SOLVING INTERPOLATION PROBLEMS VIA GENERALIZED EIGENVALUE MINIMIZATION

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Abstract

A number of problems in the analysis and design of control systems may be reformulated as the problem of minimizing the largest generalized eigenvalue of a pair of symmetric matrices which depend affinely on the decision variables, subject to constraints that are linear matrix inequalities. For these generalized eigenvalue problems, there exist numerical algorithms that are guaranteed to be globally convergent, have polynomial worst-case complexity, and stopping criteria that guarantee desired accuracy. In this paper, we show how a number of important interpolation problems in control may be solved via generalized eigenvalue minimization.

1. Introduction

$\mathbf{R}$ denotes the set of real numbers. $\mathbf{C}$ denotes the set of complex numbers. $\mathbf{C}_+$ denotes the set of complex numbers with positive real part. For $c \in \mathbf{C}$, $\Re c$ is the real part of c. The set of $p \times q$ matrices with complex entries is denoted $\mathbf{C}^{p\times q}$. P^* stands for the complex conjugate transpose of P. I denotes the identity matrix, with size determined from context. $\|P\|$ denotes the spectral norm (maximum singular value) of $P \in \mathbf{C}^{p\times q}$, defined as the square-root of the maximum eigenvalue of P^*P. The matrix inequalities $A > B$ and $A \geq B$ mean A and B are square, Hermitian, and that $A - B$ is positive definite and positive semi-definite, respectively.

A linear matrix inequality or an LMI is a matrix inequality of the form $D(x) = D_0 + \sum_{i=1}^{N} x_i D_i > 0$, where D_i are given real symmetric matrices, and the x_is are the decision variables. We will use the term EVP (GEVP) for the optimization problem of minimizing the largest eigenvalue (generalized eigenvalue) of a symmetric matrix (pair of symmetric matrices) that depends affinely on the decision variables, subject to linear matrix inequality constraints. EVPs (GEVPs) are convex (quasiconvex) nondifferentiable optimization problems; therefore there exist readily derived necessary and sufficient optimality conditions and a well-developed duality theory. Moreover, the fundamental computational complexity of these problems is low, in particular, polynomial-time. Thus EVPs and GEVPs are *tractable*.

There exist a number of general algorithms such as the ellipsoid method that are guaranteed to work for EVPs and GEVPs. More recently, a number of very efficient interior point methods have been proposed for these problems. We refer the reader to [1, 2] for more details.

Several important problems in the analysis and design of control systems may be reformulated as EVPs and GEVPs (see [3] and the references therein). In this paper, we will show how a number of important interpolation problems in control are equivalent to generalized eigenvalue minimization over linear matrix inequalities.

2. Tangential Nevanlinna-Pick problem

Given $\lambda_1, \ldots, \lambda_m$ with $\lambda_i \in \mathbf{C}_+$, $u_1, \ldots, u_m$, with $u_i \in \mathbf{C}^q$ and $v_1, \ldots, v_m$, with $v_i \in \mathbf{C}^p$, $i = 1, \ldots, m$, and $M > 0$, the tangential Nevanlinna-Pick problem is to find, if possible, a function $H : \mathbf{C} \to \mathbf{C}^{p\times q}$ which is analytic in $\mathbf{C}_+$, and satisfies

$$H(\lambda_i)u_i = v_i, \quad i = 1, \ldots, m \text{ with } \|H\|_\infty \leq M.$$

($\|H\|_\infty$ denotes the $\mathbf{H}_\infty$-norm of H, which is defined as $\sup_{\omega\in\mathbf{R}} \|H(j\omega)\|$.) This problem, which arises in multi-input multi-output $\mathbf{H}_\infty$-control theory [4], has a solution if and only if there exist $G_{\rm in} > 0$ and $G_{\rm out} > 0$ such that the following equations and inequality hold.

$$\begin{array}{rcl} A^*G_{\rm in} + G_{\rm in}A - U^*U &=& 0, \\ A^*G_{\rm out} + G_{\rm out}A - V^*V &=& 0, \\ M^2G_{\rm in} - G_{\rm out} &\geq& 0, \end{array} \qquad (1)$$

where $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$, $U = [u_1 \ \cdots \ u_m]$ and $V = [v_1 \ \cdots \ v_m]$. Finding the smallest M such that there exist $G_{\rm in} > 0$ and $G_{\rm out} > 0$ satisfying (1) is a GEVP with variables $G_{\rm in}$ and $G_{\rm out}$. In fact, it is a simple generalized eigenvalue calculation.

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3. Nevanlinna-Pick problem with scaling

Given $\lambda_1, \ldots, \lambda_m$ with $\lambda_i \in \mathbf{C}_+$, $u_1, \ldots, u_m$, with $u_i \in \mathbf{C}^p$ and $v_1, \ldots, v_m$, with $v_i \in \mathbf{C}^p$, $i = 1, \ldots, m$, the problem is to find

$$\gamma_{\rm opt} = \inf \left\{ \|DHD^{-1}\|_\infty \;\middle|\; \begin{array}{c} H \text{ analytic in } \mathbf{C}_+ \\ D = D^* > 0 \\ D \in \mathcal{D} \\ H(\lambda_i)u_i = v_i \\ i = 1, \ldots, m \end{array} \right\},$$

where $\mathcal{D}$ is typically the set of diagonal or block-diagonal matrices.

This problem corresponds to finding the smallest scaled $\mathbf{H}_\infty$ norm of all interpolants. This problem arises in multi-input multi-output $\mathbf{H}_\infty$-control synthesis for systems with structured perturbations [5].

With a change of variables $P = D^*D$, $\gamma_{\rm opt}$ may be determined as the smallest positive M such that there exist $P > 0$, $P \in \mathcal{D}$, $G_{\rm in} > 0$ and $G_{\rm out} > 0$ such that the following equations and inequality hold.

$$\begin{array}{rcl} A^*G_{\rm in} + G_{\rm in}A - U^*PU & = & 0, \\ A^*G_{\rm out} + G_{\rm out}A - V^*PV & = & 0, \\ M^2G_{\rm in} - G_{\rm out} & \geq & 0, \end{array}$$

where $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$, $U = [u_1 \;\; \cdots \;\; u_m]$ and $V = [v_1 \;\; \cdots \;\; v_m]$. This is a GEVP with variables P, $G_{\rm in}$ and $G_{\rm out}$.

4. Frequency response identification

Consider the problem of identifying the transfer function H of a single-input single-output linear system from noisy measurements of its frequency response at a list of frequencies [6]: Given ω_i and $f_i = H(j\omega_i) + n_i$, $i = 1, \ldots, m$, where n_i are the (unknown) noise values, the problem is to identify $H(s)$.

H may be required to satisfy some constraints, which may arise from information about the system known *a priori*, or may reflect properties desired of the identified system, consistent with measurements, such as

1. H is analytic in $\Re s > -\alpha$ with $\alpha > 0$.
2. The α-shifted $\mathbf{H}_\infty$ norm of H, defined as $\|H\|_{\alpha,\infty} \triangleq \sup_{\Re s > -\alpha} \|H(s)\|$ does not exceed M.
3. The noise values satisfy $\|n\| \leq \epsilon$. (Or $\|n\|_\infty \triangleq \max_{1 \leq i \leq m} |n_i| \leq \epsilon$.)

From Nevanlinna-Pick theory, there exist H and n with $H(\omega_i) = f_i + n_i$ satisfying conditions (1–3) if and only if there exist n satisfying (3), $G_{\rm in} > 0$ and $G_{\rm out} > 0$ such that

$$(A + \alpha I)^* G_{\rm in} + G_{\rm in}(A + \alpha I) - e^*e = 0,$$
$$(A + \alpha I)^* G_{\rm out} + G_{\rm out}(A + \alpha I) - (f+n)^*(f+n) = 0,$$
$$M^2G_{\rm in} - G_{\rm out} \geq 0,$$

where $A = \mathrm{diag}(j\omega_1, \ldots, j\omega_m)$, $e = [1 \;\; \cdots \;\; 1]$, $f = [f_1 \;\; \cdots \;\; f_m]$ and $n = [n_1 \;\; \cdots \;\; n_m]$. It can be shown that these conditions are equivalent to

$$M^2G_{\rm in} - G_{\rm out} \geq 0$$
$$(A + \alpha I)^* G_{\rm in} + G_{\rm in}(A + \alpha I) - e^*e \leq 0$$
$$(A + \alpha I)^* G_{\rm out} + G_{\rm out}(A + \alpha I) - (f+n)^*(f+n) \geq 0$$

with $n^*n \leq \epsilon^2$ (or $|n_i| \leq \epsilon, \;\; i = 1, \ldots, m$).

With this observation, we may answer a number of interesting questions in frequency response identification by solving EVPs and GEVPs.

For fixed α and ϵ, minimize M. Solving this GEVP answers the question "Given α and a bound on the noise values, what is the smallest possible α-shifted $\mathbf{H}_\infty$ norm of the system consistent with the measurements of the frequency response?"

For fixed α and M, minimize ϵ. Solving this EVP answers the question "Given α and a bound on α-shifted $\mathbf{H}_\infty$ norm of the system, what is the "smallest" possible noise consistent with the measurements of the frequency response?"

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