Multitone Signals with Low Crest Factor

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Abstract — Using some results from the recent mathematics literature, we show how to generate signals with perfect low-pass or bandpass spectra which have very low crest factors (under 6 dB). An application to multitone frequency response testing is given.

I. Notation and Preliminaries

We will be concerned with periodic signals. For a T-periodic signal u, the L^∞ and L^2 norms are defined by

$$\|u\|_\infty \triangleq \sup_t |u(t)|$$

and

$$\|u\|_2 \triangleq \left(\frac{1}{T}\int_0^T (u(t))^2\,dt\right)^{1/2}.$$

In engineering terms, $\|u\|_\infty$ is just the *peak* of the signal u; $\|u\|_2$ is its rms value.

We define the *crest factor* of a nonzero signal as the ratio of its peak to rms value

$$CF(u) \triangleq \frac{\|u\|_\infty}{\|u\|_2}.$$

Often the crest factor is given in decibels, i.e., 20 log $CF(u)$.

It is an important fact that the crest factor is always at least one. This can be seen as follows: $u(t)^2 \leqslant \|u\|_\infty^2$ for all t, hence

$$\frac{1}{T}\int_0^T (u(t))^2\,dt \leqslant \|u\|_\infty^2.$$

Taking square roots and dividing yields $CF(u) \geqslant 1$.

A signal can have a crest factor of one only if it is a switching signal, that is, takes on only the values $\pm\|u\|_\infty$. Intuitively, a signal with low crest factor spends most of its time near the values $\pm\|u\|_\infty$. This can be made precise by studying the *ampli-*

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tude distribution of a signal u

$$F_u(a) \triangleq T^{-1}\mu(\{t \in [0,T] \,|\, |u(t)| > a\})$$

where $\mu(E)$ denotes the total length (technically, Lebesgue measure) of the set E. Thus, $F_u(a)$ is the proportion of time the signal exceeds a in absolute value, and so, of course, decreases as a increases. In terms of F_u

$$\|u\|_\infty = \min\{a \,|\, F_u(a) = 0\} \tag{1}$$

and

$$\|u\|_2 = \left(\int_0^\infty 2aF_u(a)\,da\right)^{1/2} \tag{2}$$

which can be seen as follows. In general

$$\frac{1}{T}\int_0^T |u(t)|\,dt = \int_0^\infty F_u(a)\,da \tag{3}$$

(which in fact is another important norm of u, the L^1 norm, $\|u\|_1$). Noting that the amplitude distribution of u^2 is $F_{u^2}(a) = F_u(\sqrt{a})$, (3) yields

$$\frac{1}{T}\int_0^T (u(t))^2\,dt = \int_0^\infty F_{u^2}(a)\,da$$
$$= \int_0^\infty F_u(\sqrt{a})\,da = \int_0^\infty 2aF_u(a)\,da$$

which gives (2) on taking square roots.

Thus $\|u\|_2^2$, which is the total power in u, is a weighted integral of its amplitude distribution function. From (1.1) and (1.2), it is clear that the amplitude distribution which maximizes $\|u\|_2$ subject to $\|u\|_\infty = A$ is one for $a < A$ and then zero for $a \geqslant A$. Moreover the amplitude distributions of low crest factor signals must do most of their decreasing near $a = \|u\|_\infty$. Figures 1 and 4 show two unit-power signals and their amplitude distributions. The first has a large crest factor of 8 and an appropriately spread out amplitude distribution; the second has a lower crest factor of 2 and a less spread out amplitude distribution.

All of the following is independent of the period T so for notational simplicity we will take $T = 2\pi$ in the sequel.

II. Crest Factor of Multitone Signals

Consider the multitone signal

$$u(t) = \sqrt{\frac{2}{N}} \sum_{k=N_0+1}^{N_0+N} \cos(kt + \delta_k) \tag{4}$$

where $N_0 \geqslant 0$. The Fourier coefficients of u satisfy

$$|\hat{u}_k|^2 = \begin{cases} \dfrac{2}{N}, & N_0 < k \leqslant N_0 + N \\ 0, & \text{otherwise} \end{cases}. \tag{5}$$

Thus, the rms value of u is $\|u\|_2 = \sqrt{\frac{1}{2}\Sigma|\hat{u}_k|^2} = 1$, regardless of the phases δ_k. For this reason, we call u a *unit power signal* with perfect *bandpass spectrum* (if $N_0 > 0$) or perfect *low-pass* or *boxcar spectrum* (if $N_0 = 0$).[1]

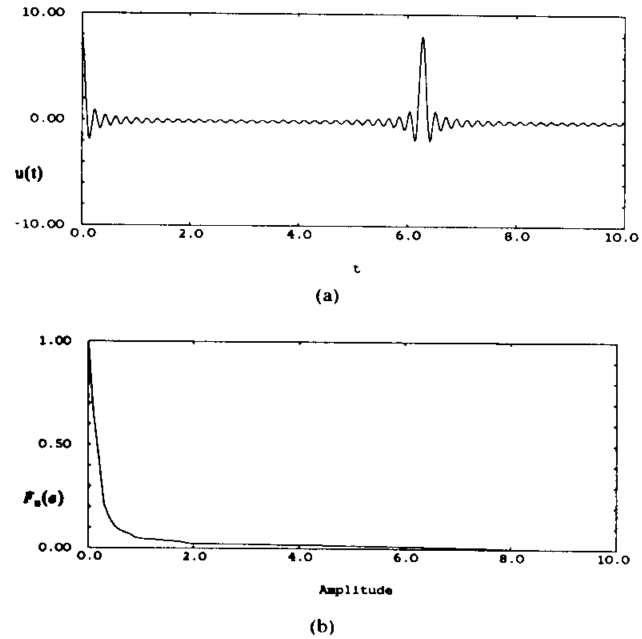

Fig. 1. (a) 32-tone zero-phase signal. (b) Amplitude distribution of 32-tone zero-phase signal.

While the rms level is independent of the phases, the peak (which is also $CF(u)$) changes dramatically with the phases. The question we address is: How do we choose the phases δ_k in (4) to minimize the crest factor of u, or at least to make it small?

The worst choice of phases is $\delta_k = 0$, $k = 1, \cdots, N$, which yields a peak $\|u\|_\infty = \sqrt{2N}$ (which occurs at $t = 0$). Fig. 1 shows the 32-tone multitone signal with zero phases, and its amplitude distribution.

If the δ's vary linearly, e.g., $\delta_k = \tau k$, then the peak remains $\sqrt{2N}$. It is clear that some sort of random pattern of phases is needed so that cancellations always keep $|u(t)|$ small. In fact, random phases yield crest factors on the order of $\sqrt{\log N}$, which is much smaller than $\sqrt{N}$, but still grows with N [1].

In the sequel, we will show that the crest factor can be made quite small, under 6 dB, for arbitrarily large N, that is, arbitrarily many tones. It is surprising that signals can be designed which satisfy both the frequency domain constraint (5) and the time-domain constraint that $CF(u)$ be small.

Two choices of phases will be presented. Neither choice yields the global minimum crest factor achievable, but both yield very small crest factors, at most a few decibels over the minimum achievable (which, although unknown, must exceed 0 dB). Both choices of phases have the advantage of being very easy to compute or generate.

The first choice of phases, due to Shapiro and Rudin, can be *proved* to have crest factor under 6 dB when the number of tones is a power of two (a very common situation, e.g., in FFT-based frequency response measurement), and to have a reasonably small crest factor in all cases. The Shapiro-Rudin phases $\delta_k^{(S-R)}$ are in fact either 0 or π, and do not depend on N, the number of tones.

The second choice of phases is suggested in Newman [2] in connection with a different problem. Numerical investigations show that Newman's phases yield smaller crest factors (about 4.6 dB, decreasing as N increases) than the Rudin-Shapiro phases. At the moment, there is no *proof* that the Newman phases always yield very low crest factors, but the numerical evidence suggests so.

[1] If desired, the low-pass spectrum signal can start with a dc term, for example,

$$u(t) = \sqrt{\frac{2}{N+1}} \sum_{k=0}^{N-1} \cos(kt + \delta_k)$$

with $\delta_0 = 0$.

III. The Shapiro-Rudin Phases

Using a result due to Shapiro and Rudin, [3], [4], we now show that whenever N is a power of 2, there are phases which yield a low crest factor of 2. To do this, we will need to define a certain sequence of signs, that is, a sequence of 1's and -1's. Start with the string $p=11$, and repeatedly perform the following: concatenate to p a copy of p with its second half negated. Fig. 2 shows the first few strings constructed. Note that the strings constructed extend each other, and so form the initial strings of a whole sequence of 1's and -1's.

Definition: The kth *Rudin sign* r_k is the kth element of the sequence generated.

Thus, e.g., $r_1 = r_{11} = 1$ and $r_4 = r_7 = -1$.

There are other ways to describe the Rudin signs r_k; for example, in Appendix II, we give a very short program which computes r_k.

Theorem: Let

$$u_N^{(S-R)}(t) \triangleq \sqrt{\frac{2}{N}} \sum_{k=1}^{N} r_k \cos((k+N_0)t). \tag{6}$$

Then for $N = 2^l$, we have

$$CF\left(u_N^{(S-R)}\right) = \|u_N^{(S-R)}\|_\infty \leq 2.$$

Remark: If $N_0 = -1$, so there is a dc term in (6), we still have $CF(u^{(S-R)}) \leq 2$.

The proof is given in Appendix II.

Thus, with the phases

$$\delta_k^{(S-R)} = \begin{cases} 0, & r_k = 1 \\ \pi, & r_k = -1 \end{cases}$$

the signal (4) has a crest factor at most 6 dB. Note that the theorem means that there are multitone signals with an arbitrarily large number of tones which have a crest factor at most 6 dB; in particular, with proper choice of phases, the growth of a crest factor with N can be avoided.

If N is not a power of 2, it turns out that the signal $u_N^{(S-R)}$ defined in (6) still has a relatively low crest factor, especially if N is close to a power of two. Fig. 3 shows $CF(u_N^{(S-R)})$ for $n = 1, \cdots, 100$.

Note that the crest factor dips to two when the number of tones is a power of two, and rises a bit in between. For larger N, the crest factor behaves the same way.

Fig. 4 shows the 32-tone Shapiro-Rudin signal and its amplitude distribution. Note that the amplitude distribution is nearly *linear*, that is, the signal is nearly *uniformly distributed*. This is true for larger N as well.

IV. The Newman Phases

The phases suggested by Newman are, for N tones

$$\delta_k^{(NEW)} = \frac{\pi(k-1)^2}{N}.$$

Numerical investigation shows that the crest factor of the Newman multitones

$$u_N^{(NEW)}(t) \triangleq \sqrt{2N^{-1}} \sum_{k=1}^{N} \cos((k+N_0)t + \delta_k^{(NEW)})$$

is always very small, about 4.6 dB for moderate N (up to a few hundred) and decreasing slightly for larger N.[2] In all cases checked, the Newman multitones had crest factors smaller than the Shapiro-Rudin multitones, and we suspect that this is true for all N. It is interesting that while phases which vary *linearly* yield the worst crest factor possible, the Newman phases, which vary *quadratically*, yield very close to the minimum achievable.

Fig. 5 shows $CF(u_N^{(NEW)})$ for $N = 1, \cdots, 100$. Note the small variation in crest factor with N; for N larger, the variation is even smaller.

Fig. 6 shows the 32-tone Newman multitone and its amplitude distribution. Note the curious resemblance to a swept frequency signal (we remind the reader that the spectrum of the signal is perfect, i.e., given by (5)). The Newman signal is also nearly uniformly distributed on $[-\|u\|_\infty, \|u\|_\infty]$.

[2] Once again we can set $N_0 = -1$ if desired.

11
111-1
111-111-11
111-111-11111-1-1-11-1
111-111-11111-1-1-11-1111-111-11-1-1-1111-11

Fig. 2. First few strings in construction of Shapiro-Rudin sequence.

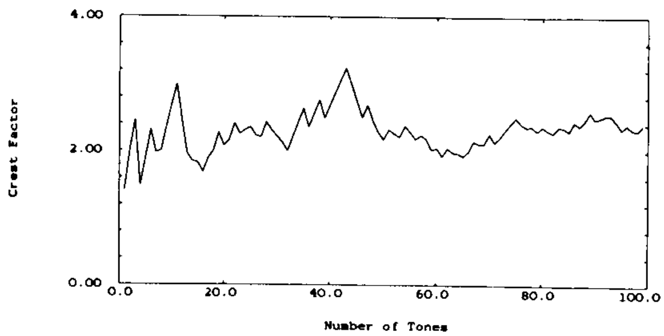

Fig. 3. Crest factor of Shapiro-Rudin signals.

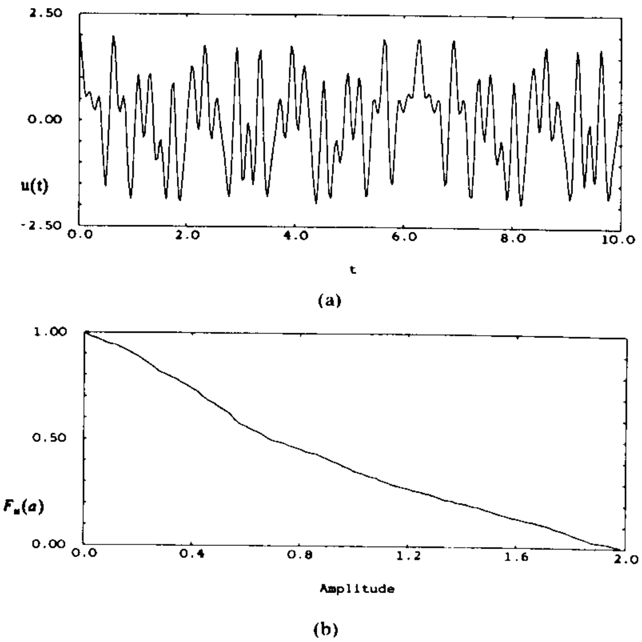

Fig. 4. (a) 32-tone Shapiro-Rudin signal. (b) Amplitude distribution of 32-tone Shapiro-Rudin signal.

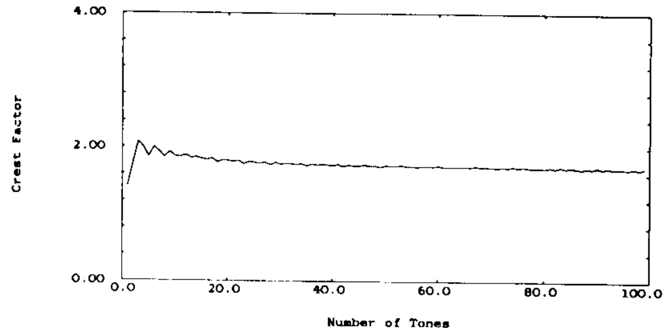

Fig. 5. Crest factor of Newman signals.

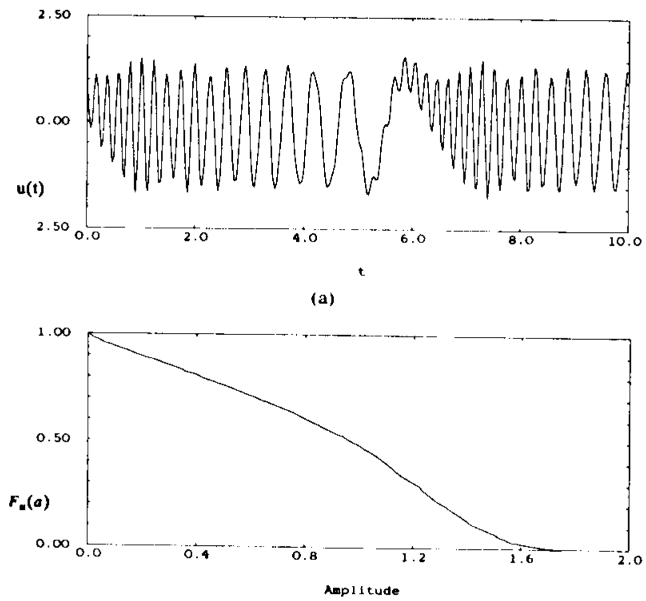

Fig. 6. (a) 32-tone Newman signal. (b) Amplitude distribution of 32-tone Newman signal.

V. Multitone Frequency Response Testing

The signal (4) is an appropriate probing signal for making nonparametric measurements of the frequency response of a linear system, specifically, measuring $H(jkT^{-1})$ for $k=1,\cdots,N$. Let us consider the S/N ratio we can achieve using a B-bit D/A (quantizer) to generate the probing signal (4). Let Δ be the resolution of the D/A, so that its output range is $\pm 2^{B-1}\Delta$. To avoid clipping, we must have

$$CF(u)=\|u\|_\infty \leqslant 2^{B-1}\Delta.$$

Thus

$$\Delta \geqslant 2^{1-B}CF(u).$$

An excellent approximation is that the quantization error has the amplitude distribution function which decreases linearly from one at $a=0$ to zero at $a=\Delta/2$. Hence, the quantizer noise has rms value $\Delta/\sqrt{12}$. Thus

$$\mathrm{S/N} \leqslant \frac{2^B\sqrt{3}}{CF(u)}$$

or approximately

$$20\log \mathrm{S/N} \leqslant 6B+5-20\log CF(u).$$

The important point is that an n decibel decrease in the crest factor of the probing signal yields an n decibel increase in quantizer S/N ratio. For example, if we use the Newman phases, the S/N ratio of our probing signal exceeds $6B$ dB. The Newman phases are used in Boyd, Tang, and Chua [5] for multitone measurements of nonlinear systems.

Of course, there is also the problem of A/D quantizer noise. While little precise can be said, we make these general remarks. It is *possible* that the output of our device under test has a high crest factor, even though its input has a low crest factor, but most systems encountered in practice do not drastically increase the crest factor of a signal. For example, if the system whose frequency response we are measuring in an allpass with phases the negative of the phases we use in our probing signal, then its output signal has a high ($\sqrt{2N}$) crest factor, and, hence, our A/D quantization noise is high. But such a situation is very unlikely. This pathological system has a response which varies drastically with k, and so it is inappropriate to make the nonparametric measurements above, that is, the frequency spacing T^{-1} is not fine enough.

VI. Conclusion

We have shown two simple choices of phases which yield a nearly minimal crest factor for a multitone signal.

We close with two comments. First, standard optimization techniques do not help much in the design of low crest factor signals. The crest factor is a very complicated function of the phases, with very many local minima. A descent routine, started either from Newman's or Shapiro and Rudin's phases, yields minimal decrease in the crest factor.

Our second comment concerns the question, how small can the crest factor be made? According to a recent result of Kahane [6], there exist choices of phases which yield crest factors approaching $\sqrt{2}$ (3 dB) as N gets large. Kahane's proof uses a probabilistic argument, so, unfortunately, it is not possible to directly construct these phases. Note that Newman's phases yield a crest factor only a decibel and a half greater.

It is an open question whether the crest factor can be made smaller than 3 dB, but, of course, for most applications (for example, frequency response testing), a crest factor of 6 dB or 4.6 dB is fine.

Appendix I
Generating the Rudin-Shapiro Signs

Although the construction by which we defined the Rudin signs can be used directly to generate the r_k, the following fact simplifies the computation.

Fact: $r_k=(-1)^L$, where L is the number of pairs of consecutive ones in the binary expansion of $k-1$.

This fact is readily turned into an algorithm to compute r_k. For example, the following C-language function computes r_k, given k:

```
rudin_sign(k)
{
int previous_bit, sign = 1;

k = k - 1;
while (k > 0)
      {
      previous_bit = k%2;
      if ((k = k/2) = = 0) break;
      if (previous_bit = = 1 && k%2 = = 1) sign = -sign;
      }
return(sign);
}
```

Appendix II
Proof of Theorem

The following short proof follows Rudin [4].

For $l=1,2,\cdots$, define polynomials

$$P_l(z) \triangleq \sum_{k=1}^{2^l} r_k z^k$$

$$Q_l(z) \triangleq \sum_{k=1}^{2^{l-1}} r_k z^k - \sum_{k=2^{l-1}+1}^{2^l} r_k z^k.$$

Note that the coefficients of P_l are exactly the lth string in the construction of the r_k given in Section III, and the coefficients of Q_l are the lth string, with its second half negated. Thus, we have

$$P_{l+1} = P_l + z^{2^l}Q_l \quad Q_{l+1} = P_l - z^{2^l}Q_l.$$

Hence, for $|z|=1$

$$|P_{l+1}|^2 + |Q_{l+l}|^2 = |P_l + z^{2^l}Q_l|^2 + |P_l - z^{2^l}Q_l|^2 \tag{A1a}$$

$$= 2|P_l|^2 + 2|z^{2^l}Q_l|^2 = 2(|P_l|^2 + |Q_l|^2). \tag{A1b}$$

Since $|P_1|^2 + |Q_1|^2 = 4$, (A1) tells us that for all l and $|z|=1$

$$|P_l|^2 + |Q_l|^2 = 2^{l+1}.$$

Consequently, for $|z|=1$

$$|P_l| \leqslant 2^{(l+1)/2}.$$

Thus, for $|z|=1$, $|\mathrm{Re}\,P_l(z)| \leqslant 2^{(l+1)/2}$. We finish the proof by noting that for $N=2^l$

$$u_N^{(\mathrm{S-R})}(t) = \sqrt{\frac{2}{N}}\,\mathrm{Re}\,P_l(e^{it})$$

so that

$$|u_N^{(\mathrm{S-R})}(t)| \leqslant \sqrt{\frac{2}{N}}\,2^{(l+1)/2} = 2. \qquad \square$$

Remark: The Shapiro-Rudin phases really yield a *complex* signal with crest factor $\sqrt{2}$; its real (or imaginary) part is a real signal with crest factor 2.

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