CONTROL SYSTEM ANALYSIS AND SYNTHESIS VIA LINEAR MATRIX INEQUALITIES[1]

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Abstract

A wide variety of problems in systems and control theory can be cast or recast as convex problems that involve linear matrix inequalities (LMIs). For a few very special cases there are "analytical solutions" to these problems, but in general they can be solved numerically very efficiently. In many cases the inequalities have the form of simultaneous Lyapunov or algebraic Riccati inequalities; such problems can be solved in a time that is comparable to the time required to solve the same number of Lyapunov or Algebraic Riccati equations. Therefore the computational cost of extending current control theory that is based on the solution of algebraic Riccati equations to a theory based on the solution of (multiple, simultaneous) Lyapunov or Riccati inequalities is modest.

Examples include: multicriterion LQG, synthesis of linear state feedback for multiple or nonlinear plants ("multi-model control"), optimal transfer matrix realization, norm scaling, synthesis of multipliers for Popov-like analysis of systems with unknown gains, and many others. Full details can be found in the references cited.

1. Motivation

This paper is motivated by two recent developments: the dramatic and continuing growth in computer power, and the advent of very powerful algorithms (and associated theory) for convex optimization. As a result of these developments, we can now solve very rapidly many convex optimization problems for which no traditional "analytic" or "closed-form" solutions are known (or likely to exist). Indeed, the solution to many convex optimization problems can now be computed in a time which is comparable to the time required to evaluate a "closed-form" solution for a similar problem. In our opinion, this fact has far-reaching implications for engineers: it changes our fundamental notion of what we should consider as a solution to a problem. In the past, a "solution to a problem" generally meant a "closed-form" or "analytic" solution. We believe that in the future, our concept of "solution" should be extended to include many forms of convex programming.

As an example, a control engineering problem that reduces to solving two algebraic Riccati equations is now generally regarded as "solved." Our thesis is that a control engineering problem that reduces to solving even a large number of convex algebraic Riccati inequalities (a problem which has no "analytic" solution) should also be regarded as "solved", even though there is no "analytic" solution.

A number of problems that arise in Systems and Control such as optimal matrix scaling, digital filter realization, interpolation problems that arise in system identification, robustness analysis and state-feedback synthesis via Lyapunov functions, can be reduced to a handful of standard convex and quasiconvex problems that involve matrix inequalities. Extremely efficient interior point algorithms have recently been developed for and tested on these standard problems; further development of algorithms for these standard problems is an area of active research. In this paper, we first briefly describe these optimization problems based on linear matrix inequalities. We will then discuss a few examples of problems from systems and control that can be cast as convex optimization problems over LMIs.

2. Standard problems involving LMIs

A linear matrix inequality is a matrix inequality of the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0, \qquad (1)$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n\times n}$, $i = 0, \ldots, m$ are given. The set $\{x \mid F(x) > 0\}$ is convex, and need not have smooth boundary. (We've used strict inequality mostly as a convenience; inequalities of the form $F(x) \geq 0$ are also readily handled.)

[1]Research supported in part by AFOSR grant F49620-92-J-0013-P00001 and DARPA grant AF F49620-90-C-0014.

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Multiple LMIs $F_1(x) > 0, \ldots, F_n(x) > 0$ can be expressed as the single LMI $\mathbf{diag}(F_1(x), \ldots, F_n(x)) > 0$. Therefore we will make no distinction between a set of LMIs and a single LMI, *i.e.*, "the LMI $F_1(x) > 0, \ldots, F_n(x) > 0$" will mean "the LMI $\mathbf{diag}(F_1(x), \ldots, F_n(x)) > 0$".

When the matrices F_i are diagonal, the LMI $F(x) > 0$ is just a set of linear inequalities. Nonlinear (convex) inequalities are converted to LMI form using Schur complements. The basic idea is as follows: the LMI

$$\left[\begin{array}{cc} Q(x) & S(x) \\ S(x)^T & R(x) \end{array}\right] > 0 \qquad (2)$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x, is equivalent to

$$R(x) > 0, \; Q(x) - S(x)R(x)^{-1}S(x)^T > 0. \qquad (3)$$

In other words, the set of nonlinear inequalities (3) can be represented as the LMI (2).

The matrix norm constraint $\|Z(x)\| < 1$, where $Z(x) \in \mathbf{R}^{p\times q}$ and depends affinely on x, is represented as the LMI

$$\left[\begin{array}{cc} I & Z(x) \\ Z(x)^T & I \end{array}\right] > 0$$

(since $\|Z\| < 1$ is equivalent to $I - ZZ^T > 0$). Note that the case $q = 1$ reduces to a general convex quadratic inequality on x.

The constraint $c(x)^T P(x)^{-1} c(x) < 1$, $P(x) > 0$, where $c(x) \in \mathbf{R}^n$ and $P(x) = P(x)^T \in \mathbf{R}^{n\times n}$ depend affinely on x, is expressed as the LMI

$$\left[\begin{array}{cc} P(x) & c(x) \\ c(x)^T & 1 \end{array}\right] > 0.$$

More generally, the constraint

$$\mathbf{Tr}\, S(x)^T P(x)^{-1} S(x) < 1, \quad P(x) > 0,$$

where $P(x) = P(x)^T \in \mathbf{R}^{n\times n}$ and $S(x) \in \mathbf{R}^{n\times p}$ depend affinely on x, is handled by introducing a new (slack) matrix variable $X = X^T \in \mathbf{R}^{p\times p}$, and the LMI (in x and X):

$$\mathbf{Tr}\, X < 1, \quad \left[\begin{array}{cc} X & S(x)^T \\ S(x) & P(x) \end{array}\right] > 0.$$

We often encounter problems in which the variables are matrices, *e.g.*,

$$A^T P + PA < 0 \qquad (4)$$

where $A \in \mathbf{R}^{n\times n}$ is given and $P = P^T$ is the variable. In this case we will not write out the LMI explicitly in the form $F(x) > 0$, but instead make clear which matrices are the variables. The phrase "the LMI $A^T P + PA < 0$ in P" means that the matrix P is a variable. (Of course, the Lyapunov inequality (4) is readily put in the form (1), as follows. Let $P_1, \ldots, P_m$ be a basis for symmetric $n \times n$ matrices ($m = n(n+1)/2$). Then take $F_0 = 0$ and $F_i = -A^T P_i - P_i A$.) Leaving LMIs in a condensed form such as (4), in addition to saving notation, leaves open the possibility of more efficient computation.

As another related example, consider the algebraic Riccati inequality

$$A^T P + PA + PBR^{-1}B^T P + Q < 0, \quad R > 0 \qquad (5)$$

where A, B, $Q = Q^T$, $R = R^T$ are given matrices of appropriate size, and $P = P^T$ is the variable. Inequality (5) can be expressed as the LMI in P,

$$\left[\begin{array}{cc} -A^T P - PA - Q & PB \\ B^T P & R \end{array}\right] > 0.$$

(Note that it can also be considered an LMI in P, Q, and R.)

2.1. LMI feasibility problems

Given an LMI $F(x) > 0$, the corresponding LMI Problem (LMIP) is to find x^{feas} such that $F(x^{\mathrm{feas}}) > 0$ or determine that the LMI is infeasible. (By duality, this means: find a nonzero $G \geq 0$ such that $\mathbf{Tr}\, GF_i = 0$ for $i = 1, \ldots, m$ and $\mathbf{Tr}\, GF_0 \leq 0$.) Of course, this is a convex feasibility problem. We will say "solving the LMI $F(x) > 0$" to mean solving the corresponding LMIP.

2.2. Eigenvalue problems

The eigenvalue problem (EVP) is to minimize the maximum eigenvalue of a matrix, subject to an LMI:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda I - A(x) > 0, \quad B(x) > 0. \end{array}$$

Here, A and B are symmetric matrices that depend affinely on the optimization variable x. This is a convex optimization problem.

2.3. Generalized eigenvalue problems

The generalized eigenvalue problem (GEVP) is to minimize the maximum generalized eigenvalue of a pair of matrices that depend affinely on a variable, subject to an LMI constraint. The general form of a

GEVP is:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda B(x) - A(x) > 0 \\ & B(x) > 0 \\ & C(x) > 0 \end{array}$$

where A, B and C are affine functions of x. This is a quasiconvex problem.

Note that when the matrices are all diagonal, this problem reduces to the general linear fractional programming problem. Many nonlinear quasiconvex functions can be represented in the form of a GEVP with appropriate A, B, and C (see [1]).

3. LMI problems in systems and control

3.1. Matrix scaling problem

The problem of similarity-scaling a matrix to minimize its norm appears in several control applications [2, 3, 4] (see also [5] and [6] for a related problem). Given $M \in \mathbf{C}^{m\times m}$, the optimal diagonally scaled norm of M is defined as

$$\nu(M) \triangleq \inf_{D\in\mathcal{P}} \left\| DMD^{-1} \right\|,$$

where $\mathcal{P}$ is the set of diagonal non-singular matrices of size m.

Note that $\nu(M) < \gamma$ if and only if there exists nonsingular $D \in \mathcal{P}$ such that $\left(DMD^{-1}\right)^* \left(DMD^{-1}\right) < \gamma^2 I$, or $M^*D^*DM < \gamma^2 D^*D$. Therefore,

$$\nu(M) = \inf\left\{\gamma \;\middle|\; M^*PM < \gamma^2 P,\ P = P^* > 0 \in \mathcal{P}\right\}$$

Therefore $\nu(M)$ is the optimal value of the GEVP

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & P > 0, \quad P \in \mathcal{P} \\ & M^*PM < \gamma^2 P \end{array}$$

3.2. Lyapunov function search

Consider the differential inclusion (DI)

$$\frac{dx}{dt} = A(t)x(t), \;\; A(t) \in \mathbf{Co}\{A_1, \ldots, A_L\} \qquad (6)$$

where **Co** denotes the convex hull. We ask whether the DI is stable, *i.e.*, whether all trajectories of the system (6) converge to zero as $t \longrightarrow \infty$. A sufficient condition for this is the existence of a quadratic positive function $V(z) = z^T Pz$ such that $dV(x(t))/dt < 0$ for any trajectory of (6). Since

$$\frac{d}{dt}V(x(t)) = x(t)^T \left(A(t)^T P + PA(t)\right) x(t),$$

a sufficient condition for stability is the existence of $P > 0$ such that

$$A(t)^T P + PA(t) < 0, \;\; A(t) \in \mathbf{Co}\{A_1, \ldots, A_L\}. \quad (7)$$

If there exists such a P, we say the DI (6) is *quadratically stable*.

Condition (7) is equivalent to

$$P > 0, \quad A_i^T P + PA_i < 0, \;\; i = 1, \ldots, L,$$

which is a linear matrix inequality in P (see for example [7, 8, 9, 10]). Thus, determining quadratic stability is an LMIP.

V is sometimes called a *simultaneous quadratic Lyapunov function* since it proves stability of each of $A_1, \ldots, A_L$.

3.3. Lyapunov functions and state feedback

Consider the system (6) with state feedback:

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t), \;\; u(t) = Kx(t) \qquad (8)$$

where

$$[A(t)\ B(t)] \in \mathbf{Co}\left\{[A_1\ B_1], \ldots, [A_L\ B_L]\right\}.$$

Our objective is to design the matrix K such that such that (8) is quadratically stable. This is the "quadratic stabilizability" problem (see [11], and [12, 13, 14]; related references are [15, 16, 17] and [18]).

System (8) is quadratically stable for some state state-feedback K if there exist $P > 0$ and K such that

$$(A_i + B_iK)^T P + P(A_i + B_iK) < 0, \;\; i = 1, \ldots, L.$$

Note that this matrix inequality is *not* convex in P and K. However, with the linear fractional transformation $Y \triangleq P^{-1}$, $W \triangleq KP^{-1}$, we may rewrite it as

$$(A_i + B_iWY^{-1})^T Y^{-1} + Y^{-1}(A_i + B_iWY^{-1}) < 0.$$

Multiplying this inequality on the left and right by Y (such a congruence preserves the inequality) we get an LMI in Y and W:

$$YA_i^T + W^T B_i^T + A_iY + B_iW < 0, \;\; i = 1, \ldots, L.$$

If this LMIP in Y and W has a solution, then the Lyapunov function $V(z) = z^T Y^{-1} z$ proves the quadratic stability of the closed-loop system with state-feedback $u(t) = WY^{-1}x(t)$.

In other words, we can synthesize a linear state feedback for the DI (6) by solving a set of simultaneous Lyapunov inequalities. We will briefly discuss the implications of this result for robust control synthesis in section 5..

Let us also note that by synthesizing a state feedback for the DI (6), we have also synthesized a suitable state feedback for the nonlinear, time-varying, uncertain system

$$\frac{dx}{dt} = f(x, u, t, p)$$

where p is some parameter vector, provided we have

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial u} \end{array}\right] \in \mathbf{Co}\,\{[A_1\ B_1], \ldots, [A_L\ B_L]\}$$

for all x, u, t, and p.

3.4. System realization

For the discrete-time LTI system

$$\begin{array}{rcl} x(k+1) &=& Ax(k) + Bu(k) \\ y(k) &=& Cx(k), \end{array}$$

where $x : \mathbf{Z}_+ \to \mathbf{R}^n$, $u : \mathbf{Z}_+ \to \mathbf{R}^{n_u}$ $y : \mathbf{Z}_+ \to \mathbf{R}^{n_y}$, and $\{A, B, C\}$ is a minimal realization, the *system realization problem* is to find a change of state coordinates $x = T\bar{x}$ with two competing constraints: First, the input-to-state transfer matrix, $T^{-1}(zI - A)^{-1}B$ should be "small", in order to avoid state overflow in numerical implementation; and second, the state-to-output transfer matrix, $C(zI - A)^{-1}T$ should be "small", in order to minimize effects of state quantization at the output. We refer the reader to [19] and [20]; the forthcoming book [21], by M. Gevers, describes a number of digital filter realization problems.

If it is known that the RMS value of the input is bounded by α and the RMS value of the state is required to be less than, say, one, we have an $\mathbf{H}_\infty$ norm bound on the input-to-state map:

$$\alpha \left\| T^{-1}(zI - A)^{-1}B \right\|_\infty < 1. \qquad (9)$$

Next, suppose that the state quantization noise is modeled as a unit white noise sequence w (*i.e.*, $\mathbf{E}\,w(k) = 0$, $\mathbf{E}\,w(k)w(j)^T = \eta^2 \delta_{kj} I$) injected directly into the state, and its effect on the output is measured by the total noise power appearing in the output, which is just η^2 times the square of the $\mathbf{H}_2$ norm of the state-to-output transfer matrix:

$$p_{\rm noise} = \eta^2 \left\| C(zI - A)^{-1}T \right\|_2^2. \qquad (10)$$

Our problem is then to compute T to minimize the output noise power (10) subject to the overflow avoidance constraint (9). We will show that this problem can be expressed as an EVP.

The constraint (9) is equivalent to the existence of $P > 0$ such that

$$\left[\begin{array}{cc} A^T PA - P + T^{-T}T^{-1} & A^T PB \\ B^T PA & B^T PB - I/\alpha^2 \end{array}\right] < 0.$$

The output noise power can be expressed as

$$p_{\rm noise}^2 = \eta^2\, \mathbf{Tr}\, T^T W_{\rm obs} T,$$

where $W_{\rm obs}$ is the observability Gramian of the original system $\{A, B, C\}$, *i.e.*, the unique solution of the Lyapunov equation

$$A^T W_{\rm obs} A - W_{\rm obs} + C^T C = 0.$$

With $X = T^{-T}T^{-1}$, the realization problem becomes: minimize $\eta^2\, \mathbf{Tr}\, W_{\rm obs} X^{-1}$ subject to $X > 0$ and the LMI (in $P > 0$ and X)

$$\left[\begin{array}{cc} A^T PA - P + X & A^T PB \\ B^T PA & B^T PB - I/\alpha^2 \end{array}\right] < 0. \qquad (11)$$

This is a convex problem in X and P, and can be transformed into the EVP

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, Y \\ \text{subject to} & (11),\ P > 0,\ \left[\begin{array}{cc} Y & \eta W_{\rm obs}^{1/2} \\ \eta W_{\rm obs}^{1/2} & X \end{array}\right] > 0 \end{array}$$

More sophisticated realization problems are readily reduced to LMI problems. For examples, we can have a bound on the RMS value of each component of the state, and minimize the maximum quantization induced noise power of the components of the output.

We note that a very simple realization problem, in which the overflow constraint and the noise objective are both expressed as $\mathbf{H}_2$ constraints, has a well-known "analytic" solution: T is chosen so that the system is, except for a constant scaling, balanced. Our point is that more sophisticated realization problems, which reflect much more accurately the true engineering specifications, are also readily solved, not "analytically" but as LMI problems.

3.5. Inverse problem of optimal control

Given a system

$$\begin{array}{rcl} \frac{d}{dt}x(t) &=& Ax(t) + Bu(t) \\ z(t) &=& \left[\begin{array}{cc} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{array}\right] \left[\begin{array}{c} x(t) \\ u(t) \end{array}\right] \\ x(0) &=& x_0, \end{array}$$

assuming (A, B) is stabilizable, (Q, A) is detectable and $R > 0$, the LQR ("optimal control") problem is to determine the input u that minimizes the performance index

$$\int_0^\infty z(t)^T z(t)\, dt.$$

The solution of this problem can be expressed as a state-feedback $u = Kx$ with $K = -R^{-1}B^T P$, where P is the unique positive definite solution of the ARE

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

The "inverse optimal control problem" is: given a gain K, determine whether there exist $Q \geq 0$ and $R > 0$, (Q, A) being observable such that $u(t) = Kx(t)$ is the optimal control law corresponding to the corresponding LQR problem.

This inverse optimal control problem was originally considered in a famous paper of Kalman [22], who gave the solution in the case of a single actuator in terms of the loop transfer function. Anderson and Moore [23, p. 131-133] give a solution based on the singular-value plot of an appropriate loop transfer matrix in the case of multiple actuators and *known* matrix R.

The general inverse optimal control problem is readily reduced to an LMI problem. It is equivalent to finding $R > 0$ and $Q \geq 0$ such that there exists a positive P and a positive-definite W satisfying

$$(A + BK)^T P + P(A + BK) + K^T RK + Q < 0,$$

$$A^T W + WA < Q, \text{ and } B^T P + KR = 0.$$

This is an LMIP in P, W, R and Q.

4. Solving LMI-based problems

The most important point is:

LMIPs, EVPs, and GEVPs, are tractable

in a sense that can be made precise from a number of theoretical and practical viewpoints. (This is to be contrasted with much less tractable problems, *e.g.*, the general problem of robustness analysis for a system with real parameter perturbations.)

From a theoretical standpoint:

- we can immediately write down necessary and sufficient optimality conditions
- there is a well-developed duality theory (for GEVPs, in a limited sense)
- these problems can be solved in polynomial time (indeed with a variety of interpretations of the term "polynomial-time").

The most important practical implication is that there are effective and powerful algorithms for the solution of these problems, that is, algorithms that rapidly compute the global optimum, with non-heuristic stopping criteria. Thus, on exit, the algorithms can prove that the global optimum has been obtained to within some prespecified accuracy.

There are a number of general algorithms for the solution of these problems, for example, the ellipsoid algorithm (see *e.g.*, [24, 25]). The ellipsoid method has polynomial-time complexity, and works in practice for smaller problems, but can be slow for larger problems. Other algorithms specifically for LMI-based problems are discussed in, *e.g.*, [26, 27].

Recently, various researchers [28, 1, 29, 30] have developed interior point methods for solving LMI-based problems, based on the work of Nesterov and Nemirovsky [31]. Numerical experience shows that these algorithms solve LMI problems with extreme efficiency. In some specific cases (one is discussed below) these methods can solve LMI-based problems with computational effort that is comparable to that required to "evaluate" the "analytic" solutions of similar problems.

4.1. Multiple Lyapunov inequalities

In this section we consider a specific family of LMI problems for which efficient interior methods have been developed and extensively tested.

Consider the EVP: minimize (over P) $\mathbf{Tr}\, CP$ subject to:

$$A_i^T P + PA_i + B_i < 0, \quad i = 1, \ldots, L$$

where $A_i,\ B_i,\ P \in \mathbf{R}^{n \times n}$.

With new primal-dual methods [29], this problem can be solved in $O(L^{1.2}n^4)$ operations. Of course, the cost associated in solving the *L independent Lyapunov equations*

$$A_i^T P_i + P_i A_i + B_i = 0, \quad i = 1, \ldots, L$$

is $O(Ln^3)$.

Thus the ratio of the cost of solving multiple simultaneous Lyapunov inequalities to the cost of solving the same number of Lyapunov equations is $O(L^{0.2}n)$. In other words:

> the cost of solving **multiple simultaneous Lyapunov inequalities** is not much more than solving the same number of **Lyapunov equations**,

even though the former problem has no "analytic solution" while the latter does. Similar statements hold for multiple Riccati inequalities.

5. Multi-model robust control

The fact that multiple Lyapunov or Riccati inequalities can be solved very efficiently suggests the possibility of extending current control theory and practice beyond the solution of (a pair of) algebraic Riccati equations. One practical implication is that the synthesis of state-feedback for the problem discussed in section 3.3. (or really, more useful extensions) is computationally very cheap.

We describe here one possible extension. The current paradigm for robust control is:

- Develop a model of the set of possible plants in a specific form such as a nominal plant and frequency-dependent bounds on the possible plant perturbations. (Recent work on identification seeks to make the development of plant set models from empirical data more rigorous.)
- Apply various methods of "robust synthesis" to determine a controller.

The remarks above suggest the following alternative and perhaps simpler paradigm:

- Develop many models of the real plant, perhaps from experimental data taken at different times, under different operating conditions, and so on. This set of models is our model of the plant set. In particular, we do not conjecture various frequency response bounds on modeling errors or plant perturbations.
- Use the method of section 3.3. (or really, extensions of it that include bounds on actuator effort, etc.) to synthesize a state feedback that is appropriate for the DI given by our plant models.

We suspect that a state feedback designed in this way has a very high probability of success with the real plant. We should note immediately an important limitation: it is only for *state feedback* that these robust synthesis problems can be reduced to LMI problems; *output feedback* cannot be handled.

6. Conclusion

We have shown that many problems in systems and control can be cast as convex optimization problems involving LMIs. These problems do not have "analytic solutions" but can be solved extremely efficiently. The list of problems we have presented is by no means exhaustive. Other problems include:

- synthesis of gain-scheduled state-feedback
- linear controller design via Q-parametrization
- multi-criterion LQG
- interpolation problems involving scaling
- synthesis of Lyapunov functions with Popov terms for nonlinearities
- synthesis of multipliers for analysis of systems with unknown constant parameters
- analysis and design for randomly varying systems
- synthesis of quadratic Lyapunov functionals for delay systems
- problems in robust identification

We refer the reader to the forthcoming monograph [33] for details.

7. History

Perhaps the most famous LMI in control is the Lyapunov inequality for the stability of LTI systems $A^TP + PA < 0$ (see for example, [34, p.277]), which was originally considered about 100 years ago. Yakubovich was the first to make systematic use of LMIs along with the "$\mathcal{S}$-procedure" for proving stability of nonlinear control systems (see references [35, 36, 37, 38]). The works of Popov [39] and Willems [40] on optimal control outlined the relationship between the problem of absolute stability of automatic control, $\mathbf{H}_\infty$ theory and LMIs. Willems [41], in particular, mentions LMIs as potentially powerful tools for systems analysis:

> The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example.

—Jan Willems, 1971.

More recent work on LMIs includes:

- Barmish, Hollot [11, 13]: *quadratic stabilizability*
- Khargonekar & Rotea [16, 42]: *mixed* $\mathbf{H}_2/\mathbf{H}_\infty$
- Doyle, Packard & associates [18, 45, 46, 47, 46]: *LMIs,* μ *problems and robust control*
- Geromel, Peres & Bernussou [48, 49]: *robust control*
- Safonov [50, 51, 52]: K_m*-synthesis*
- Nesterov & Nemirovskii [31], Vandenberghe & Boyd [29], Haeberly & Overton [53], Jarre [54], Fan [55, 56]: *algorithms*
- Boyd, El Ghaoui, Feron & Balakrishnan [33]: *monograph in preparation*

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