Analysis of Linear Systems with Saturation using Convex Optimization

Haitham Hindi[1] Stephen Boyd[2]
Dept. of Electrical Engineering, Stanford University, CA 94305, USA.

Abstract

We show how Linear Matrix Inequalities (LMIs) can be used to perform *local* stability and performance analysis of linear systems with saturating elements. This leads to less conservative information on stability regions, disturbance rejection, and L_2-gain than standard global stability and performance analysis. The Circle and Popov criteria are used to obtain Lyapunov functions whose sublevel sets provide regions of guaranteed stability and performance within a restricted state space region. Our LMI formulation leads directly to simple convex optimization problems that can be solved efficiently as Semidefinite Programs. The results cover both single and multiple saturation elements and can be immediately extended to discrete time systems. An obvious application of these techniques is in the analysis of control systems with saturating control inputs.

1 Introduction

Linear systems with saturation nonlinearities, see Fig.1, occur very often in practice. A typical example is a control system for a plant with saturating control inputs. Such systems exhibit nonlinear behaviour such as local stability, finite disturbance rejection, and performance degradation, when operating in saturation. Yet there is todate no standard method for analyzing them. The objective of this paper is to develop some simple LMI-based computational tools for the *local* stability and performance analysis of linear systems with saturating elements.

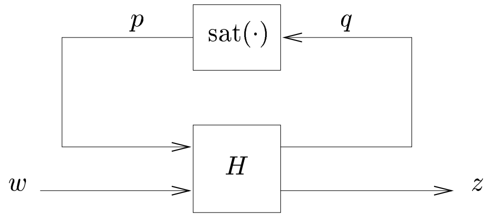

Figure 1: LTI system H with saturation.

[1]Contact author, was also at Stanford Linear Accelerator Center. Work supported by DOE contract # DE-AC0376SF00515. Email: hhindi@kona.stanford.edu

[2]Email: boyd@kona.stanford.edu)

The tools we present are essentially local versions of the standard Circle and Popov criteria, and have the following features: By performing local analysis rather than global analysis, they can provide useful information even about systems that are not globally stable, and the information they yield is always less than that obtained from corresponding global analysis. They can handle systems of any order and multiple saturation nonlinearities. They can easily be extended to discrete time systems. They provide information not only about local stability but local performance as well. Other approaches in the literature which do not use the circle or Popov criteria, see [1, 7, 5, 13, 16] for example, are not able to offer all of the features mentioned above.

Our approach has its origins in work that was done about three decades ago [18, 17, 4], on computing guaranteed regions of attraction for linear systems with locally sector bounded nonlinearities, by constructing quadratic and Lur'e type Lyapunov functions. Since then, various authors have made use of the the circle criterion to analyze the local stability of systems with saturation, see for example [6, 9, 8]. A similar approach to ours for computation of regions of attraction has been reported independently in [12].

Note that none of the references above treat local performance analysis. Yet in practice, local performance analysis can give very valuable information on how the performance of a system degrades as it operates in saturation. We will see that local performance analysis is a straight forward matter, using the LMI framework.

The contribution of this paper is a thorough presentation in the LMI framework, fully generalized to the case of multiple saturation elements using the $\mathcal{S}$-procedure [3, 14], extended to performance analysis, with explicit formulas for the semidefinite programs that must be solved for each case. In reference [11], the authors attempt to extend these ideas to output feedback synthesis.

Experience has shown that these methods can work quite well, and they are simple to implement. It is probably fair to say that the main reason why they have not been widely used is because until recently, it was not appreciated that these problems, due to their convexity and structure, are extremely tractable and that they could be solved efficiently even for systems

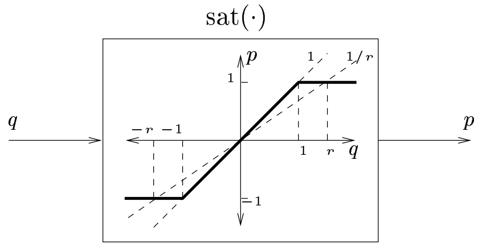

Figure 2: Model for saturation nonlinearity.

of moderate order (*e.g.*30), where the number of parameters of the Lyapunov functions is in the hundreds.

In this paper, due to space limitations, we will focus on presenting the results. Most of the proofs are omitted, but they follow in a straight forward way from the discussion in Preliminaries.

1.1 Model

Consider the linear system H with a (decoupled) block of saturators, see Fig.1:

$$\begin{array}{rcl} \dot{x} & = & Ax + B_p p + B_w w; \quad x(0) = x_0 \\ q & = & C_q x, \quad z = C_z x \\ p & = & \mathrm{sat}(q) \end{array} \tag{1}$$

where $x \in \mathbf{R}^{n_x}$, $p \in \mathbf{R}^{n_p}$, $q \in \mathbf{R}^{n_q}$ $(n_p = n_q)$, $w \in \mathbf{R}^{n_w}$, and $z \in \mathbf{R}^{n_z}$. The $\mathrm{sat}(\cdot)$ denotes the normalized unit saturation function, and is defined on a scalar $q_i \in \mathbf{R}^1$ by

$$\mathrm{sat}(q_i) \triangleq \left\{ \begin{array}{cl} 1 & ; q_i > 1 \\ q_i & ; |q_i| \leq 1 \\ -1 & ; q_i < -1 \end{array} \right. \tag{2}$$

and on a vector $q = (q_1, \ldots, q_{n_q})$ by

$$\mathrm{sat}(q) \triangleq \left(\mathrm{sat}(q_1), \ldots, \mathrm{sat}(q_{n_q})\right). \tag{3}$$

Note that any block of saturators can be put into this standard form by scaling the input and output of each saturator appropriately.

1.2 Local Analysis Formulation

A restriction on the amplitudes of the inputs to the saturators, translates directly into a constraint on the state of the system. Let $r \in \mathbf{R}_+^{n_q}$ be a vector whose components specify allowable input amplitudes to the saturators. Then the constraint $\{|q_i| \leq r_i, i = 1, \ldots, n_q\}$ holds if and only if the state x lies in the region $\mathcal{R}_r = \{x \,|\, |C_{q_i} x| \leq r_i, i = 1, \ldots, n_q\}$. In particular, note that whenever x lies in the region $\mathcal{R}_{\mathbf{1}}$, obtained by setting $r = \mathbf{1} = (1, \ldots, 1)$, the system behaves linearly, since all the inputs to the saturators are less than 1 in absolute value. Therefore, we will only be interested in situations where $r_i \geq 1$, in which case the r_i specify the amount by which the input amplitude can exceed the output amplitude at each saturator.

Definition 1 Any set $\hat{\mathcal{D}} \subset \mathcal{R}_r$ such that whenever $x(0) \in \hat{\mathcal{D}}$ and $w \equiv 0$ we have $\lim_{t\to\infty} x(t) = 0$ will be called an r-level guaranteed region of attraction. The largest set $\mathcal{D} \subset \mathbf{R}^{n_x}$ for which $x(t) \to 0$ whenever $x(0) \in \mathcal{D}$ and $w \equiv 0$ is called the region of attraction.

Definition 2 Any number $\hat{\alpha}_{\max,r} \in \mathbf{R}_+$ such that whenever $x(0) \equiv 0$ and $\|w\|_2^2 \leq \hat{\alpha}_{\max,r}$ we have
(i) $\lim_{t\to\infty} x(t) = 0$
(ii) $x(t) \in \mathcal{R}_r \quad \forall\, t \geq 0$,
will be called an r-level disturbance rejection bound. The largest such number will be denoted by $\alpha_{\max,r}$.

Definition 3 Any number $\hat{\gamma}_r \in \mathbf{R}_+$ such that whenever $x(0) \equiv 0$ and $\|w\|_2^2 \leq \hat{\alpha}_{\max,r}$ we have
(i) $\lim_{t\to\infty} x(t) = 0$
(ii) $x(t) \in \mathcal{R}_r \ \forall\, t \geq 0$,
(iii) $\|z\|_2/\|w\|_2 \leq \hat{\gamma}_r$,
will be called an r-level L_2-gain bound. The smallest such number will be denoted by γ_r.

1.3 Problem Statement

Given model (1) with $A + B_p C_q$ Hurwitz and a given set of allowable saturation input amplitudes specified in the vector r, we want to compute the following:

1. r-level guaranteed region of attraction $\hat{\mathcal{D}}_r$,
2. r-level disturbance rejection bound $\hat{\alpha}_{\max,r}$,
3. r-level L_2-gain bound $\hat{\gamma}_r$.

We would like these to be not overly conservative. We also require that the technique be: efficient, capable of handling multiple nonlinearities with no restriction on order of system, and easily extendable to discrete time systems. Of prime concern will be simplicity and generality.

Notation: Let P be a symmetric positive definite matrix in $\mathbf{R}^{n_x \times n_x}$, then $\mathcal{E}_P(\alpha)$ denotes the set $\{x \,|\, x^T P x \leq \alpha\}$. For a function $V : \mathbf{R}^{n_x} \to \mathbf{R}$, $\mathrm{lev}_\alpha V$ denotes the α-sublevel set of V , *i.e.*, the set $\{x \,|\, V(x) \leq \alpha\}$. For any function $\varphi : \mathbf{R}^{n_1} \to \mathbf{R}^{n_2}$, $\mathrm{gr}\,\varphi$ is the graph of the function φ, *i.e.*, it is the set $\{(x,y) \,|\, y = \varphi(x)\}$. If $a, b \in \mathbf{R}$, then $\mathrm{sect}[a,b]$ denotes the sector in $\mathbf{R}\times\mathbf{R}$ defined by $\{(q,p) \,|\, (p-aq)(p-bq) \leq 0\}$. If a and b are vectors in $\mathbf{R}^n$, then $\mathrm{sect}[a,b]$ is taken to be $\mathrm{sect}[a_1,b_1] \times \cdots \times \mathrm{sect}[a_n,b_n]$. A function $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ is said to be sector$[a,b]$ if $\varphi(q) = (\varphi_1(q_1), \ldots, \varphi_n(q_n))$, $\mathrm{gr}\,\varphi_i \in \mathrm{sect}[a_i, b_i]$.

2 Preliminaries

The following are some facts about quadratic functions, Popov functions and multiobjective optimization which we will find useful.

2.1 Quadratic Functions

Recall that if $V(x) = x^T Q^{-1} x$ is a quadratic function with $Q = Q^T > 0$, then its α-sublevel sets are ellipsoids $\mathcal{E}_{Q^{-1}}(\alpha)$ whose volume is proportional to $\alpha^n \det Q$ [10, 2].

Lemma 1 (Constrained Quadratic)
Let $V(x) = x^T Q^{-1} x$ where $Q = Q^T > 0$, C be a row vector in $\mathbf{R}^n$ and r be a nonzero scalar. Then the minimum of V along the hyperplane $\{x \,|\, Cx = r\}$ is given by

$$\alpha_r = \frac{r^2}{CQC^T}.$$

A necessary and sufficient condition for the sublevel set $\mathrm{lev}_\alpha V = \mathcal{E}_{Q^{-1}}(\alpha)$ to be contained in the region $\mathcal{R}_r$ is $\alpha \le \alpha_r$ where

$$\alpha_r \triangleq \min_{i=1,\ldots,n_q} \frac{r_i^2}{C_{q,i} Q C_{q,i}^T}$$

where $C_{q,i}$ are the rows of C_q.

2.2 Popov Functions

Definition 4 A Popov function $V : \mathbf{R}^n \to \mathbf{R}_+$ in standard form is defined as

$$V(x) = x^T P x + \textstyle\sum_{i=1}^{n_q} 2\lambda_i \int_0^{C_{q,i}x} \varphi_i(\sigma)\, d\sigma \qquad (4)$$

where $P = P^T > 0$, $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_{n_q}) \ge 0$, $\mathrm{gr}\,\varphi_i \subset \mathrm{sect}[0,1]$, $\quad i = 1, \ldots, n_q$ and $C_{q,i}$ are the rows of C_q in (1).

From now on, we assume that the φ_i are odd functions whose graphs lie in the first and third quadrants. Then we have the following properties:
(i) $\int_0^{C_{q,i}x} \varphi_i(\sigma)\, d\sigma$ is an even function
(ii) $0 \le \int_0^{C_{q,i}x} \varphi_i(\sigma)\, d\sigma \le \frac{1}{2} x^T C_{q,i}^T C_{q,i} x$
We associate with V the following functions:

$$\begin{array}{rcl} \underline{V}(x) & = & x^T P x, \\ V_i(x) & = & x^T P x + 2\lambda_i \int_0^{C_{q,i}x} \varphi_i(\sigma)\, d\sigma, \\ \overline{V}(x) & = & x^T (P + C_q^T \Lambda C_q) x. \end{array} \qquad (5)$$

Then we have the following ordering and inclusions:

$$\begin{array}{c} 0 \le \underline{V} \le V_i \le V \le \overline{V} \\ \mathcal{E}_P(\alpha) \supset \mathrm{lev}_\alpha V_i \supset \mathrm{lev}_\alpha V \supset \mathcal{E}_{(P + C_q^T \Lambda C_q)}(\alpha) \end{array} \qquad (6)$$

The first part of the following lemma is due to [17].

Lemma 2 (Constrained Popov)
Let $V(x) = x^T P x + 2\lambda \int_0^{Cx} \varphi(\sigma)\, d\sigma$, where $P = P^T > 0$, $\lambda > 0$, φ is sector$[0,1]$, C is a row vector in $\mathbf{R}^n$, and let r be some positive scalar. Then the minimum of V along the hyperplane $\{x \,|\, Cx = r\}$ is given by

$$\alpha_r = \frac{r^2}{CP^{-1}C^T} + 2\lambda \int_0^r \varphi(\sigma)\, d\sigma.$$

Let $V(x) = x^T P x + \sum_{i=1}^{n_q} 2\lambda_i \int_0^{C_{q,i}x} \varphi_i(\sigma)\, d\sigma$, where φ_i are all sector$[0,1]$ and odd, $P = P^T > 0$, and $\lambda_i \ge 0$. A sufficient (and necessary for $n_q = 1$) condition for the sublevel set $\mathrm{lev}_\alpha V$ to be contained in the region $\mathcal{R}_r$ is $\alpha \le \alpha_r$ where

$$\alpha_r \triangleq \min_{i=1,\ldots,n_q} \left\{ \frac{r_i^2}{C_{q,i} P^{-1} C_{q,i}^T} + 2\lambda_i \int_0^{r_i} \varphi_i(\sigma)\, d\sigma \right\}.$$

Remark 1 It is important to realize that the terms $\int_0^{r_i} \varphi_i(\sigma)\, d\sigma$ are simply *constants* which can be computed directly from the loop transformed nonlinearities φ_i.

2.3 Multiobjective Optimization

Definition 5 Let $F : \mathbf{R}^n \to \mathbf{R}^m$ be a vector valued objective function $F(x) = (f_1(x), \ldots, f_m(x))$. Then a point $x_{\mathrm{opt}} \in \mathbf{R}^n$ is Pareto optimal with respect to F if there is no other point $x \in \mathbf{R}^n$ such that $f_i(x) \le f_i(x_{\mathrm{opt}})$ for all i, and $f_{i_0}(x) < f_{i_0}(x_{\mathrm{opt}})$ for at least one i_0, [2].

Lemma 3 (Characterizing Maximizer)
Let $f_1 : \mathbf{R}^{n_1} \to \mathbf{R}$ and $f_2 : \mathbf{R}^{n_2} \to \mathbf{R}$, a_1 and a_2 be positive numbers, and $\mathcal{C} \subset \mathbf{R}^{n_1+n_2}$. Then the maximizer of the following optimization problem (when it exists)

$$\begin{array}{ll} \max & a_1 \left(f_1(x_1)\right)^{-1} + a_2 \left(f_2(x_2)\right)^{-1} \\ \text{s.t.} & (x_1, x_2) \in \mathcal{C} \end{array}$$

will be a Pareto Optimal point with respect to the vector objective $F(x_1, x_2) = (f_1(x_1), f_2(x_2))$.

Proof: Suppose that the maximum is achieved at a point $(x_1^\star, x_2^\star) \in \mathcal{C}$. Then we must have

$$\begin{array}{l} a_1 \left(f_1(x_1^\star)\right)^{-1} + a_2 \left(f_2(x_2^\star)\right)^{-1} \ge \\ a_1 \left(f_1(x_1)\right)^{-1} + a_2 \left(f_2(x_2)\right)^{-1} \quad \forall\, (x_1, x_2) \in \mathcal{C} \end{array} \qquad (7)$$

Now suppose that $(x_1^\star, x_2^\star)$ were not Pareto Optimal. Then from the definition there exists a point $(x_1, x_2) \in \mathcal{C}$ such that one of its objective values $f_1(x_1)$ or $f_2(x_2)$ is strictly less than that of $(x_1^\star, x_2^\star)$ while the other is at least as small. This would immediately contradict 7.

Theorem 1 (Computing Pareto)
When the component functions f_i of the objective are convex, then the set of all Pareto optimal points with respect to the vector objective F above, is the set of solutions (when they exist and when they are unique) of $\inf\{(1-\mu) f_1(x_1) + \mu f_2(x_2) \,|\, (x_1, x_2) \in \mathcal{C}\}$ for all $\mu \in [0,1]$ [2].

3 Linear Analysis

In this section, we will show how to compute stability regions and performance information by requiring the state to remain in the region $\mathcal{R}_{\mathbf{1}} = \{x \,|\, \|C_q\|_\infty \leq 1\}$. Since the system never saturates in this region, the closed loop system (1) behaves like:

$$
\begin{array}{rcl}
\dot{x} & = & A_{\rm cl}x + B_w w; \quad x(0) = x_0 \\
z & = & C_z x \\
x & \in & \mathcal{R}_{\mathbf{1}} \quad \Leftrightarrow \quad |q_i| \leq 1,\ i = 1, \ldots, n_q \\
A_{\rm cl} & = & (A + B_p C_q)
\end{array}
\tag{8}
$$

Therefore, provided that we can ensure that x remains in $\mathcal{R}_{\mathbf{1}}$, any information we can compute about system (8) must apply to system (1).

Theorem 2 (Region of Attraction)
For system (8), an $\mathbf{1}$-level region of attraction $\hat{\mathcal{D}}_{\mathbf{1}}$ is given by the maximum volume invariant ellipsoid $\mathcal{E}_{Q^{-1}}(1)$ contained in the region $\mathcal{R}_{\mathbf{1}}$. This can be computed by solving the following convex determinant maximization (maxdet) optimization problem [3, 15] in the variable $Q = Q^T \in \mathbf{R}^{n_x \times n_x}$:

$$
\begin{array}{rl}
\max & \log\det Q \\
\text{s.t.} & C_{q,i} Q C_{q,i}^T \leq 1 \ \ ; i = 1, \ldots, n_q \\
& Q > 0 \\
& A_{\rm cl} Q + Q A_{\rm cl}^T < 0.
\end{array}
\tag{9}
$$

Remark 2 The $\log\det$ objective maximizes volume. But other objectives could be just as valid depending on the application, *e.g.*, one could instead maximize the trace of Q which corresponds to maximizing the sum of the squares of the major axes of $\mathcal{E}_{Q_{-1}}(1)$.

Theorem 3 (Disturbance Rejection)
For system (8), the largest $\mathbf{1}$-level disturbance rejection bound, $\alpha_{\max,\mathbf{1}}$, can be computed as [3, 14]

$$
\begin{array}{rl}
\alpha_{\max,\mathbf{1}} = & \min_{i=1,\ldots,n_q} \frac{1}{C_{q,i} Q C_{q,i}^T} \\
\text{where} & A_{\rm cl} Q + Q A_{\rm cl}^T + B_w B_w^T = 0.
\end{array}
$$

Furthermore, $\mathcal{E}_{Q^{-1}}(\alpha_{\max,\mathbf{1}}) \subset \mathcal{R}_{\mathbf{1}}$ and whenever $\|w\|_2^2 \leq \alpha_{\max,\mathbf{1}}$ and $x(0) = 0$, then $x(t)$ will never leave $\mathcal{E}_{Q^{-1}}(\alpha_{\max,\mathbf{1}})$.

Corollary 1 (Local L_2-Gain)
For system (8), whenever $\|w\|_2^2 \leq \alpha_{\max,\mathbf{1}}$ and $x(0) = 0$, the smallest $\mathbf{1}$-level L_2-gain bound, $\gamma_{\mathbf{1}}$, can be computed as [3, 14]

$$
\begin{array}{rl}
\min & \gamma^2 \\
\text{s.t.} & Q > 0 \\
& \begin{bmatrix} A_{\rm cl} Q + Q A_{\rm cl}^T & Q C_z^T & B_w \\ C_z Q & -I & 0 \\ B_w & 0 & -\gamma^2 I \end{bmatrix} < 0.
\end{array}
\tag{10}
$$

4 Nonlinear Analysis

From fig.2, we see that when $|q_i| \leq r_i$, $i = 1, \ldots, n_q$, the system falls into the class of Lur'e systems:

$$
\begin{array}{rcl}
\dot{x} & = & Ax + B_p p + B_w w; \quad x(0) = x_0 \\
q & = & C_q x, \quad z = C_z x \\
p_i & = & \varphi_{r,i}(q_i), \quad \mathrm{gr}\, \varphi_{r,i} \subset \mathrm{sect}[\frac{1}{r_i}, 1], \quad i = 1, \ldots, n_q \\
x & \in & \mathcal{R}_r \quad \Leftrightarrow \quad |q_i| \leq r_i,\ i = 1, \ldots, n_q
\end{array}
\tag{11}
$$

This system can be analyzed using the circle and Popov criteria. Any information that we can obtain about system (11) will apply to our actual system (1), provided that we can ensure that the state x remains in $\mathcal{R}_r$. This has been noted in the past by several authors [9, 6, 8, 12].

4.1 Circle Analysis

In this section, stability and performance will be analyzed using a quadratic Lyapunov function $V(x) = x^T Q^{-1} x$, where $Q = Q^T > 0$.

Using the change of variables (loop transformation)

$$
\begin{array}{c}
\bar{p}_i = \frac{1}{\delta_{r,i}}(p_i - \rho_{r,i} q) \\
\rho_{r,i} = \frac{1}{2}(1 + \frac{1}{r_i}), \quad \delta_{r,i} = \frac{1}{2}(1 - \frac{1}{r_i})
\end{array}
\tag{12}
$$

the nonlinearity can be centered and normalized on the set $x \in \mathcal{R}_r$ to give the following model in standard form:

$$
\begin{array}{rcl}
\dot{x} & = & A_r x + B_{p,r} \bar{p} + B_w w; \quad x(0) = x_0 \\
q & = & C_q x, \quad z = C_z x \\
\bar{p}_i & = & \Delta_{r,i}(q_i), \quad \|\Delta_{r,i}\| \leq 1, \quad i = 1, \ldots, n_q \\
x & \in & \mathcal{R}_r \quad \Leftrightarrow \quad |q_i| \leq r_i,\ i = 1, \ldots, n_q \\
A_r & = & A + B_p \Gamma_r C_q, \quad B_{p,r} = B_p \Pi_r \\
\Gamma_r & = & \mathbf{diag}(\rho_{r,1}, \ldots, \rho_{r,n_q}) \\
\Pi_r & = & \mathbf{diag}(\delta_{r,1}, \ldots, \delta_{r,n_q}).
\end{array}
\tag{13}
$$

Theorem 4 (Region of Attraction)
For system (13), an r-level region of attraction $\hat{\mathcal{D}}_r$ is given by the maximum volume invariant ellipsoid $\mathcal{E}_{Q^{-1}}(1)$ contained in the region $\mathcal{R}_r$. This can be computed by solving the following maxdet convex optimization problem in the variables $Q = Q^T \in \mathbf{R}^{n_x \times n_x}$ and $S = \mathbf{diag}(s_1, \ldots, s_{n_q})$, [3, 12] :

$$
\begin{array}{rl}
\max & \log\det Q \\
\text{s.t.} & C_{q,i} Q C_{q,i}^T \leq r_i^2 \ \ ; i = 1, \ldots, n_q \\
& Q > 0, \quad S > 0, \\
& \begin{bmatrix} A_r Q + Q A_r + B_{p,r} S B_{p,r}^T & Q C_q^T \\ C_q Q & -S \end{bmatrix} < 0.
\end{array}
\tag{14}
$$

Theorem 5 (Disturbance Rejection)
For system (13), an r-level disturbance rejection bound, $\hat{\alpha}_{\max,r}$, can be computed as $\hat{\alpha}_{\max,r} = 1/t^\star$, where $t^\star$ is the optimal value of the following convex semidefinite program in the variables $t \in \mathbf{R}$,

$Q = Q^T \in \mathbf{R}^{n_x \times n_x}$ and $S = \mathbf{diag}(s_1, \dots, s_{n_q})$ [3, 14]

$$\begin{array}{ll} \min & t \\ \text{s.t.} & C_{q,i} Q C_{q,i}^T \leq r_i^2 t \;\; ; i = 1, \dots, n_q \\ & Q > 0, \quad S > 0, \end{array} \tag{15}$$
$$\begin{bmatrix} A_r Q + Q A_r^T + B_w B_w^T + B_{p,r} S B_{p,r}^T & Q C_q^T \\ C_q Q & -S \end{bmatrix} < 0.$$

Furthermore, $\mathcal{E}_{Q^{-1}}(\hat{\alpha}_{\max,r}) \subset \mathcal{R}_r$ and whenever $\|w\|_2^2 \leq \hat{\alpha}_{\max,r}$ and $x(0) = 0$, then $x(t)$ will never leave $\mathcal{E}_{Q^{-1}}(\hat{\alpha}_{\max,r})$.

Corollary 2 (Local L_2-Gain)
For system (13), whenever $\|w\|_2^2 \leq \hat{\alpha}_{\max,r}$ and $x(0) = 0$, an r-level L_2-gain bound, $\hat{\gamma}_r$, can be computed as [3, 14]

$$\begin{array}{ll} \min & \hat{\gamma}_r^2 \\ \text{s.t.} & Q > 0, \quad S > 0, \end{array}$$
$$\begin{bmatrix} A_r Q + Q A_r^T + B_w B_w^T + B_{p,r} S B_{p,r}^T & Q C_q^T & Q C_z^T \\ C_q Q & -S & 0 \\ C_z Q & 0 & -\hat{\gamma}_r^2 I \end{bmatrix} < 0. \tag{16}$$

4.2 Popov Analysis

Since saturation is a time invariant memoryless nonlinearity, the Popov criterion can often lead to less conservative information than the Circle criterion.

Using the change of variables (loop transformation)

$$\begin{array}{c} \bar{p}_i = \frac{1}{\delta_{r,i}}(p_i - \rho_{r,i} q) \\ \text{either } \rho_{r,i} = 1, \quad \delta_{r,i} = -(1 - \frac{1}{r_i}) \\ \text{or } \rho_{r,i} = \frac{1}{r_i}, \quad \delta_{r,i} = (1 - \frac{1}{r_i}) \end{array} \tag{17}$$

the nonlinearity can be put into standard form on the set $x \in \mathcal{R}_r$:

$$\begin{array}{lcl} \dot{x} & = & A_r x + B_{p,r} \bar{p} + B_w w; \quad x(0) = x_0 \\ q & = & C_q x, \quad z = C_z x \\ \bar{p}_i & = & \bar{\varphi}_{r,i}(q_i), \quad \text{gr}\, \bar{\varphi}_{r,i} \in \text{sect}[0, 1], \quad i = 1, \dots, n_q \\ x & \in & \mathcal{R}_r \quad \Leftrightarrow \quad |q_i| \leq r_i, \; i = 1, \dots, n_q \\ A_r & = & A + B_p \Gamma_r C_q, \quad B_{p,r} = B_p \Pi_r \\ \Gamma_r & = & \mathbf{diag}(\rho_{r,1}, \dots, \rho_{r,n_q}) \\ \Pi_r & = & \mathbf{diag}(\delta_{r,1}, \dots, \delta_{r,n_q}). \end{array} \tag{18}$$

Now we will use the Popov function

$$V(x) = x^T P x + \sum_{i=1}^{n_q} 2\lambda_i \int_0^{C_{q,i} x} \bar{\varphi}_{r,i}(\sigma)\, d\sigma \tag{19}$$

to investigate the local stability and performance. Note that V now depends on the loop transformed nonlinearities and hence depends on the loop transformation.

Since each saturator has three possible states, the Lyapunov function will be a piecewise quadratic function defined on 3^{n_q} regions. As a result of this, the computation of the exact volume and the tangent invariant level set along with its point of tangency with $\mathcal{R}_r$ grows exponentially in complexity. Therefore, we will provide exact solutions only for the case of a single nonlinearity. For the general multiple nonlinearity case, we will provide low complexity heuristics.

Recall from (6), that $\mathcal{E}_{(P + C_q^T \Lambda C_q)}(\alpha)$ and $\mathcal{E}_P(\alpha)$ provide, respectively, inner and outer approximations of $\text{lev}_\alpha V$. Unfortunately, $\mathbf{vol}\, \mathcal{E}_{(P + C_q^T \Lambda C_q)}(\alpha)$ and $\mathbf{vol}\, \mathcal{E}_P(\alpha)$ are not convex expressions in (P, Λ). Therefore we will use the trace, which gives the sum of the squares of the inverses of the major axes as a measure of the size of the ellipsoids.

Theorem 6 (Region of Attraction)
For system (18), an r-level region of attraction $\hat{\mathcal{D}}_r$ is given by the invariant set $\text{lev}_\alpha V$, where V is the Popov Lyapunov function obtained by solving the following convex optimization problem in the variables $P = P^T \in \mathbf{R}^{n_x \times n_x}$, $\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_{n_q})$, and $S = \mathbf{diag}(s_1, \dots, s_{n_q})$ [3, 12]:

$$\begin{array}{ll} \min & \mathbf{Tr}(P + C_q^T \Lambda C_q) \\ \text{s.t.} & \begin{bmatrix} r_i^2 & C_{q,i} \\ C_{q,i}^T & P \end{bmatrix} \geq 0 \;\; ; i = 1, \dots, n_q \\ & P > 0, \quad \Lambda > 0, \quad S > 0, \end{array}$$
$$\begin{bmatrix} A_r^T P + P A_r & P B_{p,r} + A_r^T C_q \Lambda + C_q^T S \\ B_{p,r}^T P + \Lambda C_q A_r + S C_q & \Lambda C_q B_{p,r} + B_{p,r}^T C_q^T \Lambda - 2S \end{bmatrix} < 0. \tag{20}$$

Then α can be taken to be $\max(1, \alpha_r)$ where α_r is computed using Lemma 2.

Next we consider computing $\hat{\alpha}_{\max,r}$ using the Popov criterion. Although this turns out to be more complicated than in the circle criterion case, it can often yield significantly sharper results. We will treat the single nonlinearity case first, then the multiple nonlinearity case.

Theorem 7 (Disturbance Rejection - SISO)
For system (18), an r-level disturbance rejection bound, $\hat{\alpha}_{\max,r}$, can be computed as

$$\hat{\alpha}_{\max,r} = \max_{\mu \in [0,1]} \frac{r^2}{C_q P_\mu^{-1} C_q^T} + 2\lambda_\mu \int_0^r \varphi(\sigma)\, d\sigma \tag{21}$$

where for each $\mu \in [0, 1]$, (P_μ, λ_μ) is the optimal value of the following convex semidefinite program in the variables $t_1, t_2 \in \mathbf{R}$, $P = P^T \in \mathbf{R}^{n_x \times n_x}$, $\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_{n_q})$, and $s \in \mathbf{R}$ [3]:

$$\begin{array}{ll} \min & (1 - \mu)\, t_1 + \mu\, t_2 \\ \text{s.t.} & \begin{bmatrix} t_1 & C_q \\ C_q^T & P \end{bmatrix} \geq 0, \quad \begin{bmatrix} t_2 & 1 \\ 1 & \lambda \end{bmatrix} \geq 0, \\ & P > 0, \quad \lambda > 0, \quad s > 0, \end{array}$$
$$\begin{bmatrix} A_r^T P + P A_r & P B_{p,r} + A_r^T C_q \lambda + C_q^T s & P B_w \\ B_{p,r}^T P + \lambda C_q A_r + s C_q & \lambda C_q B_{p,r} + B_{p,r}^T C_q^T \lambda - 2s & \lambda C_q B_w \\ B_w^T P & B_w^T C_q^T \lambda & -I \end{bmatrix} < 0. \tag{22}$$

Furthermore, if $\mu^\star$ is the optimizer of (21) and V is the Popov function formed from the corresponding $(P_{\mu^\star}, \lambda_{\mu^\star})$, then $\text{lev}_{\hat{\alpha}_{\max,r}} V \subset \mathcal{R}_r$. Whenever

$\|w\|_2^2 \leq \hat{\alpha}_{\max,r}$ and $x(0) = 0$, then $x(t)$ will never leave $\mathrm{lev}_{\hat{\alpha}_{\max,r}} V$.

Remark 3 Of course in practice it is impossible to search the entire segment $[0,1]$. However, taking 5 to 10 points for different μ's often yields a sufficiently good maximum.

We will not attempt to extend Theorem 7 to the case of multiple nonlinearities since it would become computationally very intensive. Instead, we offer the following heuristic method to compute $\hat{\alpha}_{\max,r}$ which is obtained by direct analogy with the circle criterion case.

Theorem 8 (Disturbance Rejection - MIMO) For system (18), an r-level disturbance rejection bound, $\hat{\alpha}_{\max,r}$, can be computed by applying Lemma 2 to the Popov function V obtained from $(P^\star, \Lambda^\star)$, where $(P^\star, \Lambda^\star)$ are the optimizers of the convex semidefinite program in the variables $t \in \mathbf{R}$, $P = P^T \in \mathbf{R}^{n_x \times n_x}$, $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_{n_q})$, and $S = \mathbf{diag}(s_1, \ldots, s_{n_q})$ [3]:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & \begin{bmatrix} r_i^2 t & C_{q,i} \\ C_{q,i}^T & P \end{bmatrix} \geq 0 \ ; i = 1, \ldots, n_q \\ & P > 0, \quad \Lambda > 0, \quad S > 0, \end{array}$$
$$\begin{bmatrix} A_r^T P + P A_r & P B_{p,r} + A_r^T C_q \Lambda + C_q^T S & P B_w \\ B_{p,r}^T P + \Lambda C_q A_r + S C_q & \Lambda C_q B_{p,r} + B_{p,r}^T C_q^T \Lambda - 2S & \Lambda C_q B_w \\ B_w^T P & B_w^T C_q^T \Lambda & -I \end{bmatrix} < 0. \tag{23}$$

Furthermore, $\mathrm{lev}_{\hat{\alpha}_{\max,r}} V \subset \mathcal{R}_r$ and whenever $\|w\|_2^2 \leq \hat{\alpha}_{\max,r}$ and $x(0) = 0$, then $x(t)$ will never leave $\mathrm{lev}_{\hat{\alpha}_{\max,r}} V$.

Corollary 3 (Local L_2-Gain) For system (18), whenever $\|w\|_2^2 \leq \hat{\alpha}_{\max,r}$ and $x(0) = 0$, an r-level L_2-gain bound, $\hat{\gamma}_r$, can be computed as [3]

$$\begin{array}{ll} \min & \hat{\gamma}_r^2 \\ \text{s.t.} & P > 0, \quad \Lambda > 0, \quad S > 0, \end{array}$$
$$\begin{bmatrix} A_r^T P + P A_r + C_z^T C_z & P B_{p,r} + A_r^T C_q \Lambda + C_q^T S & P B_w \\ B_{p,r}^T P + \Lambda C_q A_r + S C_q & \Lambda C_q B_{p,r} + B_{p,r}^T C_q^T \Lambda - 2S & \Lambda C_q B_w \\ B_w^T P & B_w^T C_q^T \Lambda & -\hat{\gamma}_r^2 I \end{bmatrix} < 0. \tag{24}$$

5 Conclusion

We have presented a set of tools for performing local stability and performance analysis for linear systems with saturation. Our results are all formulated in terms of linear matrix inequalities, and all lead to efficient computations. They are general and handle multiple nonlinearities, and can easily be extended to discrete time systems. Our experience indicates that the information obtained using these tools is often significantly better than that obtained from the linear analysis.

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