methods were developed, and are now very widely used.

Regularization. Regularization is another term for methods that modify an optimization problem to mitigate sensitivity to data. It is almost universally used in statistics and machine learning when fitting models to data. Here we fit the parameters of a model to some given training data, accounting for the fact that the training data set could have been different [TA77, HTF09]. This process of regularization can be done explicitly by adding a penalty term to the objective, and also implicitly by adding constraints to the problem that prevent extreme outcomes. Regularization can often be interpreted as a form of robust optimization; see, *e.g.*, [BV04, Chap. 6.3–6.4].

The high level story. Robust optimization and regularization both follow the same high level story, and both can be applied to the Markowitz problem. The story starts with a basic optimization-based method that relies on data that are not known precisely. We then modify the optimization problem, often by adding additional objective terms or constraints. Doing this *worsens* the in-sample performance. But if done well, it *improves* out-of-sample performance. Roughly speaking, robustification and regularization tell the optimizer to not fully trust the data, and this serves it well out-of-sample.

In portfolio construction a long-only constraint can be interpreted as a form of regularization [JM03]. A less extreme version is to impose a leverage limit, which can help avoid many of the data sensitivity issues. We will describe below some effective and simple robustification methods for portfolio construction.

Regularization can (and should) also be applied to the forecasting of the mean and covariance in Markowitz portfolio construction. The Black-Litterman approach to estimating the mean returns regularizes the estimate toward the market implied return [BL90]. A return covariance estimate can be regularized using *shrinkage*, another term for regularized estimation in statistics [LW04].

1.4 Convex optimization

Over the same 70-year period since Markowitz's original work, there has been a parallel advance in mathematical optimization, and especially convex optimization, not to mention stunning increases in available computer power. Roughly speaking, convex optimization problems are mathematical optimization problems that satisfy certain mathematical properties. They can be solved reliably and efficiently, even when they involve a very large number of variables and constraints, and involve nonlinear, even nondifferentiable, functions [BV04].

Shortly before Markowitz published his paper on portfolio selection, George Dantzig developed the simplex method [Dan51], which allowed for the efficient solution of linear programs. In 1959, Wolfe [Wol59] extended the simplex method to QP problems, citing Markowitz's work as a motivating application. This close connection between portfolio construction and optimization was no coincidence, since Dantzig and Markowitz were colleagues at RAND.

Since then, the field of convex optimization has grown tremendously. Today, convex optimization is a mature field with a large body of theory, algorithms, software, and applications [BV04]. Being able to solve optimization problems reliably and efficiently is crucial for portfolio construction, especially for back-testing or simulating a proposed method on historical or synthesized data, where portfolio construction has to be carried many times. Thus, any extension of the Markowitz objective or additional constraints should be convex to ensure tractability. As we will see, this is hardly a limitation in practice.

Solvers. The dominant convex optimization problem form is now the *cone program*, a generalization of linear programming that handles nonlinear objective terms and constraints [NN92, BV04, LVBL98a, VB96]. There are now a number of reliable and efficient solvers for such problems, including open-source ones like ECOS [DCB13], Clarabel [GC24], and SCS [OCPB16], and commercial solvers such as MOSEK [ApS20], GUROBI [Gur23], and CPLEX [Cpl09]. A recent open-source solver for QPs is OSQP [SBG+20].

Domain-specific languages. Convex optimization is also now very accessible to practitioners, even those without a strong background in the mathematics or algorithms of convex optimization, thanks to high-level domain-specific languages (DSLs) for convex optimization, such as CVXPY [AVDB18, DB16], CVX [GB14], Convex.jl [UMZ+14], CVXR [FNB17], and YALMIP [Lof04]. These DSLs make it easy to specify complex, but convex, optimization problems in a natural, human readable way. The DSLs transform the problem from the human readable form to a lower level form (often a cone program) suitable for a solver. These DSLs make it easy to develop convex optimization based methods, as well as to modify, update, and maintain existing ones. As a result, CVXPY is used at many quantitative hedge funds today, as well as in many other applications and industries. The proposed extension of Markowitz's portfolio construction method that we describe below is a good example of the use of CVXPY. It is a complex problem involving nonlinear and nondifferentiable functions, but its specification in CVXPY takes only a few tens of lines of clear readable code, given in appendix B. The overhead of translating the human readable problem specification into a cone program is typically small. Additionally, in some DSLs, such as CVXPY, problems can be parametrized [AAB+19], such that they can be solved for a range of values of the parameters, making the translation overhead negligible. Related to DSLs are modeling layers provided by some solver, such as MOSEK's Fusion API [ApS20], which provides a high-level interface to the solver. Less focused on convex optimization, there are other modeling languages such as JuMP [LDD+23] and Pyomo [HWW11, BHH+21] that do not verify convexity, but provide flexibility in modeling a wide range of optimization problems, including nonconvex ones.

Code generators. Code generators like CVXGEN [MB12] and CVXPYgen [SBD+22] are similar to DSLs. They support high level specification of a problem (family) but instead of directly solving the problem, they generate custom low level code (typically C) for the problem that is specified. This code can be compiled to a very fast and totally reliable solver, suitable for embedded real-time applications. For example, CVXGEN-generated code guides all of SpaceX's Falcon 9 and Falcon Heavy first stages to their landings [Bla16].

1.5 Previous work

The literature on portfolio construction is vast, and focusing on the practical implementation of Markowitz's ideas, we do not attempt to survey it here in detail. Instead, we highlight only a few major developments that are relevant to our work. For a detailed overview see, *e.g.*, [GK00, Chap. 14], [Nar09, Chap. 6], and [CT06, KTF14].

Building on Markowitz's framework, the field of portfolio construction has undergone substantial evolution. Notable contributions include Sharpe's Capital Asset Pricing Model [Sha63] and the Black-Litterman model [BL90]. A pivotal figure in bringing the field to the forefront of the industry was Barr Rosenberg, whose research evolved to become the Barra risk model [Ros84, She96], first used for risk modeling and then in portfolio optimization. The introduction of risk parity models [MRT10] brought a focus on risk distribution. Additionally, hierarchical risk parity, a recent advancement, offers a more intricate approach to risk allocation, considering the hierarchical structure of asset correlations [DP16]. These developments reflect the field's dynamic adaptation to evolving market conditions and analytical techniques.

Software. Dedicated software helped practitioners access the solvers and DSLs mentioned earlier, and has facilitated the wide acceptance of Markowitz portfolio construction. A wealth of software packages have been developed for portfolio optimization, many (if not most) with Python interfaces, both open-source and commercial. Examples range from simple web-based visualization tools to complex trading platforms. Here we mention only a few of these software implementations.

On the simpler end Portfolio Visualizer [Glo23] is a web-based tool that allows users to back-test and visualize various portfolio strategies. PyPortfolioOpt [Mar21] and Cvxportfolio [BBD+17] are Python packages offering various portfolio optimization techniques. PyPortfolioOpt includes mean-variance optimization, Black-Litterman allocation [BL90], and more recent alternatives like the Hierarchical Risk Parity algorithm [DP16], while Cvxportfolio [BBD+17] supports multi-period strategies. Another Python implementation is proposed in [SXD20], where the authors introduce an approach to multicriteria portfolio optimization. Quantlib [The23] is an alternate open-source software package for modeling, trading, and risk management.

The list of commercial software is also extensive. MATLAB's Financial Toolbox [Bra13, Mat23] includes functions for mathematical modeling and statistical analysis of financial data, including portfolio optimization. Another example is Axioma, which on top of its popular risk model offers a portfolio optimizer [Qon23].

Other software packages include Portfolio123 [Por23a], PortfoliosLab [Por23b], and PortfolioLab by Hudson & Thames [Tha23]. Additionally, many solvers, such as MOSEK [ApS20, ApS23], provide extensive examples of portfolio optimization problems, making them easy to use for portfolio optimization.

1.6 This paper

Our goal is to describe an extension of the basic Markowitz portfolio construction method that includes a number of additional objective terms and constraints that reflect practical issues and address the issue of sensitivity to inevitable forecasting errors. We give a minimal formulation that is both simple and practical; we make no attempt to list all possible extensions that a portfolio manager (PM) might wish to add.

While the resulting optimization problem might appear complex, containing nonlinear nondifferentiable functions, it is convex, which means it can be solved reliably and efficiently. It can also be specified in a DSL such as CVXPY in just a few tens of lines of clear simple code. We can solve even large instances of the optimization problem very quickly, making it practical to carry out extensive back-testing to predict performance or adjust parameter values. One additional advantage of our formulation is that parameters that need to be specified are generally more interpretable than those appearing in basic formulations. For example a PM specifies a target risk and a target turnover instead of some parameters that are less directly related to them.

Most of the material in this paper is not new but scattered across many sources, in different formats, and indeed in different application fields. Some of our recommendations are widely accepted and industry standard, but others are rarely discussed in the literature and even less commonly used in practice.

The authors bring a diverse set of backgrounds to this paper. Some of us have applied Markowitz portfolio construction day-to-day in research, writing, and real portfolios. Others approach Markowitz's method from the perspective of optimization and control in engineering. Control systems engineering has a long history and is widely applied in essentially all engineering applications. Most applications of control engineering use methods based on models that are either wrong or heavily simplified. While naïve implementations of these methods do not work well (or worse), simple sensible modifications, similar to the ones we describe later in this paper, work very well in practice.

These different backgrounds together can provide a new perspective and bring modern tools to the endeavor Markowitz began. These techniques have made Markowitz's method even more applicable and useful to investors.

Software. We have created two companion software packages. One is designed for pedagogical purposes, uses limited parameter testing and checking, and very closely follows the terminology and notation of the paper. It is available at

`https://github.com/cvxgrp/markowitz-reference`.

The second package is a robust and flexible implementation, which is better suited for practical use. It is available at

`https://github.com/cvxgrp/cvxmarkowitz`.

Outline. In §2 we set up our notation, define weights and trades, and describe various objective terms and constraints. Return and risk forecasts are covered in §3. In §4 we pull together the material of the previous two sections to define the (generalized) Markowitz trading problem, which we refer to as Markowitz++. In §5 we present some simple numerical experiments that illustrate how the extra terms robustify the basic Markowitz trading policy, and how parameters are tuned via back-testing to improve good performance.

2 Portfolio holdings and trades

This section introduces the notation and terminology for portfolio holdings, weights, and trades, fundamental objects in portfolio construction independent of the trading strategy. We follow the notation of [BBD+17], with the exceptions of handling the cash weight separately and dropping the time period subscript.

2.1 Portfolio weights

Universe. We consider a portfolio consisting of investments (possibly short) in n assets, plus a cash account. We refer to the set of assets we might hold as the *universe* of assets, and n as the size of the universe. These assets are assumed to be reasonably liquid, and could include, for example, stocks, bonds, or currencies.

Asset and cash weights. To describe the portfolio investments, we work with the *weights* or fractions of the total portfolio value for each asset, with negative values indicating short positions. We denote the weights for the assets as w_i, $i = 1, \dots, n$, and collect them into a portfolio weight vector $w = (w_1, \dots, w_n) \in \mathbf{R}^n$. The weights are readily interpreted: $w_i = 0.05$ means that 5% of the total portfolio value is held in asset i, and $w_k = -0.01$ means that we hold a short position in asset k, with value 1% of the total portfolio value. The dollar value of asset i held is Vw_i, where V is the total portfolio value, assumed to be positive.

We denote the weight for the cash account, *i.e.*, our cash value divided by the portfolio value, as c. If c is negative, it represents a loan. When $c > 0$ we say the portfolio is *diluted* with cash; when $c < 0$, the portfolio is *margined*. The dollar value of the cash account is Vc.

By definition the weights sum to one, so we have

$$\mathbf{1}^T w + c = 1, \tag{3}$$

where $\mathbf{1}$ is the vector with all entries one. The first term, $\mathbf{1}^T w$, is the total weight on the non-cash assets, and we refer to it as the total asset weight. The cash weight is one minus the total asset weight, *i.e.*, $c = 1 - \mathbf{1}^T w$.

Several portfolio types can be expressed in terms of the holdings. A *long-only* portfolio is one with all asset weights nonnegative, *i.e.*, $w \geq 0$ (elementwise). A portfolio with $c = 0$, *i.e.*, no cash holdings, is called *fully invested*. In such a portfolio we have $\mathbf{1}^T w = 1$, *i.e.*, the total asset weight is one. As another example, a *cash-neutral* portfolio is one with $c = 1$. For a cash-neutral portfolio we have $\mathbf{1}^T w = 0$, *i.e.*, the total (net) asset weight is zero.

Leverage. The *leverage* of the portfolio, denoted L, is

$$L = \sum_{i=1}^{n} |w_i| = \|w\|_1.$$

(Several other closely related definitions are also used. Our definition is commonly referred to as the *gross leverage* [AGv11].) The leverage does not include the cash account.

In a long-only portfolio, the leverage is equal to the total asset weight. The *130-30 portfolio* [LEB09] refers to a fully invested portfolio with leverage $L = 1.6$. For such a portfolio, the total weight of the short positions (*i.e.*, negative w_i) is -0.3 and the total weight of the long positions (*i.e.*, positive w_i) is 1.3.

Benchmark and active weights. In some cases our focus is on portfolio performance relative to a *benchmark portfolio*. We let $w^{\mathrm{b}} \in \mathbf{R}^n$ denote the weights of the benchmark. Typically the benchmark does not include any cash weight, so $\mathbf{1}^T w^{\mathrm{b}} = 1$. We refer to $w - w^{\mathrm{b}}$ as the *active weights* of our portfolio. A positive active weight on asset i, *i.e.*, $w_i - w_i^{\mathrm{b}} > 0$, means our portfolio is *over-weight* (relative to the benchmark) on asset i; a negative active weight, $w_i - w_i^{\mathrm{b}} < 0$, means our portfolio is *under-weight* on asset i.

2.2 Holding constraints and costs

Several constraints and costs are associated with the portfolio holdings w and c.

Weight limits. Asset and cash *weight limits* have the form

$$w_i^{\min} \leq w_i \leq w_i^{\max}, \quad i = 1, \ldots, n, \quad c^{\min} \leq c \leq c^{\max},$$

where $w^{\min}$ and $w^{\max}$ are given vectors of lower and upper limits on asset weights, and $c^{\min}$ and $c^{\max}$ are given lower and upper limits on the cash weight. We write the asset weight inequalities in vector form as $w^{\min} \leq w \leq w^{\max}$. We have already encountered a simple example: a long-only portfolio has $w^{\min} = 0$.

Portfolio weight limits can reflect hard requirements, for example that a portfolio must (by legal or regulatory requirements) be long-only. Portfolio weight limits can also be used to avoid excessive concentration of a portfolio, or limit short positions. For example, $w^{\max} = 0.15$ means that our portfolio cannot hold more than 15% of the total portfolio value in any one asset. (Here we adopt the convention that in a vector-scalar inequality, the scalar is implicitly multiplied by $\mathbf{1}$.) As another example, $w^{\min} = -0.05$ means that the short position in any asset can never exceed 5% of the total portfolio value. For large portfolios it is reasonable to also limit holdings relative to the asset capitalization, *e.g.*, to require that our portfolio holdings of each asset are no more than 10% of the asset capitalization.

Weight limits can also be used to capture the portfolio manager's views on how the market will evolve. For example, she might insist on a long position for some assets, and a short position for some others.

When a benchmark is used, we can impose limits on active weights. For example $|w - w^{\mathrm{b}}| \leq 0.10$ means that no asset in the portfolio can be more than 10% over-weight or under-weight.

Leverage limit. In addition to weight limits, we can impose a leverage limit,

$$L \leq L^{\mathrm{tar}}, \tag{4}$$

where L^{tar} is a specified maximum or target leverage value. (Other authors have suggested including leverage as a penalty term in the objective, to model leverage aversion [JL13].)

Holding costs. In general a fee is paid to borrow an asset in order to enter a short position. Analogously we pay a borrow cost fee for a negative cash weight. We will assume these holding costs are a linear function of the negative weights, *i.e.*, of the form

$$\phi^{\mathrm{hold}}(w, c) = (\kappa^{\mathrm{short}})^T (-w)_+ + \kappa^{\mathrm{borrow}} (-c)_+, \tag{5}$$

where $(a)_+ = \max\{a, 0\}$ denotes the nonnegative part, applied elementwise and in its first use above. Here $\kappa^{\mathrm{short}} \geq 0$ is the vector of borrow cost (rates) for the assets, and $\kappa^{\mathrm{borrow}} \geq 0$ is the borrow cost for cash.

Other holding constraints. There are many other constraints on weights that might be imposed, some convex, and others not. A *concentration limit* is an example of a useful constraint that is convex. It states that the sum of the K largest absolute weights cannot exceed some limit. As a specific example, we can require that no collection of five assets can have a total absolute weight of more than 30% [SR20, ApS23]. A *minimum nonzero holding* constraint is an example of a commonly imposed nonconvex constraint. It states that any nonzero weight must have an absolute value exceeding some given minimum, such as 0.5%. (This one is easily handled using a heuristic based on convex optimization; see §4.3.)

2.3 Trades

Trade vector. We let w^{pre} and c^{pre} denote the pre-trade portfolio weights, *i.e.*, the portfolio weights before we carry out the trades to construct the portfolio given by w and c. We need the pre-trade weights to account for transaction costs. We refer to

$$z = w - w^{\mathrm{pre}}, \tag{6}$$

the current weights minus the previous ones, as the (vector of) *trades* or the *trade list*. These trades have a simple interpretation: $z_i = 0.01$ means we buy an amount of asset i equal in value to 1% of our total portfolio value, and $z_i = -0.03$ means we sell an amount of asset i equal to 3% of the portfolio value.

Since $\mathbf{1}^T w^{\rm pre} + c^{\rm pre} = 1$, we have

$$c = c^{\rm pre} - \mathbf{1}^T z, \tag{7}$$

i.e., the post-trade cash weight is the pre-trade cash weight minus the net weight of the trades. This does not include holding and transaction costs, discussed below.

Turnover. The quantity

$$T = \frac{1}{2}\sum_{i=1}^{n} |z_i| = \frac{1}{2}\|z\|_1$$

is the *turnover*. Here too, several other different but closely related definitions are also used, for example the minimum of the total weight bought and the total weight sold [GK00, Chap. 16]. A turnover $T = 0.01$ means that the average of total amount bought and total amount sold is 1% of the total portfolio value. The turnover is often annualized, by multiplying by the number of trading periods per year.

2.4 Trading constraints and costs

We typically have constraints on the trade vector z, as well as a trading cost that depends on z.

Trade limits. Trade limits impose lower and upper bounds on trades, as

$$z^{\rm min} \leq z \leq z^{\rm max},$$

where $z^{\rm min}$ and $z^{\rm max}$ are given limits. These trade limits can be used to limit market participation, defined as the ratio of the magnitude of each trade to the trading volume, using, *e.g.*,

$$|z| \leq 0.05 v, \tag{8}$$

where $v \in \mathbf{R}^n$ is the trading volumes of the assets, expressed as multiples of the portfolio value. This constraint limits our participation for each asset to be less than 5%. (It corresponds to trade limits $z^{\rm max} = -z^{\rm min} = 0.05v$.) Since the trading volumes are not known when z is chosen, we use a forecast instead of the realized trading volumes.

Turnover limit. In addition to trade limits, we can limit the turnover as

$$T \leq T^{\rm tar}, \tag{9}$$

where $T^{\rm tar}$ is a specified turnover limit.

Trading cost. Trading cost refers to the cost of carrying out a trade. For example, if we buy a small quantity of an asset, we pay the ask price, while if we sell an asset, we receive the bid price. Since the nominal price of an asset is the midpoint between the ask and bid prices, we can think of buying or selling the asset as doing so at the nominal price, plus an additional positive cost that is the trade amount times one-half the bid-ask spread. This *bid-ask spread transaction cost* has the form

$$\sum_{i=1}^{n} \kappa_i^{\mathrm{spread}} |z_i| = (\kappa^{\mathrm{spread}})^T |z|,$$

where $\kappa^{\mathrm{spread}} \in \mathbf{R}^n$ is the vector of one-half the asset bid-ask spreads (which are all positive). This is the transaction cost expressed as a fraction of the portfolio value. For small trades this is a reasonable approximation of transaction cost.

For larger trades we 'eat through' the order book. To buy a quantity of an asset, we buy each ask lot, in order from lowest price, until we fill our order. An analogous situation occurs when selling. This means that we end up paying more per share than the ask price when buying, or receiving less than the bid price when selling. This phenomenon is called *market impact*.

A useful approximation of transaction cost that takes market impact into account is

$$\phi^{\mathrm{trade}}(z) = (\kappa^{\mathrm{spread}})^T |z| + (\kappa^{\mathrm{impact}})^T |z|^{3/2}, \tag{10}$$

where the first term is the bid-ask spread component of transaction cost, and the second models the market impact, *i.e.*, the additional cost incurred as the trade eats through the order book. The vector κ^{impact} has positive entries and typically takes the form

$$\kappa_i^{\mathrm{impact}} = a s_i v_i^{-1/2},$$

where s_i is the volatility of asset i over the trading period, v_i is the volume of market trading, expressed as a multiple of the portfolio value, and a is a constant on the order of one; see [GK00, Loe83, TLD+11] and [BBD+17, §2.3]. Evidently the transaction cost increases with volatility, and decreases with market volume. Several other approximations of transaction cost are used [AC00].

Liquidation cost. Suppose we liquidate the portfolio, *i.e.*, close out all asset positions, which corresponds to the trade vector $z = -w$. The *liquidation cost* is

$$\phi^{\mathrm{trade}}(-w) = (\kappa^{\mathrm{spread}})^T |w| + (\kappa^{\mathrm{impact}})^T |w|^{3/2}.$$

If the liquidation is carried out over multiple periods, the bid-ask term stays the same, but the market impact term decreases. For this reason a common approximation of the liquidation cost ignores the market impact term. A liquidation cost constraint has the form

$$(\kappa^{\mathrm{spread}})^T |w| \leq \ell^{\max}, \tag{11}$$

where $\ell^{\max}$ is a maximum allowable liquidation cost, such as 1%. This is a weight constraint; it limits our holdings in less liquid assets, which have higher bid-ask spreads. It can be interpreted as a liquidity-weighted leverage (taking the bid-ask spread as a proxy for liquidity). When all assets have the same bid-ask spread, the liquidation constraint reduces to a leverage constraint. For example with all bid-ask spreads equal to 0.001 (*i.e.*, 10 basis points or bps) and a maximum liquidation cost $\ell^{\max} = 0.01$ (*i.e.*, 1% of the total portfolio value), the liquidation cost limit (11) reduces to a leverage limit (4) with $L^{\text{tar}} = 10$.

Transaction cost forecasts. When the trades z are chosen, we do not know the bid-ask spreads, the volatilities, or the volumes. Instead we use forecasts of these quantities in (8), (10), and (11). Simple forecasts, such as a trailing average or median of realized values, are typically used. More sophisticated forecasts take can into account calendar effects such as seasonality, or the typically low trading volume the day after Thanksgiving.

3 Return and risk forecasts

3.1 Return

Gross portfolio return. We let r_i denote the return, adjusted for dividends, splits, and other corporate actions, of asset i over the investment period. We collect these asset returns into a return vector $r = (r_1, \ldots, r_n) \in \mathbf{R}^n$. The portfolio return from asset i is $r_i w_i$. We let r^{rf} denote the risk-free interest rate, so the return in the cash account is $r^{\text{rf}} c$. The (gross) total portfolio return is then

$$R = r^T w + r^{\text{rf}} c.$$

This gross return does not include holding or trading costs. A closely related quantity is the *excess return*, the portfolio return minus the risk-free return, $R - r^{\text{rf}} = r^T w + r^{\text{rf}}(c - 1)$.

Net portfolio return. The net portfolio return is the gross return minus the holding costs and transaction costs,

$$R^{\text{net}} = R - \phi^{\text{hold}}(w) - \phi^{\text{trade}}(z). \tag{12}$$

Active return. The *active portfolio return* is the return relative to a benchmark portfolio,

$$r^T w + r^{\text{rf}} c - r^T w^{\text{b}} = r^T (w - w^{\text{b}}) + r^{\text{rf}} c.$$

If we subtract holding and trading costs we obtain the *net active portfolio return*.

Cash as slack. Since we do not know but only forecast the bid-ask spread, volatility, and volume, which appear in the transaction cost (10) (which is itself only an approximation) we should consider the post-trade cash c in (7) as an approximation that uses a forecast of holding and transaction costs, not the realized holding and transaction costs. We do not expect the realized post-trade cash weight to be exactly c.

3.2 Probabilistic asset return model

When we choose the trades z we do not know the asset returns r. Instead, we model r as a multivariate random variable with mean $\mu \in \mathbf{R}^n$ and covariance matrix $\Sigma \in \mathbf{S}^n_{++}$ (the set of symmetric positive definite $n \times n$ matrices),

$$\mathbf{E}\, r = \mu, \qquad \mathbf{E}(r-\mu)(r-\mu)^T = \Sigma.$$

The entries of the mean μ are often referred to as *trading signals* [Isi21]. The asset return mean and covariance are forecasts, as described below. The asset return volatilities $s \in \mathbf{R}^n$ appearing in the transaction cost model (10) can be expressed as $s = \mathbf{diag}(\Sigma)^{1/2}$, where the squareroot is elementwise.

Expected return and risk. With this statistical model of r, the portfolio return R is a random variable with mean $\bar{R} = \mathbf{E}\, R$ and variance $\sigma^2 = \mathbf{var}\, R$ given by

$$\bar{R} = \mu^T w + r^{\mathrm{rf}} c, \qquad \sigma^2 = w^T \Sigma w.$$

The *risk* of the portfolio is defined as the standard deviation of the portfolio return, *i.e.*, σ.

Similarly, the active return R^{a} is a random variable with mean and variance

$$\bar{R}^{\mathrm{a}} = \mu^T (w - w^{\mathrm{b}}) + r^{\mathrm{rf}} c = \bar{R} - \mu^T w^{\mathrm{b}}, \qquad (\sigma^{\mathrm{a}})^2 = (w - w^{\mathrm{b}})^T \Sigma (w - w^{\mathrm{b}}),$$

and the *active risk* is σ^{a}. The risk and active risk are often given in annualized form, obtained by multiplying them by the squareroot of the number of periods per year.

The parameters μ and Σ are estimates or forecasts of the statistical model of asset returns, which is itself an approximation. For this reason the risk σ is called the *ex-ante* risk, to distinguish it from the standard deviation of the realized portfolio returns when trading, the *ex-post* risk. Similarly we refer to σ^{a} as the ex-ante active risk.

Optimizing expected return and risk. We have two objectives, high expected return and low risk. Perhaps the most common method for combining these objectives is to form a *risk-adjusted return*,

$$\bar{R} - \gamma \sigma^2,$$

where $\gamma > 0$ is the *risk aversion parameter*. Maximizing risk-adjusted return (possibly with other objective terms, and subject to constraints) gives the desired portfolio. Increasing γ gives us a portfolio with lower risk and also lower expected return. The risk aversion parameter allows us to explore the risk-return trade-off. This risk-adjusted return approach became popular in part because the resulting optimization problem is typically a quadratic program (QP), for which reliable solvers were developed even in the 1960s.

Another approach is to maximize expected return (possibly with other objective terms), subject to a *risk budget* or *risk target* constraint

$$\sigma \leq \sigma^{\mathrm{tar}}. \tag{13}$$

(The corresponding optimization problem is not a QP, but is readily handled by convex optimization solvers developed in the 1990s [LVBL98b, NN94, Stu99, TTT99].) This formulation seems more natural, since a portfolio manager will often have a target risk in her mind, *e.g.*, 8% annualized. This is the basic formulation that we recommend.

There are many other ways to combine expected return and risk. For example, we can maximize the return/risk ratio, called the *Sharpe ratio* (with no benchmark) or *information ratio* (with a benchmark). This problem too can be solved via convex optimization, at least when the constraints are simple [BV23].

3.3 Factor model

In practice, and especially for large universes, it is common to use a *factor model* for the returns. The factor return model, with k factors (typically with $k \ll n$), has the form

$$r = Ff + \epsilon, \tag{14}$$

where $F \in \mathbf{R}^{n \times k}$ is the *factor loading matrix*, $f \in \mathbf{R}^k$ is the vector of *factor returns*, and $\epsilon \in \mathbf{R}^n$ is the *idiosyncratic return*. The term Ff is interpreted as the component of asset returns explainable or predicted by the factor returns.

At portfolio construction time the factor loading matrix F is known, and the factor return f and idiosyncratic return ϵ are modeled as uncorrelated random variables with means and covariance matrices

$$\mathbf{E}\, f = \bar{f}, \qquad \mathbf{cov}\, f = \Sigma^{\mathrm{f}}, \qquad \mathbf{E}\, \epsilon = \bar{\epsilon}, \qquad \mathbf{cov}\, \epsilon = D,$$

where D is diagonal (with positive entries). The entries $\bar{\epsilon}$, the means of the idiosyncratic returns, are also referred to as the *alphas*, especially when there is only one factor which is the overall market return. They are the part of the asset returns not explained by the factor returns.

With the factor model (14) the asset return mean and covariance are

$$\mu = F\bar{f} + \bar{\epsilon}, \qquad \Sigma = F\Sigma^f F^T + D.$$

The return covariance matrix in a factor model has a special form, low rank plus diagonal. The portfolio return mean and variance are

$$\bar{R} = (F\bar{f})^T w + \bar{\epsilon}^T w + r^{\mathrm{rf}} c, \qquad \sigma^2 = (F^T w)^T \Sigma^{\mathrm{f}} (F^T w) + w^T D w.$$

The factor returns are constructed to have explanatory power for the returns of assets in our universe. For equities, they are typically the returns of other portfolios, such as the overall market (with weights proportional to capitalization), industries, and style portfolios like the celebrated Fama-French factors [FF92, FF93]. For bonds, the factors are typically constructed from yield curves, interest rates, and spreads. These traditional factors are interpretable.

Factors can also can be constructed directly from previous realized asset returns using methods such as principal component analysis (PCA) [BN08, Bai03, LP20a, LP20b, PX22b, PX22a]. Aside from the first principal component, which typically is close to the market return, these factors are less interpretable.

Factor and idiosyncratic returns. A factor model gives an alternative method to specify the expected return as $\mu = F\bar{f} + \bar{\epsilon}$, where $\bar{f}$ is a forecast of the factor returns and $\bar{\epsilon}$ is a forecast of the idiosyncratic returns, *i.e.*, the asset alphas. One common method uses only a forecast of the factor returns, with $\bar{\epsilon} = 0$, so $\mu = F\bar{f}$. A complementary method assumes zero factor returns, so we have $\mu = \bar{\epsilon}$, *i.e.*, the mean asset returns are the same as the idiosyncratic asset mean returns.

Factor betas and neutrality. Under the factor model (14), the covariance of the portfolio return R with the factor returns f is the k-vector

$$\mathbf{cov}(R, f) = \Sigma^{\mathrm{f}} F^T w.$$

The *betas* of the portfolio with respect to the factors divide these covariances by the variance of the factors,

$$\beta = \mathbf{diag}(s^{\mathrm{f}})^{-2} \Sigma^{\mathrm{f}} F^T w,$$

where $s^{\mathrm{f}} = \mathbf{diag}(\Sigma^{\mathrm{f}})^{-1/2}$ is the vector of factor return volatilities.

The constraint that our portfolio return is uncorrelated (or has zero beta) with the ith factor return f_i, under the factor model (14), is

$$\mathbf{cov}(R, f)_i = (\Sigma^f F^T w)_i = 0. \tag{15}$$

This is referred to as *factor neutrality* (with respect to the ith factor). It is a simple linear equality constraint, which can be expressed as $a^T w = 0$, where a is the ith column of $F\Sigma^{\mathrm{f}}$. Factor neutrality constraints are typically used with active weights. In this case, factor neutrality means that the portfolio beta matches the benchmark beta for that factor.

Advantages of a factor model. Especially with large universes, the factor model (specified by F, Σ^{f}, and D) can give a better estimate of the return covariance, compared to methods that directly estimate the $n \times n$ matrix Σ [JOP+23]. Another substantial advantage is computational. By exploiting the low-rank plus diagonal structure of the return covariance with a factor model, we can reduce the computational complexity of solving the Markowitz optimization problem from $O(n^3)$ flops (without exploiting the factor model) to $O(nk^2)$ flops (exploiting the factor form). These computational savings can be dramatic, *e.g.*, for a whole world portfolio with $n = 10000$ and $k = 100$, where we obtain a 10000 fold decrease in solve time; see §5.6.

3.4 Return and risk forecasts

Here we briefly discuss the forecasting of μ and Σ (or F, Σ^{f}, and D in a factor model). Markowitz himself did not address the question of estimating μ and Σ; when asked by practitioners how one should choose these forecasts, his reply was [SS23]

“*That's your job, not mine.*”

It is well documented that poor or naïve estimates of these, *e.g.*, the sample mean and covariance, can yield poor portfolio performance [Mic89]. But even reasonable forecasts will have errors, which can degrade performance. We show some methods to mitigate this forecast uncertainty in §3.5.

Asset returns estimate. The expected returns vector μ is by far the most important parameter in the portfolio construction process, and methods for estimating it, or the factor and idiosyncratic return means are for obvious reasons in general proprietary. It is also the most challenging data to estimate. There is no consensus on how to estimate the mean returns, and the literature is vast.

Regularization methods can improve mean estimates. As an example, the Black-Litterman model [BL90] allows a portfolio manager to incorporate her views on how the expected returns differ from the market consensus, and in essence acts as a form of regularization of the portfolio toward the market portfolio. Another method that serves implicitly as regularization is winsorization, where the mean estimates are clipped when they go outside a specified range [WZ07], [GK00, Chap. 14]. Yet another method is cross-sectionalization, where the preliminary estimate of returns μ is replaced with $\tilde{\mu}$, the same values monotonically mapped to (approximately) a Gaussian distribution [GK00, Chap. 14].

Return covariance estimate. There are many ways to estimate the covariance matrix, with or without a factor model. Approaches that work well in practice include the exponentially weighted moving average (EWMA) [OS96], dynamic conditional correlation (DCC) [Eng02], and iterated EWMA [BB22]. For a detailed discussion on how to estimate a covariance matrix for financial return data, see [JOP+23] and the references therein.

3.5 Making return and risk forecasts robust

In this section we address methods to mitigate the impact of forecast errors in return and covariance estimation, which can lead to poor performance. This directly addresses one of the main criticisms of the Markowitz method, that it is too sensitive to estimation errors. Here, we briefly review how to address robust return mean and covariance estimation, and refer the reader to [BBD+17, §4.3] and [FKPF07, TK04] for more detailed discussions.

Robust return forecast. We model our uncertainty in the mean return vector by giving an interval of possible values for each return mean. We let $\mu \in \mathbf{R}^n$ denote our nominal estimate of the return means, and we take the nonnegative vector $\rho \in \mathbf{R}^n_+$ to describe the half-width or radius of the uncertainty intervals. Thus we imagine that the return can be any vector of the form $\mu + \delta$, where $|\delta_i| \leq \rho_i$. For example $\mu_i = -0.0010$ and $\rho_i = 0.0005$ means that the mean return for asset i lies in the range $[-15, -5]$ bps.

We define the *worst-case mean portfolio return* as the minimum possible mean portfolio return consistent with the given ranges of asset return means:

$$R^{\mathrm{wc}} = \min\{(\mu + \delta)^T w \mid |\delta| \leq \rho\}.$$

We can think of this as an adversarial game. The portfolio manager (PM) chooses the portfolio w, and an adversary then chooses the worst mean return consistent with the given uncertainty intervals. This second step has an obvious solution: We choose $\mu_i - \delta_i$ when $w_i \geq 0$, and we choose $\mu_i + \delta_i$ when $w_i < 0$. In words: For long positions the worst return is the minimum possible; for short positions the worst return is the maximum possible. With this observation, we obtain a simple formula for the worst-case portfolio mean return,

$$R^{\text{wc}} = \bar{R} - \rho^T |w|. \tag{16}$$

The first term is the nominal mean return; the second term, which is nonpositive, gives the degradation of return induced by the uncertainty. We refer to $\rho^T|w|$ as the portfolio *return forecast error penalty* in our return forecast. The return forecast error penalty has a nice interpretation as an uncertainty-weighted leverage.

When the portfolio is long-only, so $w \geq 0$, the worst-case asset returns are obvious: they simply take their minimum values, $\mu - \rho$. In this case the worst-case portfolio mean return (16) is the usual mean portfolio return, with each nominal asset return reduced by its uncertainty.

The return forecast uncertainties ρ can be chosen by several methods. One simple method is to set all entries the same, and equal to some quantile of the entries of $|\mu|$, such as the 20th percentile. A more sophisticated method relies on multiple forecasts of the returns, and sets μ as the mean or median forecast, and ρ as some measure of spread, such as standard deviation, of the forecasts.

Robust covariance forecast. We can also consider uncertainty in the covariance matrix. We let Σ denote our nominal estimate of the covariance matrix. We imagine that the covariance matrix has the form $\Sigma + \Delta$ where $\Delta \in \mathbf{S}^n$ (the set of symmetric $n \times n$ matrices) where the perturbation Δ satisfies

$$|\Delta_{ij}| \leq \varrho(\Sigma_{ii}\Sigma_{jj})^{1/2},$$

where $\varrho \in [0, 1)$ defines the level of uncertainty. For example, $\varrho = 0.04$ means that the diagonal elements of the covariance matrix can change by up to 4% (so the volatilites can change by around 2%), and the asset return correlations can change by up to around 4%. (You should not trust anyone who claims that his asset return covariance matrix estimate is more accurate than this.)

We define the *worst-case portfolio risk* as the maximum possible risk over covariance matrices consistent with our uncertainty set,

$$(\sigma^{\text{wc}})^2 = \max\{w^T(\Sigma + \Delta)w \mid |\Delta_{ij}| \leq \varrho(\Sigma_{ii}\Sigma_{jj})^{1/2}\}.$$

This can be expressed analytically as [BBD+17, §4.3]

$$(\sigma^{\text{wc}})^2 = \sigma^2 + \varrho \left(\sum_{i=1}^{n} \Sigma_{ii}^{1/2} |w_i| \right)^2. \tag{17}$$

The second term is the *covariance forecast error penalty.* It has a nice interpretation as an additive regularization term, the square of a volatility-weighted leverage. The worst-case risk can be expressed using Euclidean norms as

$$\sigma^{\mathrm{wc}} = \left\| \left(\sigma, \sqrt{\varrho}(\mathbf{diag}(\Sigma)^{1/2})^T |w_i|\right) \right\|_2. \tag{18}$$

When the portfolio is long-only, the worst-case risk (17) can be simplified. In this case, the worst-case risk is the risk using the covariance matrix $\Sigma + \varrho s s^T$, where $s = \mathbf{diag}(\Sigma)^{1/2}$ is vector of asset volatilities, under the nominal covariance.

4 Convex optimization formulation

4.1 Markowitz problem

In this section we assemble the objective terms and constraints described in §2 and §3 into one convex optimization problem. We obtain the Markowitz problem

$$\begin{array}{ll} \text{maximize} & R^{\mathrm{wc}} - \gamma^{\mathrm{hold}}\phi^{\mathrm{hold}}(w) - \gamma^{\mathrm{trade}}\phi^{\mathrm{trade}}(z) \\ \text{subject to} & \mathbf{1}^T w + c = 1, \quad z = w - w^{\mathrm{pre}}, \\ & w^{\min} \le w \le w^{\max}, \quad L \le L^{\mathrm{tar}}, \quad c^{\min} \le c \le c^{\max}, \\ & z^{\min} \le z \le z^{\max}, \quad T \le T^{\mathrm{tar}}, \\ & \sigma^{\mathrm{wc}} \le \sigma^{\mathrm{tar}}, \end{array} \tag{19}$$

with variables $w \in \mathbf{R}^n$ and $c \in \mathbf{R}$, and positive parameters γ^{hold} and γ^{trade} that allow us to scale the holding and transaction costs, respectively. Despite the nonlinear and nondifferentiable functions appearing in the objective and constraints, this is a convex optimization problem, which can be very reliably and efficiently solved. We can add other convex objective terms to this problem, such as factor neutrality or liquidation cost limit, or work with active risk and return with a benchmark.

The objective is our forecast of the (robustified, worst-case) net portfolio return, with the holding and transaction costs scaled by the parameters γ^{hold} and γ^{trade}, respectively. The first line of constraints relate the pre-trade portfolio, which is given, and the post-trade portfolio, which is to be chosen. The second line of constraints are weight limits, and the third line contains the trading constraints. The last line of constraints is the (robustified, worst-case) risk limit.

Data. We divide the constants that need to be specified in the problem (19) into two groups, *data* and *parameters*, although the distinction is not sharp. Data are quantities we *observe* (such as the previous portfolio weights) or forecast (such as return means, market volumes, or bid-ask spreads):

- *Pre-trade portfolio weights* w^{pre} and c^{pre}.
- *Asset return forecast* μ.

- *Risk model* Σ, or for a factor model, F, Σ^{f}, and D.
- *Holding cost parameters* κ^{short} and κ^{borrow}.
- *Trading cost parameters* κ^{spread} and κ^{impact} (which in turn depend on the forecast bid-ask spreads, asset volatilities, and market volume).

Parameters. Parameters are quantities that we *choose* in order to obtain good investment performance, or to reflect portfolio manager preferences, or to comply with legal requirements or regulations. These are

- *Target risk* σ^{tar}.
- *Holding and trading scale factors* γ^{hold} and γ^{trade}.
- *Weight and leverage limits* $w^{\min}$, $w^{\max}$, $c^{\min}$, $c^{\max}$, and L^{tar}.
- *Trade and turnover limits* $z^{\min}$, $z^{\max}$, and T^{tar}.
- *Mean and covariance forecast uncertainties* ρ and ϱ.

We list the mean and covariance forecast uncertainties as parameters since they are closer to being chosen than measured or estimated. When the mean return uncertainties are chosen as described above from a collection of return forecasts, they would be closer to data.

Initial default choices for parameters. The target risk, and the weight, leverage, trade, and turnover limits are interpretable and can be assigned reasonable values by the PM. The return and risk uncertainty parameters ρ and ϱ can be chosen as described above. The hold and trade scale factors can be chosen to be around one.

To improve performance the PM will want to adjust or tune these parameter values around their natural or default values, as discussed in §5.4.

4.2 Softening constraints

The Markowitz problem (19) includes a number of constraints. This can present two challenges in practice. First, it can lead to substantial trading, for example to satisfy our leverage or ex-ante worst-case risk limits, even when they would have been violated only slightly, which can lead to poor performance due to excessive trading. Second, the problem can be infeasible, meaning there is no choice of the variables that satisfy all the constraints. This can complicate back-tests or simulations, as well as running the trading policy in production, where such infeasiblilities naturally occur most frequently during periods of market stress, putting the PM under additional pressure.

Soft constraints. Here we explain a standard method in optimization, in which some of the constraints can be softened, which means we allow them to be somewhat violated, if needed. In optimization, softness refers to how much we care about different values of an objective. We can think of the objective as infinitely soft: We will accept any objective value, but we prefer larger values (if we are maximizing). We can think of constraints as infinitely hard: We will not accept any violation of them, even if it is only by a small amount. *Soft constraints*, described below, are in between. They should normally act as constraints, but when needed, they can be violated. When a soft constraint is violated, and by how much, depends on our priorities, with high priority meaning that the constraint should be violated only when absolutely necessary.

Consider a (hard) constraint such as $f \leq f^{\max}$. This means that we will not accept any choice of the variables for which $f > f^{\max}$. To make it a *soft constraint*, we remove the constraint from the problem and form a penalty term

$$\gamma(f - f^{\max})_+$$

which we subtract from the objective, when we are maximizing. The number $(f - f^{\max})_+$ is the *violation* of the original constraint $f \leq f^{\max}$. The positive parameter γ is called the *priority parameter* associated with the softened constraint. In this context, we refer to the parameter $f^{\max}$ as a *target* for the value of f, not a limit. With the softened constraint, we can accept variable choices for which $f > f^{\max}$, but the optimizer tries to avoid this given the penalty paid (in the objective) when this occurs. Softening constraints preserves convexity of a problem.

Markowitz problem with soft constraints. A number of constraints in (19) should be left as (hard) constraints. These include the constraints relating the proposed and previous weights, *i.e.*, the first line of constraints in (19). When the portfolio is long-only, the constraint $w \geq 0$ should be left as a hard constraint, and similarly for a constraint such as $c \geq 0$, *i.e.*, that we do not borrow cash. When a leverage limit is strict or imposed by a mandate, it should be left as a hard constraint; when it is imposed by the portfolio manager to improve performance, or more likely, to help her avoid poor outcomes, it can be softened.

The other constraints in (19) are candidates for softening. Weight and trade limits, including leverage and turnover limits, should be softened (except in the cases described above). The worst-case risk limit $\sigma \leq \sigma^{\text{tar}}$ should be softened, with a risk penalty term

$$\gamma^{\text{risk}}(\sigma - \sigma^{\text{tar}})_+$$

subtracted from the objective. When the associated priority parameter γ^{risk} is chosen appropriately, this allows us to occasionally violate our risk limit a bit when the violation is small. We refer to the softened Markowitz problem as the Markowitz++ problem.

One nice attribute of the Markowitz++ problem is that it is always feasible; the choice $z = 0$, *i.e.*, no trading, is always feasible, even when it is a poor choice. This means that the softened Markowitz problem can be used to define a trading policy that runs with little or no human intervention (with, however, any soft constraints that exceed their targets reported to the portfolio manager).

Priority parameters. When we soften the worst-case risk, leverage, and turnover constraints, we gain several more parameters,

$$\gamma^{\text{risk}}, \quad \gamma^{\text{lev}}, \quad \gamma^{\text{turn}}.$$

Evidently the larger each of these priority parameters is, the more reluctant the optimizer is to violate it. (Here we anthropomorphize the optimization problem solver.) When the priority parameters are large, the associated soft constraints are effectively hard. Beyond these observations, however, it is hard to know what values should be used.

Choosing priority parameters. Here we describe a simple method to obtain a reasonable useful initial values for the priority parameters associated with softened constraints. Our method is based on Lagrange multipliers or dual variables. Suppose we solve a problem with hard constraints, and obtain optimal Lagrange multipliers for each of the constraints. If we use these Lagrange multipliers as priorities in a softened version of the problem, all the original constraints will be satisfied. Roughly speaking, the Lagrange multipliers give us values of priorities for which the soft constraints are effectively hard. We would want to use priority values a bit smaller, so that the original constraints can occasionally be violated.

Now we describe the method in detail. We start by solving multiple instances of the problem with hard constraints, for example in a back-test, recording the values of the Lagrange multipliers for each problem instance (when the problem is feasible). We then set the priority parameters to some quantile, such as the 80th percentile, of the Lagrange multipliers. With this choice of priority parameters, we expect (very roughly) the original constraints to hold around 80% of the time. For hard constraints that are only occasionally tight, another method for choosing the priority parameters is as a fraction of the maximum Lagrange multiplier observed.

Using this method we can obtain reasonable starting values of the priority parameters. The final choice of priority parameters is done by back-testing and parameter tuning, starting from these reasonable values, as discussed in §4.4.

4.3 Nonconvex constraints and objectives

All objective terms and constraints discussed so far are convex, and the Markowitz problem (19), and its softened version, are convex optimization problems. They can be reliably and efficiently solved.

Some other constraints and objective terms are not convex. The most obvious one is that the trades must ultimately involve an integer number of shares. As a few other practical examples, we might limit the number of nonzero weights, or insist on a minimum nonzero weight absolute value. When these constraints are added to (19), the problem becomes nonconvex. Great advances have been made in solvers that handle so-called mixed-integer convex problems [KBLG18], and these can be used to solve these portfolio construction problems. The disadvantage is longer solve time, compared to a similar convex problem, and sometimes, dramatically longer solve time if we insist on solving the problem to global optimality. A convex portfolio construction problem that can be solved in a small fraction of a second can take many seconds, or even minutes or more, to solve when nonconvex constraints are added.

For production, where the problem is solved daily, or even hourly, this is fine. The slowdown incurred with nonconvex optimization is however very bad for back-testing and validation, where many thousands, or hundreds of thousands, of portfolio construction problems are to be solved. One sensible approach is to carry out back-testing using a convex formulation, so as to retain the speed and reliability of a convex optimization, and run a nonconvex version in production. As a variant on this, back-tests using convex optimization can be used for parameter search, and one final back-test with a nonconvex formulation can be used to be sure the results are close. Running backtests using only convex constraints works because the nonconvex constraints typically only have a small impact on the portfolio and its performance.

Heuristics based on convex optimization. Essentially all solvers for nonconvex problems that attempt to find a global solution rely on convex optimization under the hood [HT13]. The issue is that a very large number of convex optimization problems might need to be solved to find a global solution.

But many nonconvex constraints can be handled heuristically by solving just a few convex optimization problem. As a simple example we might simply round the numbers of shares in a trade list to an integer. This rounding should have little effect unless the portfolio value is very small.

Other nonconvex constraints are readily handled by heuristics that involve solving just a handful of convex problems. One general method is called relax-round-solve [DTB18]. We illustrate this method to handle the constraint that the minimum nonzero weight absolute value is 0.001 (10 bps). First we solve the problem ignoring this constraint. If the weights satisfy the constraint, we are done (and the choice is optimal). If not, we set a threshold and divide the assets into those with absolute weight smaller than the threshold, those with weights larger than the threshold, and those which are less than minus the threshold. We then add constraints to the original problem, setting the weights to zero, more than 0.001, and less than -0.001, depending on the weights found in the first problem. These are convex constraints, and when we solve the second time we are guaranteed to satisfy the nonconvex constraint. We thus solve two convex problems. In the first one, we essentially decide which weights will be zero, which will be more than the minimum nonzero long weight, and which will be short more than the minimum. In the second one we adjust all the weights, ensuring that the minimum absolute nonzero weight constraint holds.

4.4 Back-testing and parameter tuning

Back-testing. Back-testing refers to simulating a trading strategy using historical data. To do this we provide the forecasts for all quantities needed, including the mean return and covariance, for Markowitz portfolio construction in each period. In each period these forecasts, together with the parameters, are sent in to the Markowitz portfolio construction method, which determines a set of trades. We then use the *realized* values of return, volatility, bid-ask spread, and market volume to compute the (simulated) realized net return R_t^{net}, where the subscript gives the time period. Note that while the Markowitz trading engine uses forecasts of various quantities, the simulation uses the realized historical values. This gives a reasonably realistic approximation of what the result would have been, had we actually carried out this trading. (It is still only an approximation, since it uses our particular trading cost model. Of course a more complex or realistic trading cost model could be used for simulation.) The back-test simulation can also include practical aspects like trading only an integer number of shares or blocks of shares. The simulation can also include external cash entering or leaving the portfolio, such as liabilities that must be paid each period.

In the simulation we log the trajectory of the portfolio. We can compute various quantities of interest such as the realized return, volatility, Sharpe or information ratio, turnover, and leverage, all potentially over multiple time periods such as quarters or years. We can determine the portfolio value versus periods, given by

$$V_t = V_1 \prod_{\tau=1}^{t-1} (1 + R_\tau^{\text{net}}),$$

where V_1 is the portfolio value at period $t = 1$. From this we can evaluate quantities like the average or maximum value of drawdown over quarters or years.

Variations. The idea of back-testing or simulating portfolio performance can be used for several other tasks. In one variation on a back-test called a *stress test*, we use historical data modified to be more challenging, *e.g.*, lower returns or higher costs than actually occurred.

Another variation called *performance forecasting* uses data that are simulated or generated, starting from the current portfolio out to some horizon like one year in the future, or the end of current fiscal year. We generate some number of possible future values of quantities such as returns, along with the corresponding forecasts of them, and simulate the performance for each of these. This gives us an idea of what we can expect our future performance to be, for example as a range of values or quantiles.

Yet another variation is a *retrospective what-if* simulation. Here we take a live portfolio and go back, say, three months. Starting from the portfolio holdings at that time, we simulate forward to the present, after making some changes to our trading method, *e.g.*, modifying some parameters. The fact that the current portfolio value would be higher (according to our simulation) if the PM had reduced the target risk three months ago is of course not directly actionable. But it still very useful information for the PM.

Parameter tuning. Perhaps the most important use of back-testing is to help the PM choose parameter values in the Markowitz portfolio construction problem. While some parameters, like the target risk, are given, others are less obvious. For example, how should we choose γ^{trade}? The default value of one is our best guess of what the single period transaction cost will be. But perhaps we get better performance with $\gamma^{\text{trade}} = 2$, which means, roughly speaking, that we are exaggerating trading cost by a factor of two. The result, of course, is a reduction in trading compared to the default value one. This will result in smaller realized transaction costs, but also, possibly, higher return, or smaller drawdown. The back-test will reveal what would happen in this case (to the limits of the back-test accuracy).

To choose among a set of choices for parameters, we carry out a back-test with each set, and evaluate multiple metrics, such as realized returns, volatility, and turnover. Our optimization problem contains target values for these, based on our forecasts and models; in a back-test we obtain the ex-post or realized values of these metrics.

To make a final choice of parameters, we must scalarize our metrics, *i.e.*, create one scalar metric from them, so we can choose among different sets of parameter values. For example we might choose to maximize Sharpe ratio, subject to other metrics being within specified bounds. Or we could form some kind of weighted combination of the individual metrics.

At the very minimum, a PM should always carry out back-tests in which all of her chosen parameters are, one by one, increased or decreased by, say, 20%. Even with 10 parameters, this requires only 20 back-tests. If any of these back-tests results in substantially improved performance, she will need to explain or defend her choices.

This simple method of changing one parameter at a time can be extended to carry out a crude but often effective parameter search. We cycle over the parameters, increasing or decreasing each and carrying out a back-test. When we find a new set of parameter values that has better performance than the current set of values, we take it as our new values. This continues until increasing or decreasing each parameter value does not improve performance.

Another traditional method of parameter tuning is *gridding*. We choose a small number of candidate values for each parameter, and then carry out a back-test for each combination, evaluating multiple performance metrics. Of course this is practical only when we are choosing just a few parameters, and we consider only a few candidate values for each one. Gridding is often carried out with a first crude parameter gridding, with the candidate values spaced by a factor of ten or so; then, when good values of these parameters are found, a more refined grid search is used to focus in on parameters near the good ones found in the first crude search. In any case there is no reason to find or specify parameter values very accurately; specifying them to even 10% is not needed. For one thing, the back-test itself is only an approximation. To put in a negative light, if a back-test reveals that $\gamma^{\text{trade}} = 2.1$ works well, but that $\gamma^{\text{trade}} = 1.9$ and $\gamma^{\text{trade}} = 2.3$ work poorly, it is very unlikely that our trading method will work well in practice. Similar to the way we want our trading policy to be robust to variations in the input data, we also want it to be robust to variation in the parameters.

More sophisticated parameter search methods can also be used. Many such methods build a statistical model of the good parameter values found so far, and obtain new values to try by sampling from the distribution; see, *e.g.*, [MBK+22] for more discussion. Another option is to obtain not just the value of some composite metric, but also its gradient with respect to the parameters. This very daunting computation can be carried out by automatic differentiation systems that can differentiate through the solution of a convex optimization problem, such as CVXPYlayers [AAB+19, BAB20].

5 Numerical experiments

In this section we present numerical experiments that illustrate the ideas and methods discussed above. In the first set of experiments, described in §5.2, we show the effect of several constraints and objective terms that serve as effective regularizers and improve performance. In §5.4 we illustrate how parameter tuning via back-tests can improve performance, and in §5.6 we show how the methods we describe scale with problem size.

5.1 Data and back-tests

Data. Throughout the experiments we use the same data set, which is based on the stocks in the S&P 100 index. We use daily adjusted close price data from 2000-01-04 to 2023-09-22. We exclude stocks without data for the entire period, and acknowledge that this inherent survivorship bias in the data set would make it unsuitable for a real portfolio construction method, but it is sufficient for our experiment, which is only concerned with the relative performance of the different methods. We end up with a universe of $n = 74$ assets. In addition to the price data, we use bid-ask spread data to estimate the trading costs, as well as the effective federal funds rate [Fed23] for short term borrowing and lending. We make the data set available with the code for reproducibility and experimentation at

https://github.com/cvxgrp/markowitz-reference.

Mean prediction. Simple estimates of the means work poorly, so in the spirit of [BBD+17], we use synthetic return predictions to simulate a proprietary mean prediction method. For each asset, the synthetic returns for each day are given by

$$\hat{r}_t = \alpha(r_t + \epsilon_t),$$

where ϵ_t is a zero-mean Gaussian noise term with variance chosen to obtain a specified information coefficient and r_t is the mean return of the asset in the week starting on day t. We take the noise variance to be $\sigma^2(1/\alpha - 1)$, where α is the square of the information coefficient, and σ^2 is the variance of the return. (These mean predictions are done for each asset separately.) We choose an information coefficient of $\sqrt{\alpha} = 0.15$. Using this parameterization, the sign of the return is predicted correctly in 52.1% of all observations, with this number ranging from 50.3% to 54.1% for the individual assets.

Covariance prediction. For the covariance prediction, we use a simple EWMA estimator, *i.e.*, the covariance matrix at time t is estimated as

$$\hat{\Sigma}_t = \alpha_t \sum_{\tau=1}^{t} \beta^{t-\tau} r_\tau r_\tau^T,$$

where

$$\alpha_t = \left(\sum_{\tau=1}^{t} \beta^{t-\tau} \right)^{-1} = \frac{1-\beta}{1-\beta^t}$$

is the normalization constant, and $\beta \in (0, 1)$ is the decay factor. (We use the second moment as the covariance, since the contribution from the mean term is negligible.) We use a half-life of 125 trading days, which corresponds to a decay factor of $\beta \approx 0.994$. We note that the specific choice of the half-life does not change the results of the experiments qualitatively.

Spread. Our simulations include the transaction cost associated with bid-ask spread. In simulation we use the realized bid-ask spread; for the Markowitz problems we use a simple forecast of spread, the average realized bid-ask spread over the previous five trading days.

Shorting and leverage costs. We use the effective federal funds rate as a proxy for interest on cash for both borrowing and lending. When shorting an asset we add a 5% annualized spread over the effective federal funds rate to approximate the shorting cost in our simulation. For forecasting, we set κ^{short} to 7.5% annualized, and κ^{borrow} to the effective federal funds rate.

Back-tests. We use a simple back-test to evaluate the performance of the different methods. We start with a warm-up period of 500 trading days for our estimators leaving us with 5,686 trading days, or approximately 22 years of data. The first 1,250 trading days (five years) are used to initialize the priority parameters. This leaves us with 4,436 out-of-sample trading days, approximately 17 years. Starting with an initial cash allocation of \$1,000,000, we call the portfolio construction method each day to obtain the target weights. We then execute the trades at the closing price, rebalancing the portfolio to the new target weights, taking into account the weight changes due to the returns from the previous day. Buy and sell orders are executed at the ask and bid prices, respectively, and interest is paid on borrowed cash and short stocks, and received on cash holdings.

5.2 Taming Markowitz

In this first experiment we show how a basic Markowitz portfolio construction method can lead to the undesirable behavior that would prompt the alleged deficiencies described in §1.2. We then show how adding just one more reasonable constraint or objective term improves the performance, taming the basic Markowitz method.

Basic Markowitz. We start by solving the basic Markowitz problem (1) for each day in the data set, with the risk target set to 10% annualized volatility. Unsurprisingly the basic Markowitz problem results in poor performance, as seen in the second line of table 1. It has low mean return, high volatility (well above the target 10%), a low Sharpe ratio, high leverage and turnover, and a maximum drawdown of almost 80%. This basic Markowitz portfolio performs considerably worse than an equal-weighted portfolio, which we give as a baseline on the top line of table 1.

	Return	Volatility	Sharpe	Turnover	Leverage	Drawdown
Equal weight	14.1%	20.1%	0.66	1.2	1.0	50.5%
Basic Markowitz	3.7%	14.5%	0.19	1145.2	9.3	78.9%
Weight-limited	20.2%	11.5%	1.69	638.4	5.1	30.0%
Leverage-limited	22.9%	11.9%	1.86	383.6	1.6	14.9%
Turnover-limited	19.0%	11.8%	1.54	26.1	6.5	25.0%
Robust	15.7%	9.0%	1.64	458.8	3.2	24.7%
Markowitz++	38.6%	8.7%	4.32	28.0	1.8	7.0%
Tuned Markowitz++	41.8%	8.8%	4.65	38.6	1.6	6.4%

Table 1: Back-test results for different trading policies.

Markowitz with regularization. In a series of four experiments we show how adding just one more reasonable constraint or objective term to the basic Markowitz method can greatly improve the performance.

In the first experiment we add portfolio weight limits of 10% for long positions and -5% for short positions. We limit the cash weight to lie between -5% and 100% (which guarantees feasibility). Adding these asset and cash weight limits leads to a significant improvement in the performance of the portfolio shown in the third row of table 1, with the Sharpe ratio increasing to 1.69 (from 0.19), and the maximum drawdown decreasing to 30%. In addition the realized volatility, 11.5%, is closer to the target value 10% than the basic Markowitz trading policy. The turnover is still very high, however, and the maximum leverage is still large at above 5.

In the second experiment we add a leverage limit to the basic Markowitz problem, with $L^{\text{tar}} = 1.6$. This one additional constraint also greatly improves performance, as seen in the fourth row of table 1, but with a lower turnover and (not surprisingly) a lower maximum leverage, which is at our target value 1.6.

Our third experiment adds a turnover limit of $T^{\text{tar}} = 25$ to the basic Markowitz problem. This additional constraint drops the turnover considerably, to a value near the target, but still achieves high return, Sharpe ratio, and even lower maximum drawdown.

Our fourth experiment adds robustness to the return and risk forecasts. As simple choices we set all entries of ρ to the 20th percentile of the absolute value of the return forecast at each time step, and use $\varrho = 0.02$. This robustification also improves performance. Not surprisingly the realized risk comes in under our target, since we use the robust risk ex-ante; we could achieve realized risk closer to our desired target 10% by increasing the target to something like 11.5% (which we didn't do).

5.3 Markowitz++

In the four experiments described above, we see that adding just one reasonable additional constraint or objective term to the basic Markowitz problem greatly improves the performance. In our last experiment, we include all of these constraints and terms, with parameters

$$\gamma^{\text{hold}} = 1, \quad \gamma^{\text{trade}} = 1, \quad \sigma^{\text{tar}} = 0.10$$
$$c^{\min} = -0.05, \quad c^{\max} = 1.00, \quad w^{\min} = -0.05, \quad w^{\max} = 0.10, \quad L^{\text{tar}} = 1.6$$
$$z^{\min} = -0.10, \quad z^{\max} = 0.10, \quad T^{\text{tar}} = 25.$$

The mean uncertainty parameter ρ is chosen as the 20th percentile of the absolute value of the return forecast, and $\varrho = 0.02$. We soften the risk target, leverage limit, and turnover limit, using the priority parameters

$$\gamma^{\text{risk}} = 5 \times 10^{-2}, \quad \gamma^{\text{lev}} = 5 \times 10^{-4}, \quad \gamma^{\text{turn}} = 2.5 \times 10^{-3}.$$

These were chosen as the 70th percentiles for the corresponding Lagrange multipliers of the hard constraints in the basic Markowitz problem for the risk and turnover limits, and as 25% of the maximum Lagrange multiplier for the leverage limit, over the five years leading up to the out-of-sample study. (We selected γ^{lev} this way since the corresponding constraint was active very rarely in the basic Markowitz problem.)

With this Markowitz++ method, we obtain the performance listed in the second from bottom row of table 1. It is considerably better than the performance achieved by adding just one additional constraint, as in the four previous experiments, and very much better than the basic Markowitz method. Not surprisingly it achieves good performance on all metrics, with a high Sharpe ratio, reasonable tracking of our volatility target, modest turnover and leverage, and very small maximum drawdown. When the parameters are tuned annually, as detailed in the next section, we see even more improvement, as shown in the bottom row of table 1.

The Sharpe ratios on the bottom two rows are high. We remind the reader that our data has survivorship bias and uses synthetic (but realistic) mean return forecasts, so the performance should not be thought of as implementable. But the differences in performance of the different trading methods is significant.

5.4 Parameter tuning

In this section we show how parameter tuning can be used to improve the performance of the portfolio construction method. We will tune the parameters γ^{hold}, γ^{trade}, γ^{lev}, γ^{risk}, and γ^{turn}, keeping the other parameters fixed. We start from the values used in Markowitz++.

Experimental setup. We tune the parameters at the start of every year, on the previous two years of data, and then fix the tuned parameters for the following year. To tune the parameters we use the simple cyclic tuning method described in §4.4. We cycle through the parameters one by one. Each time a parameter is encountered in the loop, we increase it by 25%; if this yields an improvement in the performance (defined below), we keep the new value and continue with the next parameter; if not, we decrease the parameter by 20% and check if this yields an improvement. We continue this process until a full loop through all parameters does not yield any improvement. By improvement in performance we mean that all the following are satisfied:

- The in-sample Sharpe ratio increases.
- The in-sample annualized turnover is no more than 50.
- The in-sample maximum leverage is no more than 2.
- The in-sample annualized volatility is no more than 15%.

Results. Tuning the parameters every year yields the performance given in the last row of table 1. We see a modest but significant boost in performance over untuned Markowitz++.

The tuned parameters over time are shown in figure 1. We can note several intuitive patterns in the parameter values. For example, γ^{risk} increases during 2008 to account for the high uncertainty in the market during this period. Similarly, γ^{turn} decreases during the same period, likely to allow us to trade more freely to satisfy the other constraints; interestingly γ^{trade} increases during the same period, likely to push us toward more liquid stocks when trading increases. During the same period γ^{lev} increases to reduce leverage. Similar patterns can be observed in 2020.

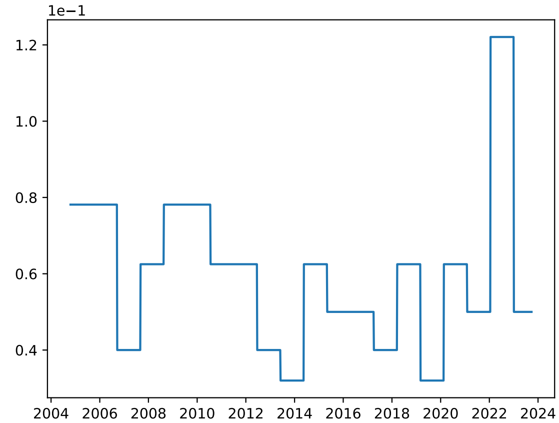
Tuning evolution. Here we show an example of the evolution of tuning, showing both in- and out-of-sample values of Sharpe ratio, turnover, leverage, and volatility. The in-sample period is April 19, 2016 to March 19, 2018, and the out-of-sample period March 20, 2018 to March 4, 2019. These are shown in figure 2. This tuning process converged after 45 back-tests to the parameter values

$$\gamma^{\text{risk}} = 4 \times 10^{-2}, \quad \gamma^{\text{hold}} = 0.64, \quad \gamma^{\text{trade}} = 0.64, \quad \gamma^{\text{lev}} = 5 \times 10^{-4}, \quad \gamma^{\text{turn}} = 1.6 \times 10^{-3}.$$

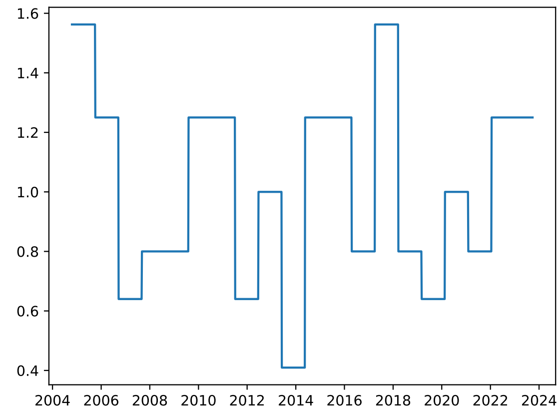
We can see that tuning increases the Sharpe ratio both in- and out-of-sample, while keeping the leverage, turnover, and volatility reasonable. In this example we end up changing 4 of our 5 adjustable parameters, although not by much, which shows that our initial default parameter values were already quite good. Still, we obtain a significant boost in performance.

5.5 Annual performance

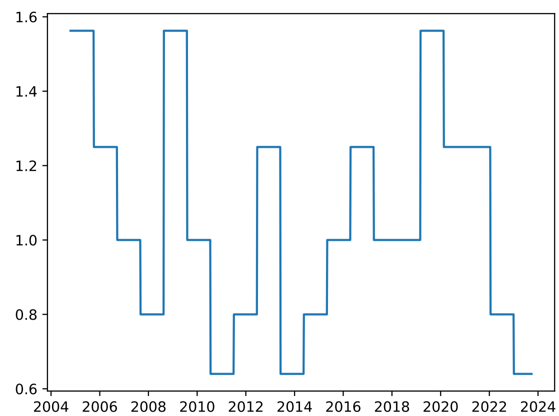
The performance analyses described above and summarized in table 1 give aggregate metrics over a 17 year out-of-sample period, long enough to include multiple distinct market regimes as a well as a few market crashes. For such a long back-test, it is interesting to see how the performance in individual years varies with different market regimes. The realized annual return, volatility, and Sharpe ratio are shown in figure 3, for basic Markowitz, equal weights, and tuned Markowitz++. Here we see that Markowitz++ not only gives the performance improvements seen in table 1, but in addition has less variability in performance across different market regimes.



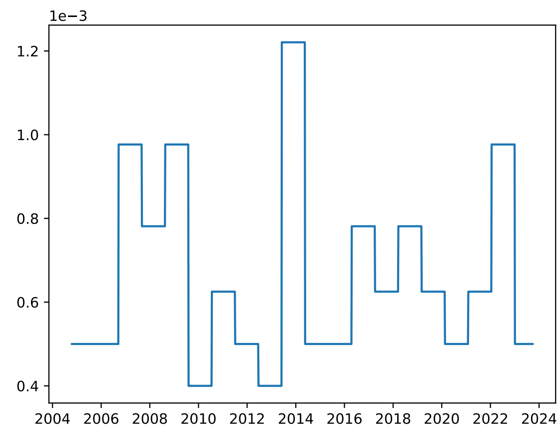
(a) γ^{risk}



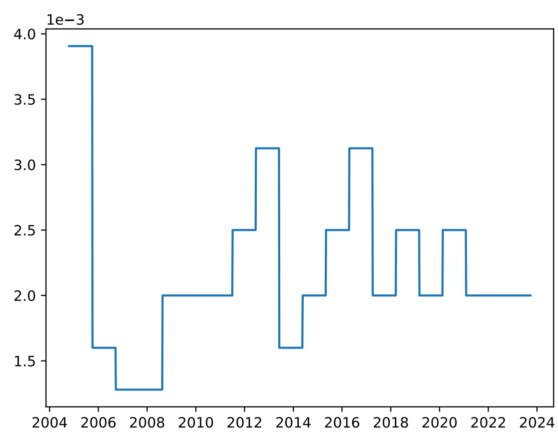
(b) γ^{hold}



(c) γ^{trade}

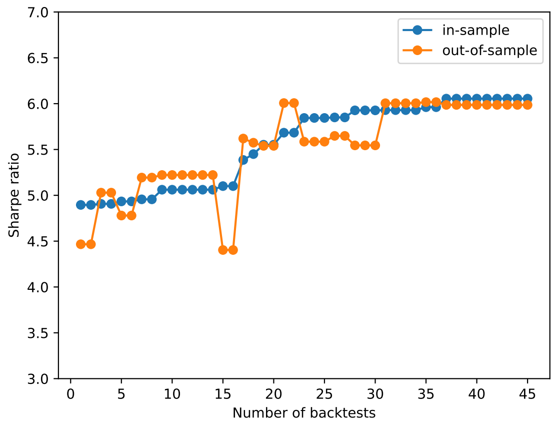


(d) γ^{lev}

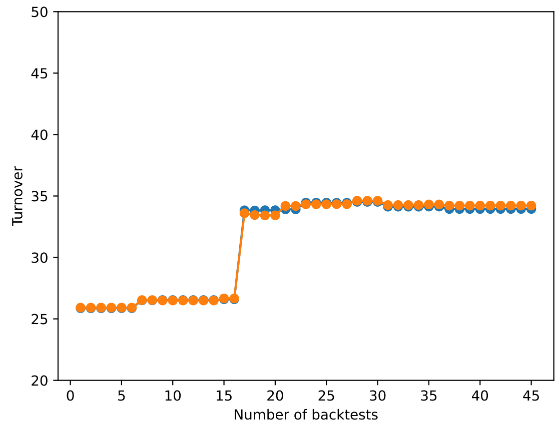


(e) γ^{turn}

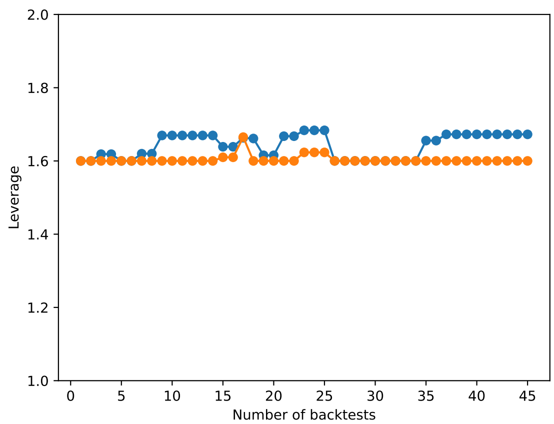
Figure 1: Tuned parameters over time.



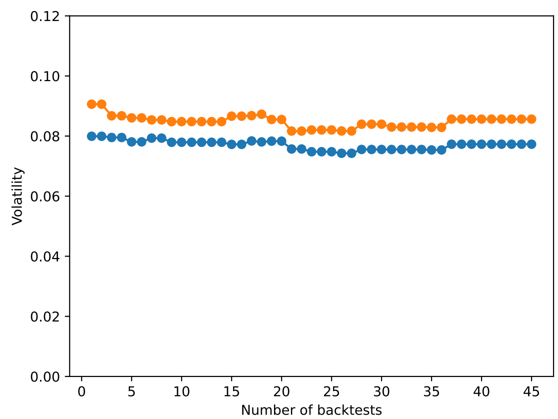
(a) Sharpe ratio.



(b) Turnover.

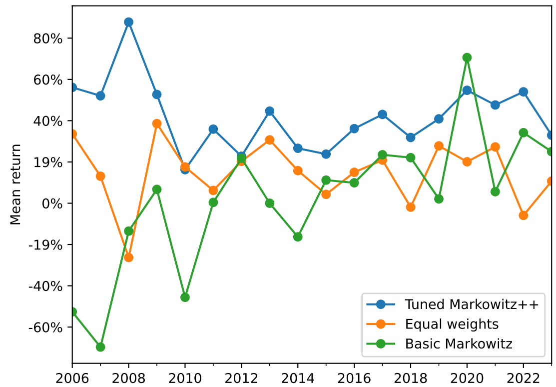


(c) Leverage.

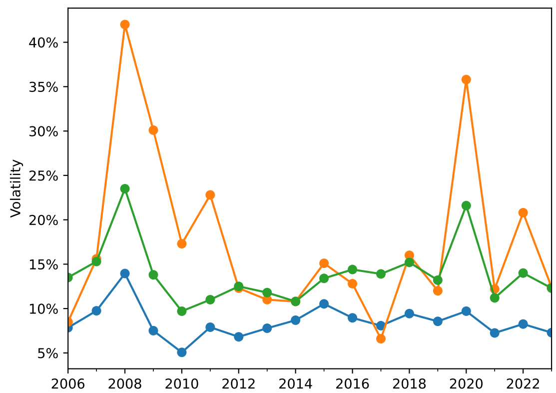


(d) Volatility.

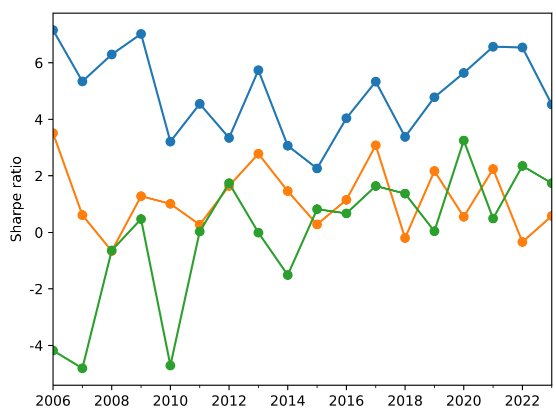
Figure 2: Parameter tuning results.



(a) Yearly annualized returns.



(b) Yearly annualized volatilities.



(c) Yearly annualized mean Sharpe ratios.

Figure 3: Yearly annualized metrics for the equal weight portfolio, basic Markowitz, and tuned Markowitz++.

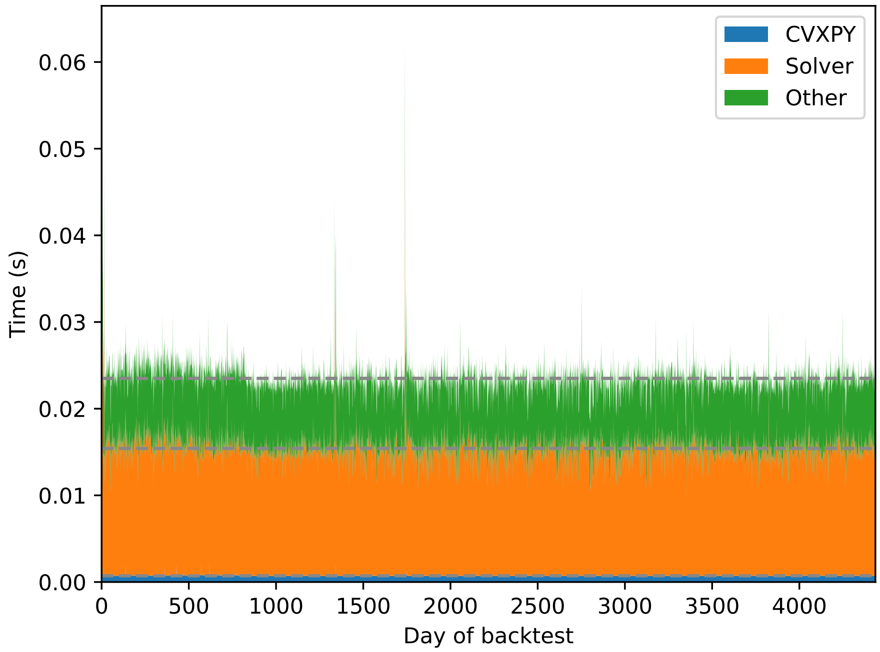


Figure 4: Timing results for the Markowitz problem with a single backtest setting.

5.6 Scaling

We now turn from the performance of the portfolios to the algorithmic performance of the portfolio construction method itself.

Small problems. We start with the small problem used in the previous section, with $n = 74$ assets, and without a factor risk model. Figure 4 shows the time required for each of the 4,436 days in the back-test, broken down into updating and logging (shown in green), CVXPY overhead (shown in blue, negligible), and solver time, the time required to solve the resulting cone program. (We do not count factorizing the covariance matrix, or computing the mean forecasts, since these are done ahead of time, and the time is amortized across all back-tests.)

The 17 year back-test, which involve solving 4,436 problems, takes around 104 seconds on a MacBook Pro with an M1 Pro processor, or about 23ms per day on average. About 63% of the time is spent in the solver, which in this case is MOSEK [ApS20], with other solvers giving similar results, including open-source solvers such as ECOS [DCB13], Clarabel [GC24], and SCS [OCPB16]. Only 3% of the time is spent in the compilation step using CVXPY. The averages for each component of the timings are indicated by the horizontal lines in the figure.

Assets n	Factors k	Solve time (s)
100	10	0.01
500	20	0.07
500	50	0.10
2,000	50	0.23
2,000	100	0.22
10,000	50	0.65
10,000	100	0.89
50,000	200	9.00
50,000	500	17.77

Table 2: Average solve times for Markowitz++ problem for MOSEK, for different problem sizes.

For a small problem like this one, we can carry out a one-year back-test (around 250 trading days) in around six seconds, on a single thread. A single processor with 32 threads can carry out round 20,000 one-year back-tests in an hour. There is little excuse for a PM who does not carry out many back-tests, even if only to vary the parameters around their chosen values.

Large problems. We now investigate the scaling of the method with problem size. As outlined in §3.3, a factor model improves the scaling from $O(n^3)$ to $O(nk^2)$. To illustrate this, we solve the Markowitz problem for different values of n and k using randomly generated but realistic data. Table 2 shows the average solve time for each problem size across 30 instances using MOSEK. (Solve times with open-source solvers such as Clarabel were a bit longer.) We can see that even very large problems can be solved with stunning speed.

We solved many more problems than those shown in table 2, and used the solve times to fit a log-log model, approximating the solve time as an^bk^c, with parameters a, b, c. We obtained coefficients $b = 0.79$ and $c = 1.72$, consistent with the theoretical scaling of $O(nk^2)$.

When the problems are even larger, generic software reaches its limits. In such cases, users may consider switching to first-order methods like the Alternating Direction Method of Multipliers (ADMM) [MGBK22, Fou23, FB18, PB13, BPC+11], which can offer better scalability and efficiency for very large problems.

6 Conclusions

It was Markowitz's great insight to formulate the choice of an investment portfolio as an optimization problem that trades off multiple objectives, originally just expected return and risk, taken to be the standard deviation of the portfolio return. His original proposal yielded an optimization problem with an analytical solution for the long-short case, and a QP for the long-only case, both of which were tractable to solve (for very small problems) even in the 1950s. Since then, stunning advances in computer power, together with advances in optimization, now allow us to formulate and solve much more complex optimization problems, that directly handle various practical constraints and mitigate the effects of forecasting errors. We can solve these problems fast enough that very large numbers of back-tests can be carried out, to give us a good idea of the performance we can expect, and to help choose good values of the parameters. It is hardly surprising that these methods are widely used in quantitative trading today.

While we have vastly more powerful computers, far better software, and easier access to data, than Markowitz did in 1952, we feel that the more complex Markowitz++ optimization problem simply realizes his original idea of an optimization-based portfolio construction method that takes multiple objectives into account.

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A Coding tricks

The problem described in §4.1 can essentially be typed directly into a DSL such as CVXPY, with very few changes. In this section we mention a few simple tricks in formulating the problem (for a DSL) that lead to better performance.

Quadratic forms versus Euclidean norms. Traditional portfolio construction optimization formulations use quadratic forms such as $w^T \Sigma w$. Modern convex optimization solvers can directly handle the Euclidean norm without squaring to obtain a quadratic form. Using norm expressions instead of quadratic forms is often more natural, and has better numerical properties. For example a risk limit, traditionally expressed using a quadratic form as

$$w^T \Sigma w \leq (\sigma^{\mathrm{tar}})^2,$$

is better expressed using a Euclidean norm as

$$\|L^T w\|_2 \leq \sigma^{\mathrm{tar}},$$

where L is the Cholesky factor of Σ, *i.e.*, $LL^T = \Sigma$, with L lower triangular with positive diagonal entries.

Exploiting the factor model. To exploit the factor model, it is critical to *never* form the covariance matrix $\Sigma = F \Sigma^{\mathrm{f}} F^T + D$. The first disadvantage of doing this is that we have to (needlessly) store an $n \times n$ matrix, which can be a challenge when n is on the order of tens of thousands. In addition, the solver will be slowed by a dramatic factor as mentioned in §3.3.

To exploit the factor model, we introduce the data matrix $\tilde{F} = FL$, where L is the Cholesky factor of Σ^{f}, so $\tilde{F}\tilde{F}^T = F \Sigma^{\mathrm{f}} F^T$. The portfolio variance is

$$\sigma^2 = w^T \tilde{F}\tilde{F}^T w + w^T D w = \|\tilde{F}^T w\|_2^2 + \|D^{1/2} w\|_2^2,$$

so the risk can be expressed using Euclidean norms as

$$\sigma = \left\| \left(\|\tilde{F}^T w\|_2, \|D^{1/2} w\|_2 \right) \right\|_2 .$$

In this expression, the outer norm is of a 2-vector; the inner lefthand norm is of a k-vector, and the inner righthand norm is of an n-vector. Here we should be careful to express D as a diagonal matrix, or to express $D^{1/2} w$ as the elementwise (Hadamard) product of two vectors.

B CVXPY code listing

We provide a reference implementation for the problem described in §4. This implementation is not optimized for performance, contains no error checking, and is provided for illustrative purposes only. For a more performant and robust implementation, we refer the reader to the cvxmarkowitz package [Gro23]. Below, we assume that the data and parameters are already defined in corresponding data structures. The complete code for the reference implementation is available at

https://github.com/cvxgrp/markowitz-reference.

```
import cvxpy as cp

w, c = cp.Variable(data.n_assets), cp.Variable()

z = w - data.w_prev
T = cp.norm1(z) / 2
L = cp.norm1(w)

# worst-case (robust) return
factor_return = (data.F @ data.factor_mean).T @ w
idio_return = data.idio_mean @ w
mean_return = factor_return + idio_return + data.risk_free * c
return_uncertainty = param.rho_mean @ cp.abs(w)
return_wc = mean_return - return_uncertainty

# worst-case (robust) risk
factor_risk = cp.norm2((data.F @ data.factor_covariance_chol).T @ w)
idio_risk = cp.norm2(cp.multiply(data.idio_volas, w))
risk = cp.norm2(cp.hstack([factor_risk, idio_risk]))
risk_uncertainty = param.rho_covariance**0.5 * data.volas @ cp.abs(w)
risk_wc = cp.norm2(cp.hstack([risk, risk_uncertainty]))

asset_holding_cost = data.kappa_short @ cp.pos(-w)
cash_holding_cost = data.kappa_borrow * cp.pos(-c)
holding_cost = asset_holding_cost + cash_holding_cost

spread_cost = data.kappa_spread @ cp.abs(z)
impact_cost = data.kappa_impact @ cp.power(cp.abs(z), 3 / 2)
trading_cost = spread_cost + impact_cost

objective = (
    return_wc
```

```
33     - param.gamma_hold * holding_cost
34     - param.gamma_trade * trading_cost
35 )
36
37 constraints = [
38     cp.sum(w) + c == 1,
39     param.w_min <= w, w <= param.w_max,
40     L <= param.L_tar,
41     param.c_min <= c, c <= param.c_max,
42     param.z_min <= z, z <= param.z_max,
43     T <= param.T_tar,
44     risk_wc <= param.risk_target,
45 ]
46
47 problem = cp.Problem(cp.Maximize(objective), constraints)
48 problem.solve()
```

We start by importing the CVXPY package in line 1 and define the variables of the problem in line 3. The variable `w` is the vector of asset weights, and `c` is the cash weight. We then define the trade vector `z`, turnover `T`, and leverage `L` in lines 5–7 to simplify the notation in the remainder of the code.

In the next block we first define the mean return in lines 10–12, taking into account the factor and idiosyncratic returns, as well as the risk-free rate. We then define the uncertainty in the mean return in line 13, which then reduces the mean return to the worst-case return in line 14.

Similarly, the robust risk is obtained in lines 17–21 by first defining the factor and idiosyncratic risk components, which are combined to the portfolio risk. The uncertainty in the risk, which depends on the asset volatilities, is combined with the portfolio risk to obtain the worst-case risk in line 21. The holding cost is defined in lines 23–25, followed by the trading cost in lines 27–29.

We form the objective function in lines 31–35 by combining the worst-case return with the holding and trading costs, weighted by the corresponding parameters. The constraints are collected in lines 37–45, starting with the budget constraint, followed by the holding and trading constraints, and ending with the risk constraint.

Finally, the problem is defined in line 47, combining the objective and constraints. It is solved in line 48 by simply calling the `.solve()` method on the problem instance, with a suitable solver being chosen automatically.

In only 48 lines of code we have defined and solved the Markowitz problem with all the constraints and objectives described in §4. This underlines the power of using a DSL such as CVXPY to specify convex optimization problems in a way that closely follows the mathematical formulation.

Parameters. Using parameters can provide both a convenient way to specify the problem, as well as a way to reduce the overhead of CVXPY when solving multiple instances. To obtain this speedup requires some restrictions on the problem formulation. For a precise definition we refer the reader to [AAB+19]. Here we only mention that we require expressions to additionally be linear, or affine, in the parameters. For example, we can use CVXPY parameters to easily and quickly change the mean return by writing to the the `.value` attribute of the `mean` and `risk_free` parameters.

```
mean = cp.Parameter(n_assets)
risk_free = cp.Parameter()

mean_return = w @ mean + risk_free * c
```

In some cases, it is necessary to reformulate the problem to satisfy the additional restrictions required to obtain the speedup, *e.g.*, by introducing auxiliary variables. For convenience, we provide a parametrized implementation of the Markowitz problem in the code repository, where these reformulations have already been carried out.