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ON PARAMETRIC H^∞ OPTIMIZATION*

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ABSTRACT

The problem of optimizing the H^∞ norm of a rational transfer matrix with respect to a finite number of design parameters is considered. The H^∞ norm is characterized as a value of a parameter for which a certain Hamiltonian matrix has multiple eigenvalues. A coprimeness test for polynomials is used to algebraically characterize the H^∞ norm as an implicit function of the design parameters. In the case of a single design parameter, necessary conditions for optimality are obtained in the form of a system of two algebraic equations with two unknowns.

1. Introduction

The problem treated in this note is as follows. We are given affine real matrix functions of a real parameter vector ξ:

$$(A,B,C,D): \Xi \subset \mathbb{R}^q \to \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} : \xi \in \Xi \to$$
$$\left(A(\xi), B(\xi), C(\xi), D(\xi)\right) = (A_o, B_o, C_o, D_o) + \sum_{i=1}^{q} (A_i, B_i, C_i, D_i)\, \xi_i \,. \quad (1.1)$$

We assume that Ξ is a compact subset of $\mathbb{R}^q$ and that for all $\xi \in \Xi$, all the eigenvalues of the matrix $A(\xi)$ have negative real parts. Forevery $\xi \in \Xi$, we define the transfer matrix

$$H(s,\xi) = C(\xi)\,[sI - A(\xi)]^{-1} B(\xi) + D(\xi). \quad (1.2)$$

The problem is then simply to find $\xi \in \Xi$ which minimizes the H^∞ norm of $H(s,\xi)$, or formally:

$$\min_{\xi\in\Xi} J(\xi) \ , \quad (1.3)$$

where

$$J(\xi) \triangleq \|H(s,\xi)\|_\infty \triangleq \sup_{\omega\in\mathbb{R}} \bar{\sigma}\,(H(j\omega,\xi))\,, \quad (1.4)$$

and $\bar{\sigma}(.)$ denotes the maximum singular value of a matrix.

2. Algebraic Viewpoint

For every $\gamma > 0$, not singular value of $D(\xi)$, define the $2n \times 2n$ Hamiltonian matrix

$$M(\gamma,\xi) = \begin{bmatrix} A(\xi) & 0 \\ 0 & -A^T(\xi) \end{bmatrix} + \begin{bmatrix} B(\xi) & 0 \\ 0 & -C^T(\xi) \end{bmatrix} \begin{bmatrix} -D(\xi) & \gamma I \\ \gamma I & -D(\xi) \end{bmatrix}^{-1} \begin{bmatrix} C(\xi) & 0 \\ 0 & B^T(\xi) \end{bmatrix} =$$
$$= \begin{bmatrix} A(\xi) - B(\xi)R(\gamma,\xi)^{-1}D^T(\xi)C(\xi) & -\gamma B(\xi)R(\gamma,\xi)^{-1}B^T(\xi) \\ \gamma C^T(\xi)S(\gamma,\xi)^{-1}C(\xi) & -A^T(\xi) + C^T(\xi)D(\xi)R^{-1}(\xi)B^T(\xi) \end{bmatrix} \quad (2.1)$$

where

$$R(\gamma,\xi) = [D^T(\xi)D(\xi) - \gamma^2 I],\ S(\gamma,\xi) = [D(\xi)D^T(\xi) - \gamma^2 I]. \quad (2.2)$$

We also define the following polynomial function of s

$$\pi(s,\gamma,\xi) \triangleq \det\left(sI - M(\gamma,\xi)\right), \quad (2.3)$$

which satisfies

$$\pi(s,\gamma,\xi) = \pi(-s,\gamma,\xi) \quad (2.4)$$
$$= \pi(s,-\gamma,\xi) \quad (2.5)$$

Notice that as $\gamma \to +\infty$, $M(\gamma,\xi) \to$ Block Diag$(A(\xi),-A^T(\xi))$ which has no imaginary eigenvalue. Therefore for all $\xi \in \Xi$ we can define

$$\gamma^*(\xi) \triangleq \inf\{\gamma \ge \bar{\sigma}(D(\xi)) \mid M(\gamma,\xi) \text{ has no imaginary eigenvalue}\}$$
$$\triangleq \inf\{\gamma \ge \bar{\sigma} D(\xi)) \mid \pi(s,\gamma,\xi) \text{ has no imaginary s-root}\} \quad (2.6)$$

Proposition 2.1 [1] For all $\xi \in \Xi$, $\|H(s,\xi)\|_\infty = \gamma^*(\xi)$. |||

Proposition 2.2 Let $j\omega^*$ be an imaginary s-root of $\pi(s,\gamma^*(\xi),\xi)$. Then $j\omega^*$ must be a double root, i.e.

$$\pi(j\omega^*,\gamma^*(\xi),\xi) = 0\ , \quad (2.7)$$
$$\frac{\partial}{\partial s}\,\pi(s,\gamma^*(\xi),\xi)\,\big|_{s=j\omega^*} = 0\,. \quad (2.8)$$

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Propositions 2.1 and 2.2 are useful because they characterize $\|H(s,\xi)\|_\infty$ as a value of γ for which the polynomial $\pi(s,\gamma,\xi)$ has a double root, i.e., for which the two polynomials $\pi(s,\gamma,\xi)$ and $\partial\pi(s,\gamma,\xi)/\partial s$ have a common root. We may, therefore, use any number of coprimeness tests for polynomials to derive an algebraic characterization of $\|H(s,\xi)\|_\infty$.

Definition 2.1 The vector ξ is called *nondegenerate* if there exists $\gamma \in \mathbb{R}$ such that the matrix $M(\gamma,\xi)$ of (2.1) has 2n distinct eigenvalues. Otherwise it is *degenerate.* |||

Proposition 2.3 For every nondegenerate ξ, the function $\gamma^*(\xi)$ of (2.6) satisfies

$$P(\gamma^*(\xi),\xi) = 0 \quad (2.9)$$

where $P(\gamma,\xi)$ is the resultant [2] of the two polynomials in s : $\pi(s,\gamma,\xi)$ and $\partial\pi(s,\gamma,\xi)/\partial s$. |||

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Proposition 2.4 Suppose that

1. $q = 1$
2. The pair (γ^*,ξ^*) solves Problem (1.3)
3. $\xi^* \in \overset{\circ}{\Xi}$ and is nondegenerate

Then, either

$$\begin{cases} P(\gamma^*,\xi^*) = 0 \;, & (2.10.a) \\ \dfrac{\partial P(\gamma,\xi)}{\partial \gamma}\Big|_{(\gamma,\xi) = (\gamma^*,\xi^*)} = 0, & (2.10.b) \end{cases}$$

or

$$\begin{cases} P(\gamma^*,\xi^*) = 0 \;, & (2.11.a) \\ \dfrac{\partial P(\gamma,\xi)}{\partial \xi}\Big|_{(\gamma,\xi) = (\gamma^*,\xi^*)} = 0 \;. & (2.11.b) \end{cases}$$

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3. Example

The simplest example to illustrate the concepts of Section 2 is possibly that of optimal zero[th] order model reduction of a first order system. Given a first order transfer function $H_1(s) = 1/(s+1)$, we want to find the best zero[th] order approximant $H_2(s,\xi) = \xi$ in the sense

$$\min_{\xi} J(\xi) = \left\| \frac{1}{s+1} - \xi \right\|_\infty . \qquad (3.1)$$

Thus, $H(s,\xi) = [1-\xi(s+1)]/(s+1)$. Following the development of Section 2, we have

$$\pi(s,\gamma,\xi) = \left(1 - \frac{\xi^2}{\gamma^2}\right) s^2 + \frac{(1-\xi)^2}{\gamma^2} - 1 \qquad (3.2)$$

The Routh table without division based on $\pi(s,\gamma,\xi)$ and $\partial\pi(s,\gamma,\xi)/\partial s$ yields the resultant

$$P(\gamma,\xi) = - \frac{\xi^2(1-\xi)^2}{\gamma^4} + \frac{[\xi^2 + (1-\xi)^2]}{\gamma^2} - 1 \;, \qquad (3.3)$$

which has the following interpretation: for every ξ, $J(\xi)$ in (4.1) is a value of γ for which $P(\gamma,\xi) = 0$. For instance, if $\xi=0$, we recover the well known fact $\|1/(s+1)\|_\infty = 1$.

Since $P(\gamma,\xi)$ is a polynomial in γ^{-1}, it is more convenient to work with

$$P(\mu,\xi) = P(\gamma,\xi)\Big|_{\gamma=\mu^{-1}}$$

$$= - \gamma^2(1-\xi)\mu^4 + \left[x^2 + (1-\xi)^2\right]\mu^2 - 1 \,. \qquad (3.4)$$

Then Proposition 2.4 implies that the optimal design satisfies either

$$\begin{cases} \overline{P}(\mu,\xi) = 0 & (3.5.a) \\ \dfrac{\partial \overline{P}}{\partial \mu} = - 4\xi^2 (1-\xi)\mu^3 + 2\left[\xi^2 + (1-\xi)^2\right]\mu = 0 & (3.5.b) \end{cases}$$

or

$$\begin{cases} \overline{P}(\mu,) = 0 & (3.6.a) \\ \dfrac{\partial \overline{P}}{\partial \xi} = 2(2\xi - 1)\left[\gamma^4 \xi (1-\xi) + \gamma^2\right] = 0 & (3.6.b) \end{cases}$$

If (3.5) hold, then the polynomials in μ (3.4) and (3.5.b) have a common root. A Routh table without division based on these two polynomials yields the resultant

$$\rho_1(\xi) = \xi^4(1-\xi)^4 (2\xi - 1)^2 \qquad (3.7)$$

whose roots are candidate optimal designs. If, on the other hand, (3.6) hold, then the two polynomials in μ (3.4) and (3.6.b) have a common root. There also, a Routh table without division yields the resultant

$$\rho_2(\xi) = \xi^2 (1-\xi)^2 (2\xi - 1)^4 (\xi^2 - \xi + 2) \;, \qquad (3.8)$$

whose roots are also candidate optimal designs.

In this very simple example, it is easily seen that the unique optimal design is $\xi^* = 1/2$ yielding $\gamma^* = 1/2$, which is one of the candidates given by (3.7) and (3.8).

References

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