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Title: Global Optimization of $\mathbf{H}_\infty$-norm of Parameter-dependent Linear Systems

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Abstract: For linear systems with unspecified parameters that lie in intervals, we present a branch and bound algorithm for computing the maximum and the minimum possible $\mathbf{H}_\infty$ norm of any transfer matrix of interest. We extend this branch and algorithm further so as to compute the "minmax" of the $\mathbf{H}_\infty$ norm, where the choice of parameters is sought that minimizes the maximum $\mathbf{H}_\infty$ norm over another set of parameters.

Keywords: Branch and bound algorithm, Global optimization, $\mathbf{H}_\infty$-norm, Parameter dependent linear systems

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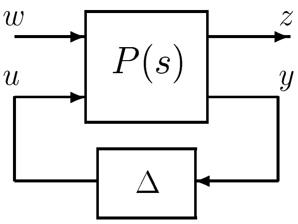

Figure 1: System in standard form.

1 Introduction

We consider the family of linear time-invariant systems described by

$$
\begin{array}{rclllll}
\dot{x} & = & Ax & + & B_u u & + & B_w w, \qquad\qquad x(0) = x_0, \\
y & = & C_y x & + & D_{yu} u & + & D_{yw} w, \\
z & = & C_z x & + & D_{zu} u & + & D_{zw} w, \\
u & = & \Delta\, y, & & & &
\end{array}
\tag{1.1}
$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{n_i}$, $z(t) \in \mathbf{R}^{n_o}$, $u(t), y(t) \in \mathbf{R}^p$, and A, B_u, B_w, C_y, C_z, D_{yu}, D_{yw}, D_{zu} and D_{zw} are real matrices of appropriate sizes. Δ is a diagonal *perturbation matrix.* In the sequel, we will assume that Δ is parametrized by a vector of parameters $q = [q_1, q_2, \ldots, q_m]$, and is given by

$$\Delta = \mathrm{diag}(q_1 I_1, q_2 I_2, \ldots, q_m I_m), \tag{1.2}$$

where I_i is an identity matrix of size p_i. Of course, $\sum_i^m p_i = p$. We will also assume that q lies in a rectangle $\mathcal{Q}_{\mathrm{init}} = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_m, u_m]$. A block diagram of the above family of linear systems is given in figure 1.

For future reference, we define

$$
\begin{array}{rcl}
P_{yu} & = & C_y (sI - A)^{-1} B_u + D_{yu} \\
P_{yw} & = & C_y (sI - A)^{-1} B_w + D_{yw} \\
P_{zu} & = & C_z (sI - A)^{-1} B_u + D_{zu} \\
P_{zw} & = & C_z (sI - A)^{-1} B_w + D_{zw}.
\end{array}
$$

We may now write down an expression for the closed-loop transfer matrix from w to z:

$$P_{\mathrm{cl}}(q) = P_{zw} + P_{zu}\Delta(I - P_{yu}\Delta)^{-1} P_{yw}.$$

Loosely speaking, system (1.1) represents linear systems which have unknown gains q_i that lie in intervals. The transfer function from w to z consists of all transfer functions of interest, and is typically required to be "small" – for example,

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when w consists of disturbances and z of error signals. The measure of "smallness" that we will use in this report is the $\mathbf{H}_\infty$ norm.

Depending on the interpretation of Δ, different questions arise regarding system (1.1). If the parameters q_i are thought of as *uncertainties*, a natural question is "how large can the transfer function from w to z be over the uncertainties?" This quantity is the maximum $\mathbf{H}_\infty$ norm of the system, denoted by $\mathcal{H}_{\max}$ and defined as

$$\mathcal{H}_{\max}(\mathcal{Q}_{\text{init}}) = \max_{q\in\mathcal{Q}_{\text{init}}} \left\{ \max_{w(t)\neq 0} \frac{\|z\|_{RMS}}{\|w\|_{RMS}} \right\} = \max_{q\in\mathcal{Q}_{\text{init}}} \|P_{\text{cl}}(q)\|_\infty,$$

where $\|\cdot\|_\infty$ refers to the $\mathbf{H}_\infty$-norm:

$$\|G\|_\infty = \sup_{\text{Re}\, s>0} \sigma_{\max}(G(s)).$$

($\sigma_{\max}(M)$ is the maximum singular value of M). $\mathcal{H}_{\max}$ is just the *worst-case root mean square gain* (RMS-gain) of the system between the input $w(t)$ and the output $z(t)$. We note that $\mathcal{H}_{\max}$ serves as a stability measure for the system (1.1). If the system has lightly damped eigenvalues for some value of the parameter vector, the RMS gain between w and z, under simple controllability and observability conditions, would be large for some particular input $w(t)$. Therefore a high $\mathcal{H}_{\max}$ could imply that the system is "not very stable".

Next, if Δ consists of *design parameters*, one might seek the choice of parameters that minimizes the $\mathbf{H}_\infty$ norm between w and z. In this case, the quantity of interest is the minimum $\mathbf{H}_\infty$ norm, denoted $\mathcal{H}_{\min}$ and defined as

$$\mathcal{H}_{\min}(\mathcal{Q}_{\text{init}}) = \min_{q\in\mathcal{Q}_{\text{init}}} \left\{ \max_{w(t)\neq 0} \frac{\|z\|_{RMS}}{\|w\|_{RMS}} \right\} = \min_{q\in\mathcal{Q}_{\text{init}}} \|P_{\text{cl}}(q)\|_\infty,$$

This situation may arise, for example, in parametric controller design.

Finally, if Δ contains both uncertainties *and* design parameters, the so-called *minmax* problem arises. Here, we seek the choice of design parameters that minimizes the $\mathcal{H}_{\max}$ over the uncertain parameters. More precisely, let the first m_1 parameters be design parameters and remaining m_2 parameters be uncertainties ($m_1 + m_2 = m$). For convenience, let us rename the m_1 design parameters as $\underline{q} = [\underline{q}_1,\ \underline{q}_2,\ ...,\ \underline{q}_{m_1}]$ and the m_2 uncertain parameters as $\overline{q} = [\overline{q}_1,\ \overline{q}_2,\ ...,\ \overline{q}_{m_2}]$. Let

$$\overline{\mathcal{Q}}_{\text{init}} = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_{m_1}, u_{m_1}],$$

and

$$\underline{\mathcal{Q}}_{\text{init}} = [l_{m_1+1}, u_{m_1+1}] \times [l_{m_1+2}, u_{m_1+2}] \times \cdots \times [l_m, u_m].$$

Then the minimax problem is the computation of

$$\begin{aligned} \mathcal{H}_{\text{minmax}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}}) &= \min_{\underline{q}\in\underline{\mathcal{Q}}_{\text{init}}} \max_{\overline{q}\in\overline{\mathcal{Q}}_{\text{init}}} \left\{ \max_{w(t)\neq 0} \frac{\|z\|_{RMS}}{\|w\|_{RMS}} \right\} \\ &= \min_{\underline{q}\in\underline{\mathcal{Q}}_{\text{init}}} \max_{\overline{q}\in\overline{\mathcal{Q}}_{\text{init}}} \|P_{\text{cl}}(q)\|_\infty, \end{aligned}$$

Computation of each of the three quantities above exactly is a hard problem. However, there exist a host of methods that yield useful bounds. Local optimization methods, Monte Carlo methods *etc.* give bounds in one direction, while analytical methods such as Small Gain Theorem or Lyapunov methods give bounds in the other. In the following, we describe branch and bound algorithms that use these bounds in order to compute each of these quantities $\mathcal{H}_{\max}$, $\mathcal{H}_{\min}$ and $\mathcal{H}_{\mathrm{minmax}}$ to within an absolute accuracy $\epsilon > 0$ in a finite number of iterations.

2 The Branch and Bound Algorithms

2.1 The branch and bound Algorithm for maximization

The branch and bound algorithm we use finds the maximum of a function $f : \mathbf{R}^m \to \mathbf{R}$ over an m-dimensional rectangle $\mathcal{Q}_{\mathrm{init}}$ (the subscript "init" stands for *initial* rectangle).

For a rectangle $\mathcal{Q} \subseteq \mathcal{Q}_{\mathrm{init}}$ we define

$$\Phi_{\max}(\mathcal{Q}) = \max_{q \in \mathcal{Q}} f(q).$$

Then, the algorithm computes $\Phi_{\max}(\mathcal{Q}_{\mathrm{init}})$ to within an arbitrary absolute accuracy of $\epsilon > 0$. The algorithm uses two functions $\Phi_{\mathrm{lb}}(\mathcal{Q})$ and $\Phi_{\mathrm{ub}}(\mathcal{Q})$ defined over $\{\mathcal{Q} : \mathcal{Q} \subseteq \mathcal{Q}_{\mathrm{init}}\}$ which are easier to compute than $\Phi_{\max}(\mathcal{Q})$. These two functions must satisfy the two following conditions, which we roughly describe:

(R1) $$\Phi_{\mathrm{lb}}(\mathcal{Q}) \leq \Phi_{\max}(\mathcal{Q}) \leq \Phi_{\mathrm{ub}}(\mathcal{Q}).$$

(R2) As the maximum half-length of the sides of $\mathcal{Q}$, denoted by size$(\mathcal{Q})$, goes to zero, the difference between upper and lower bounds *uniformly* converges to zero, *i.e.,*

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \ \text{ such that } \forall\ \mathcal{Q} \subseteq \mathcal{Q}_{\mathrm{init}},\ \mathrm{size}(\mathcal{Q}) \leq \delta \Longrightarrow \Phi_{\mathrm{ub}}(\mathcal{Q}) - \Phi_{\mathrm{lb}}(\mathcal{Q}) \leq \epsilon.$$

The algorithm starts by computing $\Phi_{\mathrm{lb}}(\mathcal{Q}_{\mathrm{init}})$ and $\Phi_{\mathrm{ub}}(\mathcal{Q}_{\mathrm{init}})$. If $\Phi_{\mathrm{ub}}(\mathcal{Q}) - \Phi_{\mathrm{lb}}(\mathcal{Q}) \leq \epsilon$, the algorithm terminates. Otherwise we partition $\mathcal{Q}_{\mathrm{init}}$ as a union of sub-rectangles as $\mathcal{Q}_{\mathrm{init}} = \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \cdots \cup \mathcal{Q}_N$, and compute $\Phi_{\mathrm{lb}}(\mathcal{Q}_i)$ and $\Phi_{\mathrm{ub}}(\mathcal{Q}_i)$, $i = 1, 2, \cdots, N$. Then

$$\max_{1 \leq i \leq N} \Phi_{\mathrm{lb}}(\mathcal{Q}_i) \leq \Phi_{\max}(\mathcal{Q}_{\mathrm{init}}) \leq \max_{1 \leq i \leq N} \Phi_{\mathrm{ub}}(\mathcal{Q}_i),$$

i.e., we have new bounds on $\Phi_{\max}(\mathcal{Q}_{\mathrm{init}})$. If the difference between the new bounds is less than or equal to ϵ, the algorithm terminates. Otherwise, the partition of $\mathcal{Q}_{\mathrm{init}}$ is further refined and the bounds updated. It is also possible to *prune* those rectangles over which we can establish that $\Phi_{\max}$ cannot be achieved (see [**?**] for details).

The general branch and bound algorithm for maximization

In the following description, k stands for the iteration index. $\mathcal{L}_k$ denotes the list of rectangles, L_k the lower bound and U_k the upper bound for $\Phi_{\max}(\mathcal{Q}_{\text{init}})$, at the end of k iterations.

The Algorithm

$k = 0;$
$\mathcal{L}_0 = \{\mathcal{Q}_{\text{init}}\};$
$L_0 = \Phi_{\text{lb}}(\mathcal{Q}_{\text{init}});$
$U_0 = \Phi_{\text{ub}}(\mathcal{Q}_{\text{init}});$
while $U_k - L_k > \epsilon$, $\{$
pick $\mathcal{Q} \in \mathcal{L}_k$ *such that* $\Phi_{\text{ub}}(\mathcal{Q}) = U_k;$
split $\mathcal{Q}$ *along one of its longest edges into* $\mathcal{Q}_I$ *and* $\mathcal{Q}_{II};$
$\mathcal{L}_{k+1} := (\mathcal{L}_k - \{\mathcal{Q}\}) \cup \{\mathcal{Q}_I, \mathcal{Q}_{II}\};$
$L_{k+1} := \max_{\mathcal{Q}\in\mathcal{L}_{k+1}} \Phi_{\text{lb}}(\mathcal{Q});$
$U_{k+1} := \max_{\mathcal{Q}\in\mathcal{L}_{k+1}} \Phi_{\text{ub}}(\mathcal{Q});$
$k = k + 1;$
$\}$

At the end of k iterations, U_k and L_k are upper and lower bounds respectively for $\Phi_{\max}(\mathcal{Q}_{\text{init}})$. Since $\Phi_{\text{lb}}(\mathcal{Q})$ and $\Phi_{\text{ub}}(\mathcal{Q})$ satisfy condition (R2), $U_k - L_k$ is guaranteed to converge to zero. We will prove this rigorously in the appendix.

2.2 The branch and bound algorithm for minimization

The branch and bound algorithm of the previous subsection may be directly applied to minimize f by maximizing $-f$. For convenience, we will present a version of the previous algorithm that directly minimizes a function $f : \mathbf{R}^m \to \mathbf{R}$. Our description below suffers from some abuse of notation; we will continue to use Φ_{lb} and Φ_{ub} as lower and upper bounds, now for the *minimum*. However, the meaning of these symbols should be clear from context.

Given a rectangle $\mathcal{Q} \subseteq \mathcal{Q}_{\text{init}}$ we define

$$\Phi_{\min}(\mathcal{Q}) = \min_{q\in\mathcal{Q}} f(q).$$

Then, the algorithm computes $\Phi_{\min}(\mathcal{Q}_{\text{init}})$ to within an arbitrary absolute accuracy of $\epsilon > 0$. $\Phi_{\text{lb}}(\mathcal{Q})$ and $\Phi_{\text{ub}}(\mathcal{Q})$ denote upper and lower bounds

$$\Phi_{\text{lb}}(\mathcal{Q}) \le \Phi_{\min}(\mathcal{Q}) \le \Phi_{\text{ub}}(\mathcal{Q}).$$

We suppose that the two bounds satisfy requirement (R2). Again, the algorithm starts by computing $\Phi_{\text{lb}}(\mathcal{Q}_{\text{init}})$ and $\Phi_{\text{ub}}(\mathcal{Q}_{\text{init}})$. If $\Phi_{\text{ub}}(\mathcal{Q}) - \Phi_{\text{lb}}(\mathcal{Q}) \le \epsilon$, the algorithm terminates. Otherwise we partition $\mathcal{Q}_{\text{init}}$ as previously and obtain

$$\min_{1\le i\le N} \Phi_{\text{lb}}(\mathcal{Q}_i) \le \Phi_{\min}(\mathcal{Q}_{\text{init}}) \le \min_{1\le i\le N} \Phi_{\text{ub}}(\mathcal{Q}_i),$$

If the difference between the new bounds is less than or equal to ϵ, the algorithm terminates. Otherwise, we further partition $\mathcal{Q}_{\text{init}}$, and continue with the computation.

The general branch and bound algorithm for minimization

Here k and $\mathcal{L}_k$ have the same meaning as before. L_k denoted the lower bound and U_k the upper bound for $\Phi_{\min}(\mathcal{Q}_{\text{init}})$, at the end of k iterations.

The Algorithm

$k = 0;$
$\mathcal{L}_0 = \{\mathcal{Q}_{\text{init}}\};$
$L_0 = \Phi_{\text{lb}}(\mathcal{Q}_{\text{init}});$
$U_0 = \Phi_{\text{ub}}(\mathcal{Q}_{\text{init}});$
while $U_k - L_k > \epsilon$, $\{$
  pick $\mathcal{Q} \in \mathcal{L}_k$ *such that* $\Phi_{\text{lb}}(\mathcal{Q}) = L_k;$
  split $\mathcal{Q}$ *along one of its longest edges into* $\mathcal{Q}_I$ *and* $\mathcal{Q}_{II};$
  $\mathcal{L}_{k+1} := (\mathcal{L}_k - \{\mathcal{Q}\}) \cup \{\mathcal{Q}_I, \mathcal{Q}_{II}\};$
  $L_{k+1} := \min_{\mathcal{Q}\in\mathcal{L}_{k+1}} \Phi_{\text{lb}}(\mathcal{Q});$
  $U_{k+1} := \min_{\mathcal{Q}\in\mathcal{L}_{k+1}} \Phi_{\text{ub}}(\mathcal{Q});$
  $k = k + 1;$
$\}$

At the end of k iterations, U_k and L_k are upper and lower bounds respectively for $\Phi_{\min}(\mathcal{Q}_{\text{init}})$. Since $\Phi_{\text{lb}}(\mathcal{Q})$ and $\Phi_{\text{ub}}(\mathcal{Q})$ satisfy condition (R2), $U_k - L_k$ is guaranteed to converge to zero.

2.3 The branch and bound algorithm for minmax-problems

We now present an extension of the branch and bound algorithm of the previous sections which minimizes, over a set of parameters, the maximum of the function over another set of parameters. More precisely, for a function $g(\underline{q}, \overline{q})$ we seek

$$\Psi_{\text{minmax}} = \min_{\underline{q}\in\underline{\mathcal{Q}}} \max_{\overline{q}\in\overline{\mathcal{Q}}} g(\underline{q}, \overline{q}).$$

The extended branch and bound algorithm needs two functions $\Psi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ and $\Psi_{\text{ub}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ defined over $\underline{\mathcal{Q}} \subseteq \underline{\mathcal{Q}}_{\text{init}}$, $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{Q}}_{\text{init}}$ which are easier to compute than $\Psi_{\text{minmax}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$. These two functions must satisfy the two following conditions:

(R3) $$\Psi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}}) \le \Psi_{\text{minmax}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}}) \le \Psi_{\text{ub}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}}).$$

(R4) As the maximum half-length of the sides of $\overline{\mathcal{Q}}$ and $\underline{\mathcal{Q}}$ denoted by size$(\overline{\mathcal{Q}})$ and size$(\underline{\mathcal{Q}})$ respectively go to zero, the difference between upper and lower bounds *uniformly* converges to zero, *i.e.*,

$$\forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ such that } \forall\, \underline{\mathcal{Q}} \subseteq \underline{\mathcal{Q}}_{\text{init}} \text{ and } \overline{\mathcal{Q}} \subseteq \overline{\mathcal{Q}}_{\text{init}},$$
$$\text{size}(\underline{\mathcal{Q}}) \le \delta \text{ and size}(\overline{\mathcal{Q}}) \le \delta \Longrightarrow \Psi_{\text{ub}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}}) - \Psi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}}) \le \epsilon.$$

As with the simpler branch and bound algorithm for maximization or minimization, the algorithm starts by computing $\Psi_{\text{lb}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}})$ and $\Psi_{\text{ub}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}})$. If the difference $\Psi_{\text{lb}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}}) - \Psi_{\text{ub}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}}) \le \epsilon$, the algorithm terminates. Otherwise $\underline{\mathcal{Q}}_{\text{init}}$ is partitioned as a union of subrectangles as $\underline{\mathcal{Q}}_{\text{init}} = \underline{\mathcal{Q}}_1 \cup \underline{\mathcal{Q}}_2 \cup \cdots \cup \underline{\mathcal{Q}}_N$, and compute $\Psi_{\text{lb}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{\text{init}})$ and $\Psi_{\text{ub}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{\text{init}})$, $i = 1, 2, \cdots, N$ are computed. Then

$$\min_{1\le i\le N} \Psi_{\text{lb}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{\text{init}}) \le \Psi_{\text{minmax}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}}) \le \min_{1\le i\le N} \Psi_{\text{ub}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{\text{init}}),$$

If the difference between these two bounds is small enough, we stop, otherwise we partition any of the subrectangles $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{\text{init}}$ into smaller subrectangles as $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{\text{init}} = \underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{i1} \cup \underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{i2} \cup \cdots \cup \underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{iM_i}$, and compute $\Psi_{\text{lb}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{ij})$ and $\Psi_{\text{ub}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{ij})$. Then

$$\min_{1\le i\le N} \left\{ \max_{1\le j\le M_i} \Psi_{\text{lb}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{ij}) \right\} \le \Psi_{\text{minmax}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}}) \le \min_{1\le i\le N} \left\{ \max_{1\le j\le M_i} \Psi_{\text{ub}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{ij}) \right\}$$

Once more, if the difference between the new bounds is less than or equal to ϵ, the algorithm terminates. Otherwise we either partition $\underline{\mathcal{Q}}_{\text{init}}$ into smaller rectangles, or we partition a subrectangle $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{\text{init}}$ into smaller subrectangles: in both cases we can then update the bounds. It is also possible to prune those rectangles over which we can establish that $\Psi_{\text{minmax}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ cannot be achieved.

The general branch and bound algorithm for minmax problems

In the following description, k stands for the iteration index. $\mathcal{L}_k$ denotes a list of N_k rectangle lists. Every rectangle list corresponds to a member $\underline{\mathcal{Q}}_i$ of a partition of $\underline{\mathcal{Q}}_{\text{init}}$ and is therefore denoted by $\ell(\underline{\mathcal{Q}}_i)$. Every subrectangle in $\ell(\underline{\mathcal{Q}}_i)$ is of the form $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij}$, with $\overline{\mathcal{Q}}_{ij} \subseteq \overline{\mathcal{Q}}_{\text{init}}$. $M_{i,k}$ stands for the number of subrectangles in the ith list $\ell(\underline{\mathcal{Q}}_i)$ at the end of k iterations. In other words, we have a two-dimensional list of rectangles, first partitioned along the minimizing parameters to yield the rectangle lists, and each of these lists further partitioned along the maximizing parameters. L_k and U_k are lower and upper bounds respectively for $\Psi_{\text{max}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}})$ at the end of the k-th iteration. Let

$$l_k(\underline{\mathcal{Q}}_i) = \max_{1\le j\le M_k(i)} \Psi_{\text{lb}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{ij}) \qquad \text{and} \qquad u_k(\underline{\mathcal{Q}}_i) = \max_{1\le j\le M_k(i)} \Psi_{\text{ub}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{ij}).$$

l_k and u_k are lower and upper bounds for Ψ_{minmax} over $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{\text{init}}$.

The Algorithm

$k = 0$;
$\ell(\underline{\mathcal{Q}}_{\text{init}}) = \{\underline{\mathcal{Q}}_{\text{init}} \times \overline{\mathcal{Q}}_{\text{init}}\}$;

$\mathcal{L}_0 = \{\ell(\underline{\mathcal{Q}}_{\text{init}})\}$;
$L_0 = \Psi_{\text{lb}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}})$;
$U_0 = \Psi_{\text{ub}}(\underline{\mathcal{Q}}_{\text{init}}, \overline{\mathcal{Q}}_{\text{init}})$;
while $U_k - L_k > \epsilon$ {
    pick $\ell(\underline{\mathcal{Q}}_i) \in \mathcal{L}_k$ *such that* $l_k(\underline{\mathcal{Q}}_i) = L_k$;
        pick $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij} \in \ell(\underline{\mathcal{Q}}_i)$ *such that* $\Psi_{\text{ub}}(\underline{\mathcal{Q}}_i, \overline{\mathcal{Q}}_{ij}) = u_k(\underline{\mathcal{Q}}_i)$;
        split $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij}$ *along one of the longest edges of* $\overline{\mathcal{Q}}_{ij}$ *into*
            $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij}^{'}$ *and* $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij}^{''}$;
        $\ell(\underline{\mathcal{Q}}_i) = (\ell(\underline{\mathcal{Q}}_i) - \underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij}) \cup \{\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij}^{'}, \underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij}^{''}\}$;
    split all $\underline{\mathcal{Q}}_i \times \overline{\mathcal{Q}}_{ij} \in \ell(\underline{\mathcal{Q}}_i)$ *along one of the longest edges of* $\underline{\mathcal{Q}}_i$ *into*
        $\underline{\mathcal{Q}}_i^{'} \times \overline{\mathcal{Q}}_{ij}$ *and* $\underline{\mathcal{Q}}_i^{''} \times \overline{\mathcal{Q}}_{ij}$;
    $\ell(\underline{\mathcal{Q}}_i^{'}) = \bigcup_j \left\{\underline{\mathcal{Q}}_i^{'} \times \overline{\mathcal{Q}}_{ij}\right\}$;
    $\ell(\underline{\mathcal{Q}}_i^{''}) = \bigcup_j \left\{\underline{\mathcal{Q}}_i^{''} \times \overline{\mathcal{Q}}_{ij}\right\}$;
    $\mathcal{L}_{k+1} := (\mathcal{L}_k - \ell(\underline{\mathcal{Q}}_i)) \cup \{\ell(\underline{\mathcal{Q}}_i^{''}), \ell(\underline{\mathcal{Q}}_i^{''})\}$;
    $L_{k+1} := \min_{\ell(\underline{\mathcal{Q}}_i) \in \mathcal{L}_{k+1}} l_k(\underline{\mathcal{Q}}_i)$;
    $U_{k+1} := \min_{\ell(\underline{\mathcal{Q}}_i) \in \mathcal{L}_{k+1}} u_k(\underline{\mathcal{Q}}_i)$;
    $k = k + 1$;
}

2.3.1 Using simple bounds

We now show how we may obtain bounds Ψ_{lb} and Ψ_{ub} from the bounds for the simple minimization or maximization of a function. The conditions under which these bounds can be used are stated in the following proposition.

Proposition 2.1 *Given any* $\underline{\mathcal{Q}}$ *and* $\overline{\mathcal{Q}}$ *let*

$$\Phi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{q}_o) \leq \min_{\underline{q} \in \underline{\mathcal{Q}}} g(\underline{q}, \overline{q}_o) \quad \text{and} \quad \Phi_{\text{ub}}(\underline{q}_o, \overline{\mathcal{Q}}) \geq \max_{\overline{q} \in \overline{\mathcal{Q}}} g(\underline{q}_o, \overline{q}) \tag{2.1}$$

with $\underline{q}_o$ *and* $\overline{q}_o$ *being any point in* $\underline{\mathcal{Q}}$ *and* $\overline{\mathcal{Q}}$ *respectively.* Φ_{lb} *and* Φ_{ub} *are lower and upper bounds for simple minimization and maximization problems respectively. Then*

$$\Psi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}}) = \Phi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{q}_o) \quad \text{and} \quad \Psi_{\text{ub}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}}) = \Phi_{\text{ub}}(\underline{q}_o, \overline{\mathcal{Q}}) \tag{2.2}$$

are bounds for Ψ_{minmax} *satisfying* (R3). *Moreover they satisfy* (R4) *if*

1. $g(\underline{q}, \overline{q})$ *is continuous in* $\{(\underline{q}, \overline{q}) : \underline{q} \in \underline{\mathcal{Q}},\ \overline{q} \in \overline{\mathcal{Q}}\}$.
2. $\Phi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{q})$ *and* $\min_{\underline{q} \in \underline{\mathcal{Q}}} g(\underline{\mathcal{Q}}, \overline{q})$ *satisfy* (R2) *for any* $\overline{q} \in \overline{\mathcal{Q}}$.
3. $\Phi_{\text{ub}}(\underline{q}, \overline{\mathcal{Q}})$ *and* $\max_{\overline{q} \in \overline{\mathcal{Q}}} g(\underline{q}, \overline{\mathcal{Q}})$ *satisfy* (R2) *for any* $\underline{q} \in \underline{\mathcal{Q}}$.

Proof: We start with the well-known inequality

$$\max_{\overline{q} \in \overline{\mathcal{Q}}} \min_{\underline{q} \in \underline{\mathcal{Q}}} g(\underline{q}, \overline{q}) \leq \min_{\underline{q} \in \underline{\mathcal{Q}}} \max_{\overline{q} \in \overline{\mathcal{Q}}} g(\underline{q}, \overline{q}). \tag{2.3}$$

This allows to define bounds on Ψ_{minmax} that involve only maximization or minimization. By choosing *any* $\underline{q}_o \in \underline{\mathcal{Q}}$ and $\overline{q}_o \in \overline{\mathcal{Q}}$ we directly derive

$$\min_{\underline{q}\in\underline{\mathcal{Q}}} g(\underline{q}, \overline{q}_o) \leq \min_{\underline{q}\in\underline{\mathcal{Q}}} \max_{\overline{q}\in\overline{\mathcal{Q}}} g(\underline{q}, \overline{q}) \leq \max_{\overline{q}\in\overline{\mathcal{Q}}} g(\underline{q}_o, \overline{q}) \tag{2.4}$$

From the assumptions on Φ_{ub} and Φ_{lb} and from equation (2.2) it follows directly that $\Psi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ and $\Psi_{\text{ub}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ are bounds for $\Psi_{\text{minmax}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ and therefore that they satisfy (R3). This completes the first part of the proof.

Now we have to prove that $\Psi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ and $\Psi_{\text{ub}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ satisfy (R4). We first show how condition 1. implies that the two bounds of equation (2.1) converge to each other uniformly over $\underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$. Condition 1 means

$$\text{given } \epsilon > 0,\ \exists\delta \text{ such that } |g(\underline{q}, \overline{q}) - g(\underline{q}_o, \overline{q}_o)| < \epsilon \text{ if } \|\underline{q} - \underline{q}_o\| < \delta \text{ and } \|\overline{q} - \overline{q}_o\| < \delta.$$

Now, by considering $\underline{q},\ \underline{q}_o \in \underline{\mathcal{Q}}$ and $\overline{q},\ \overline{q}_o \in \overline{\mathcal{Q}}$ we have that

$$\begin{array}{l} \forall\ \epsilon > 0\ \forall \text{ such that } \underline{\mathcal{Q}} \subseteq \underline{\mathcal{Q}}_{\text{init}} \text{ and } \overline{\mathcal{Q}} \subseteq \overline{\mathcal{Q}}_{\text{init}} \\ \text{size}(\underline{\mathcal{Q}}) \leq \delta/2 \text{ and size}(\overline{\mathcal{Q}}) \leq \delta/2 \Longrightarrow \min_{\underline{q}\in\underline{\mathcal{Q}}} g(\underline{q}, \overline{q}_o) - \max_{\overline{q}\in\overline{\mathcal{Q}}} g(\underline{q}_o, \overline{q}) \leq \epsilon \end{array}$$

which proves that the two bounds of equation (2.1) are uniformly convergent. With a similar argument, conditions 2. and 3. imply that the two quantities in the condition are uniformly convergent over $\underline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}$. Finally, as all the quantities on the two sides of the inequalities of the expression

$$\Phi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{q}_o) \leq \min_{\underline{q}\in\underline{\mathcal{Q}}} g(\underline{q}, \overline{q}_o) \leq \max_{\overline{q}\in\overline{\mathcal{Q}}} g(\underline{q}_o, \overline{q}) \leq \Phi_{\text{ub}}(\underline{q}_o, \overline{\mathcal{Q}})$$

uniformly converge over $\underline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}$ for any $\underline{q}_o \in \underline{\mathcal{Q}}$ and $\overline{q}_o \in \overline{\mathcal{Q}}$, we have proven that $\Psi_{\text{lb}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ and $\Psi_{\text{ub}}(\underline{\mathcal{Q}}, \overline{\mathcal{Q}})$ satisfy (R4). ■

3 Bound Computations

We now describe the computation of bounds for $\mathcal{H}_{\text{max}}$, $\mathcal{H}_{\text{min}}$ and $\mathcal{H}_{\text{minmax}}$, so that we may apply the branch and bound algorithms of the previous section towards their exact computation.

3.1 A Loop Transformation

Before we go on to describing the computation of bounds for $\mathcal{H}_{\text{max}}$ $\mathcal{H}_{\text{min}}$, and $\mathcal{H}_{\text{minmax}}$, we describe a loop transformation that will enable us to henceforth assume that the bounds are always calculated over the scaled unit cube $\mathcal{U} = [-1, 1]^m$. We refer the reader to [?] for a complete discussion of loop transformations.

The loop transformation is best explained through figure 2, where the symbols $\tilde{H}(s)$ and $\tilde{\Delta}$ refer to the "new" system and the "normalized" perturbation.

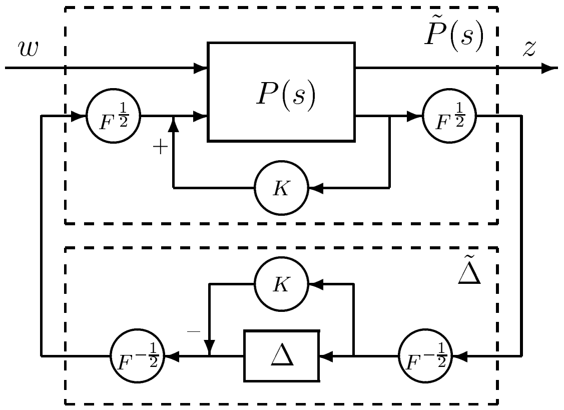

Figure 2: Loop Transformation.

The loop transformation can be interpreted as translating $\mathcal{Q}$ to the origin, and then scaling it to the cube $[-1,1]^m$.

$$K = \mathrm{diag}(\frac{u_1+l_1}{2}I_1, \frac{u_2+l_2}{2}I_2, \ldots, \frac{u_m+l_m}{2}I_m),$$
$$F = \mathrm{diag}(\frac{u_1-l_1}{2}I_1, \frac{u_2-l_2}{2}I_2, \ldots, \frac{u_m-l_m}{2}I_m)$$

are the matrices that accomplish this.

A state-space representation of the loop-transformed system $\tilde{P}(s)$ is given by $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$, where

$$\begin{aligned}
\tilde{A} &= \left[\; A+B_uTKC_y \; \right]; \\
\tilde{B} &= \begin{bmatrix} \underbrace{B_w+B_uTKD_{yw}}_{\tilde{B}_z} & \underbrace{B_uTF^{\frac{1}{2}}}_{\tilde{B}_u} \end{bmatrix}; \\
\tilde{C} &= \begin{bmatrix} \overbrace{C_z+D_{zu}TKC_y}^{\tilde{C}_z} \\ \underbrace{F^{\frac{1}{2}}(I+D_{yu}TK)C_y}_{\tilde{C}_y} \end{bmatrix}; \\
\tilde{D} &= \begin{bmatrix} \overbrace{D_{zw}+D_{zu}TKD_{yw}}^{\tilde{D}_{zw}} & \overbrace{D_{zu}TKF^{\frac{1}{2}}}^{\tilde{D}_{zu}} \\ \underbrace{F^{\frac{1}{2}}(I+D_{yu}TK)D_{yw}}_{\tilde{D}_{yw}} & \underbrace{F^{\frac{1}{2}}D_{yu}TF^{\frac{1}{2}}}_{\tilde{D}_{yu}} \end{bmatrix}.
\end{aligned} \tag{3.1}$$

$T = (I - KD_{yu})^{-1}$, and I is the identity matrix of appropriate size.

We note that performing this loop transformation automatically checks the well-posedness of the system for $\Delta = K$.

3.2 Bounds for $\mathcal{H}_{\max}$

A lower bound for $\mathcal{H}_{\max}(\mathcal{U})$ that we will use in our algorithm is very simple: we just compute the $\mathbf{H}_\infty$-norm of the closed-loop system with the parameter vector set to the midpoint of the parameter region $\mathcal{U}$. Clearly, this number is less than or equal to the *maximum* $\mathbf{H}_\infty$-norm. That is,

$$\Phi_{lb}(\mathcal{U}) = \|P_{\mathrm{cl}}(q)\|_\infty = \|P_{zw}\|_\infty \tag{3.2}$$

Computation of the upper bound uses many ideas from [?], and is based on a small gain based robust stability condition. This bound has been also proposed by Doyle [?] and Safonov [?] (see [?, p239-241]).

Theorem 3.1 *Let $P(s)$ be a stable transfer function of the form shown in figure 1 and let β be real and positive. If*

$$\left\| \begin{bmatrix} \dfrac{P_{zw}}{\beta} & \dfrac{P_{zu}}{\sqrt{\beta}} \\ \dfrac{P_{yw}}{\sqrt{\beta}} & P_{yu} \end{bmatrix} \right\|_\infty < 1,$$

then

$$\sup_{\|\Delta\|_\infty \le 1} \left\| \left[P_{zw} + P_{zu}\Delta(I - P_{yu}\Delta)^{-1} P_{yw} \right] \right\|_\infty < \beta.$$

Remark: Note that the theorem makes no assumptions regarding the structure of Δ.

Proof: The proof relies on two applications of the SGT. Consider the closed-loop system of figure 3. This system corresponds to "closing the loop" of the system in figure 1 with $u = (\hat{\Delta}/\beta)y$. We assume that $\hat{\Delta}$ satisfies $\|\hat{\Delta}\|_\infty \le 1$. Since $P(s)$ is of the form shown in figure 1, the transfer matrix $P_\beta(s)$ is given by

$$P_\beta(s) = \begin{bmatrix} \dfrac{P_{zw}}{\beta} & \dfrac{P_{zu}}{\sqrt{\beta}} \\ \dfrac{P_{yw}}{\sqrt{\beta}} & P_{yu} \end{bmatrix}. \tag{3.3}$$

We may regard the system in figure 3 as consisting of the transfer matrix $P_\beta(s)$ with the feedback matrix $\mathrm{diag}(\hat{\Delta}, \Delta)$. Since by assumption $\|\hat{\Delta}\|_\infty \le 1$, for every Δ such that $\|\Delta\|_\infty \le 1$, we have $\|\mathrm{diag}(\hat{\Delta}, \Delta)\|_\infty \le 1$. Then, from the SGT, we conclude that if $\|P_\beta(s)\|_\infty < 1$, the closed-loop system is stable.

The closed-loop system above can be viewed in yet another way, as the system $P_{\beta,\Delta}(s)$ with the feedback matrix $\hat{\Delta}$. Since the only assumption made on $\hat{\Delta}$ was that $\|\hat{\Delta}\|_\infty \le 1$ (in particular, no assumptions were made regarding its structure), the stability of the closed-loop system implies that $\|P_{\beta,\Delta}\|_\infty < 1$ for every Δ such that $\|\Delta\|_\infty \le 1$. We now observe that $P_{\beta,\Delta} = (P_{zw} + P_{zu}\Delta(I - P_{yu}\Delta)^{-1}P_{yw})/\beta$, which immediately implies that $\|P_{zw} + P_{zu}\Delta(I - P_{yu}\Delta)^{-1}P_{yw}\|_\infty < \beta$ and the conclusion of the theorem follows. ■

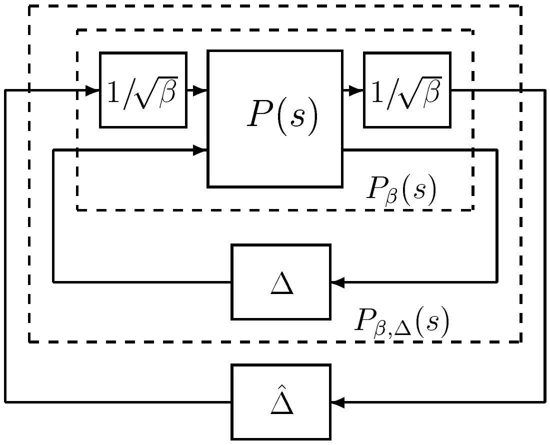

Figure 3: Closed-loop system with constant feedback $1/\beta$

This theorem suggests a direct method for the computation of the upper bound. Any $\beta > 0$ satisfying $\|P_\beta\|_\infty < 1$ is an upper bound for $\Phi_{max}(\mathcal{U})$. On the other hand, if $\|P_\beta\|_\infty \geq 1$ for all beta, we may conclude only the the upper bound is infinity. We may thus define our upper bound as:

$$\Phi_{ub}(\mathcal{U}) = \inf \{\beta : \|P_\beta\|_\infty < 1\}, \tag{3.4}$$

with the convention that the infimum of a function over an empty set is infinity.

Computational issues for the bounds

The following theorem is useful in the computation of the upper bound for $\mathcal{H}_{\max}$. In order to compute the upper bound for $\mathcal{H}_{\max}$ over a given rectangle $\mathcal{U}$, we will now establish that for $\beta > 0$, $\|P_\beta\|_\infty$ is a monotonic non-increasing function of β, approaches infinity as β approaches zero, and approaches $\|P_{yu}\|_\infty$ as β approaches infinity. Thus we may either establish that the upper bound is only infinity (in the case $\|P_{yu}\|_\infty \geq 1$), or use a bisection to find $\Phi_{ub}(\mathcal{U})$.

Theorem 3.2 *$\|P_\beta\|_\infty$ is a monotonic non-increasing function of β for $\beta > 0$, with $\lim_{\beta\to\infty} \|P_\beta\|_\infty = \|P_{yu}\|_\infty$. Provided that at least one of P_{zw}, P_{zu} and P_{yw} is nonzero, $\lim_{\beta\to 0} \|P_\beta\|_\infty = \infty$.*

Proof: For any ω and $\beta_2 > \beta_1 > 0$, with I_1 and I_2 being identity matrices of the appropriate size, and by defining $\alpha = \sqrt{\beta_1/\beta_2}$ we have

$$\begin{aligned} &\sigma_{max}(P_{\beta_2}(j\omega)) = \sigma_{max}\left(\begin{bmatrix} \alpha I_1 & 0 \\ 0 & I_2 \end{bmatrix} P_{\beta_1}(j\omega) \begin{bmatrix} \alpha I_1 & 0 \\ 0 & I_2 \end{bmatrix}\right) \\ &\leq \sigma_{max}\left(\begin{bmatrix} \alpha I_1 & 0 \\ 0 & I_2 \end{bmatrix}\right) \sigma_{max}\left(P_{\beta_1}(j\omega)\right) \sigma_{max}\left(\begin{bmatrix} \alpha I_1 & 0 \\ 0 & I_2 \end{bmatrix}\right) = \sigma_{max}(P_{\beta_1}(j\omega)) \end{aligned} \tag{3.5}$$

from which it follows directly that $\|P_{\beta_2}\|_\infty \leq \|P_{\beta_1}\|_\infty$. To prove the second part of the theorem, we note that as $\beta \to 0$, at some frequency ω, some entry of the matrix $P_\beta(j\omega)$ has magnitude that goes to infinity, which means $\|P_\beta\|_\infty \to \infty$. Finally, as $\beta \to \infty$,

$$P_\beta \to \begin{bmatrix} 0 & 0 \\ 0 & P_{yu} \end{bmatrix},$$

so that $\|P_\beta\|_\infty \to \|P_{yu}\|_\infty$. ■

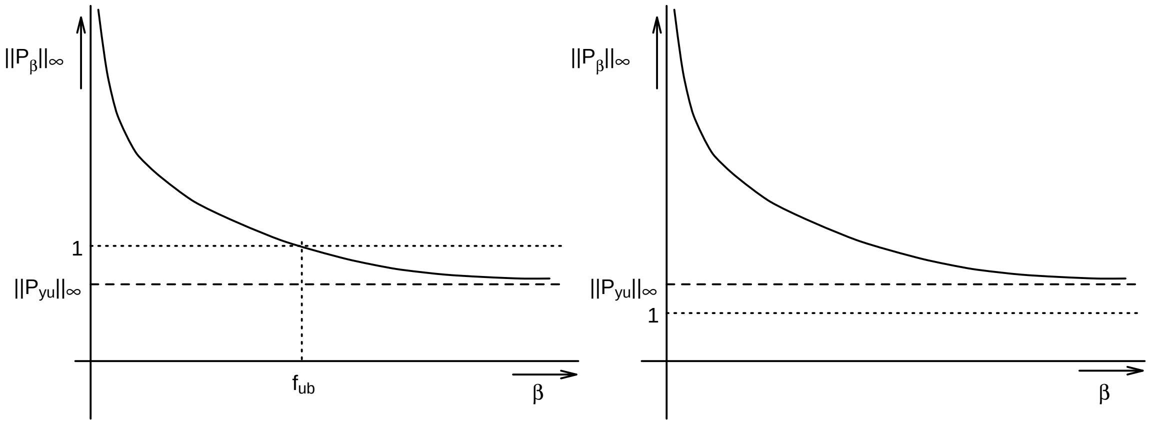

Figure 4: The figure on the left illustrates the case when a bisection may be used to compute the upper bound. For the case depicted on the right, the only guaranteed lower bound is ∞.

Thus if $\|P_{yu}\|_\infty \geq 1$, the upper bound is only ∞. In the case when $\|P_{yu}\|_\infty < 1$, we can use a bisection to compute the upper bound on the $\mathcal{H}_{\max}$. This might appear, at a first glance, a formidable task. However, it is shown in [**?**] that checking if the $\mathbf{H}_\infty$-norm of a transfer matrix is greater than one involves an eigenvalue computation of a Hamiltonian matrix; the $\mathbf{H}_\infty$-norm of the transfer matrix $C_0(sI - A_0)^{-1}B_0 + D_0$ is greater than one if and only if the matrix

$$M = \begin{bmatrix} A_0 - B_0R^{-1}D_0^TC_0 & -B_0R^{-1}B_0^T \\ C_0^TS^{-1}C_0 & -A_0^T + C_0^TD_0R^{-1}B_0^T \end{bmatrix} \tag{3.6}$$

has imaginary eigenvalues, where $R = (D_0^TD_0 - I)$ and $S = (D_0D_0^T - I)$. Thus the bisection to compute the upper bound can be performed reasonably fast. In any case, because of the conservativeness of the bound, we need not perform the bisection to any great accuracy. In practice, two to three bisection iterations will suffice.

Of course, more sophisticated bounds can be used. A local optimization procedure can be used to search for a (locally) worst parameter value, which would give a good lower bound. The upper bound can be vastly improved by scaling (see Doyle [**?**], Safonov [**?**]) or other techniques for approximating the structured singular value (see Fan and Tits [**?**]).

3.3 Bounds for $\mathcal{H}_{\min}$

The bounds we present here make the (unrealistic) assumption that system 1 is robustly stable. In terms of design, this requires the designer to apply the algorithm only over parameter ranges where the system is guaranteed to be stable. Removing this unreasonable assumption is currently under investigation.

The upper bound for $\mathcal{H}_{\min}(\mathcal{U})$ is the lower bound for $\mathcal{H}_{\max}(\mathcal{U})$, *i.e.* the $\mathbf{H}_\infty$-norm of the closed-loop system with the parameter vector set ot the midpoint of the parameter region $\mathcal{U}$. Clearly, this number is larger than or equal to the *minimum* $\mathbf{H}_\infty$-norm. That is,

$$\Phi_{ub}(\mathcal{U}) = \|P_{\mathrm{cl}}(q)\|_\infty = \|P_{zw}\|_\infty \tag{3.7}$$

Computation of the lower bound is derived using some simple norm inequalities. We can determine it in a constructive way:

$$\begin{aligned}
\|P_{\mathrm{cl}}(q)\|_\infty &= \|P_{zw} + P_{zu}\Delta(I - P_{yu}\Delta)^{-1}P_{yw}\|_\infty \\
&\geq \|P_{zw}\|_\infty - \|P_{zu}\|_\infty\|\Delta\|_\infty\|(I - P_{yu}\Delta)^{-1}\|_\infty\|P_{yw}\|_\infty \\
&\geq \|P_{zw}\|_\infty - \frac{\|P_{zu}\|_\infty\|P_{yw}\|_\infty\|\Delta\|_\infty}{1 - \|P_{yu}\|_\infty\|\Delta\|_\infty}
\end{aligned}$$

where last step requires that $\|P_{yu}\Delta\|_\infty \leq 1$. Finally, by remarking that Δ is described by a set of parameters in the m-dimensional hypercube, we have the bound

$$\Phi_{lb}(\mathcal{U}) = \begin{cases} \max\left\{\|P_{zw}\|_\infty - \dfrac{\|P_{zu}\|_\infty\|P_{yw}\|_\infty}{1 - \|P_{yu}\|_\infty},\ 0\right\} & \text{if } \|P_{yu}\|_\infty \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

We remark that the computation of the two bounds requires four $\mathbf{H}_\infty$-norm computations.

It is not difficult to show that $\|P_{zu}\|_\infty$, $\|P_{yw}\|_\infty$ and $\|P_{yu}\|_\infty$ all go to zero when the size of the rectangle goes to zero: condition (R2) is satisfied by just looking at the definitions (3.7) and (3.8).

In the appendix we show that there exist positive real numbers M_2 and δ such that for every rectangle $\mathcal{Q}$ with $\mathrm{size}(\mathcal{Q}) < \delta$ the gap $(\Phi_{ub}(\mathcal{Q}) - \Phi_{lb}(\mathcal{Q}))$ between upper and lower bounds is bounded by $M_2\mathrm{size}(\mathcal{U})$. This allows to determine a bound on the number of iterations necessary to obtain a desired precision.

Again, more sophisticated bounds can be used. Also the lower bound for $\mathcal{H}_{\min}$ can be improved by scaling or other techniques for approximating the structured singular value as it was the case for the upper bound for the $\mathcal{H}_{\max}$ computation.

For a more detailed analysis of the convergence of the branch and bound algorithm for this bounds we refer to the appendix.

3.4 Bounds for $\mathcal{H}_{\mathrm{minmax}}$

We will use $\underline{\mathcal{U}}$ and $\overline{\mathcal{U}}$ to denote unit cubes of the same dimensions as $\underline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}$. Note that the loop transformation of subsection 3.1 enables us to assume that the bounds are now computed over $\underline{\mathcal{Q}} = \underline{\mathcal{U}}$ and $\overline{\mathcal{Q}} = \overline{\mathcal{U}}$.

A lower bound for $\mathcal{H}_{\text{minmax}}(\underline{\mathcal{U}},\overline{\mathcal{U}})$ can be obtained by using the lower bound for $\mathcal{H}_{\min}$ of equation (3.8). From equation (2.2) of the previous section we can choose $\overline{q}_o = 0$, that is the center of the box $\overline{\mathcal{U}}$. and compute directly from equation (3.8)

$$\Psi_{lb}(\underline{\mathcal{U}},\overline{\mathcal{U}}) = \begin{cases} \max\left\{\|P_{zw}\|_\infty - \dfrac{\|P_{zu_1}\|_\infty \|P_{y_1 w}\|_\infty}{1-\|P_{y_1u_1}\|_\infty},\ 0\right\} & \text{if } \|P_{y_1u_1}\|_\infty \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

An upper bound for $\mathcal{H}_{\text{minmax}}(\underline{\mathcal{U}},\overline{\mathcal{U}})$ can be obtained by using the upper bound for $\mathcal{H}_{\max}$ of equation (3.4). From equation (2.2) of the previous section we can choose $\underline{q}_o = 0$ that is the center of the box $\underline{\mathcal{U}}$. and compute directly from equations (3.3) and (3.4)

$$\Psi_{ub}(\underline{\mathcal{U}},\overline{\mathcal{U}}) = \inf\left\{\beta : \left\| \begin{bmatrix} \dfrac{P_{zw}}{\beta} & \dfrac{P_{zu_2}}{\sqrt{\beta}} \\ \dfrac{P_{y_2w}}{\sqrt{\beta}} & P_{y_2u_2} \end{bmatrix} \right\|_\infty < 1 \right\}. \tag{3.10}$$

We can now apply the branch and bound algorithm together with these bounds to compute $\mathcal{H}_{\text{minmax}}$.

4 Examples

4.1 A transfer matrix

We will present one simple example to illustrate the application of the branch and bound algorithm on the maximum $\mathbf{H}_\infty$-norm computation. We consider a 2×2 transfer matrix $H(s)$:

$$\left\{ H(s) = \begin{bmatrix} \dfrac{q_1}{s+q_2} & \dfrac{q_1}{s+q_1} \\ q_2 & \dfrac{s}{s+q_1} \end{bmatrix} : \quad q_1 \in [1,4],\ q_2 \in [1,4] \right\}$$

We can cast the problem very easily into our setup as follows:

$$P(s) \ = \ \left[\begin{array}{cc|ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

with $\Delta = diag(q_1, q_1, q_1, q_2, q_2)$. The branch and bound algorithm terminates guaranteeing that the maximum $\mathbf{H}_\infty$-norm of the considered system lies in the interval $[4.2298, 4.2391]$.

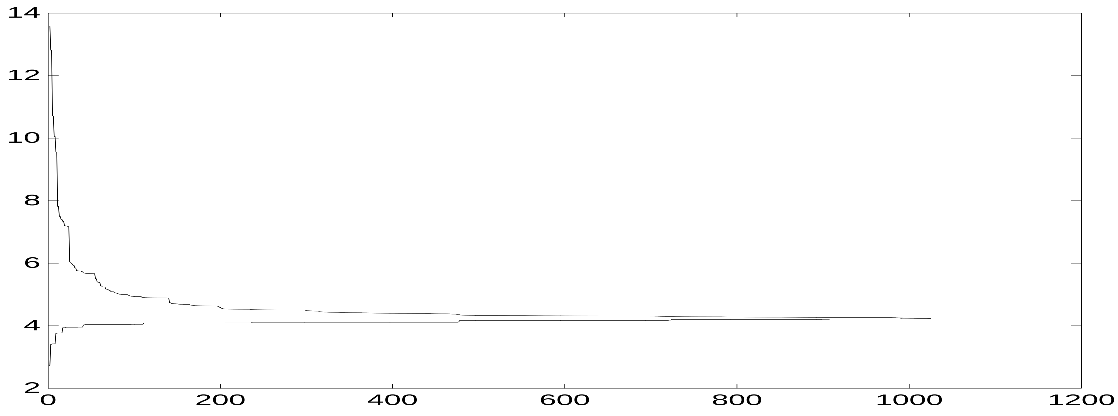

Figure 5: bounds for the maximum $\mathbf{H}_\infty$-norm of the transfer matrix example

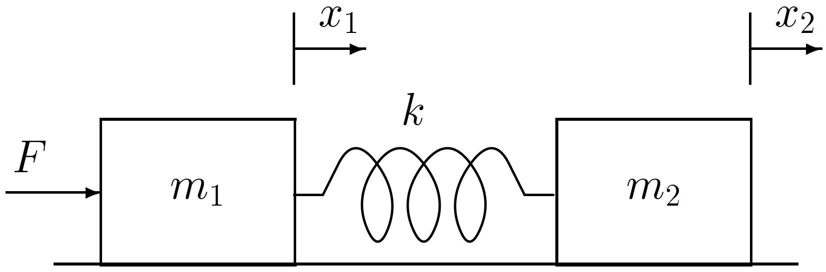

Figure 6: The plant consisting of two masses connected by a spring.

4.2 Mass-spring system

We consider a mechanical plant consisting of two masses connected by a spring with the lefthand mass driven by a force, as shown in figure 6 below.

The parameters are the masses and spring constant, each of which varies in a range between 2/3 and 3/2:

$$2/3 \leq m_1 \leq 3/2, \qquad 2/3 \leq m_2 \leq 3/2, \qquad 2/3 \leq k \leq 3/2.$$

Thus, these physical parameters can vary over a range exceeding 2 : 1.

With $[x_1 \;\; \dot{x}_1 \;\; x_2 \;\; \dot{x}_2]^T$ as the state, we employ a state-feedback law $F = -k_{\mathrm{LQR}}$ which is LQR optimal for the nominal parameter values $m_1 = m_2 = k = 1$ (with weights $Q = I$, $\rho = 1$), and consider the sensitivity transfer function (from w to z):

$$H_{\mathrm{cl}} = \frac{1}{1 + k_{\mathrm{LQR}} H}$$

where H denotes the transfer matrix from the input F to the state x, as shown in figure 7. Thus, $\mathcal{H}_{\max}$ is the worst case peak of the sensitivity transfer function induced by the parameter variations. From LQR theory we know that with the nominal parameters, $\|H_{\mathrm{cl}}\|_\infty = 1$.

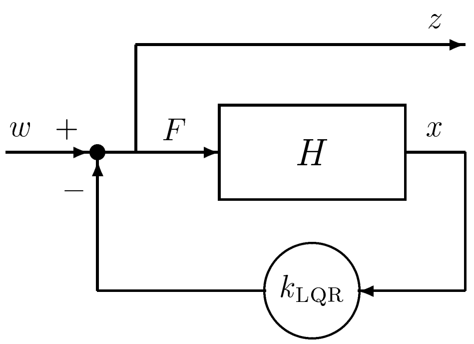

Figure 7: The closed-loop system with LQR-optimal state feedback.

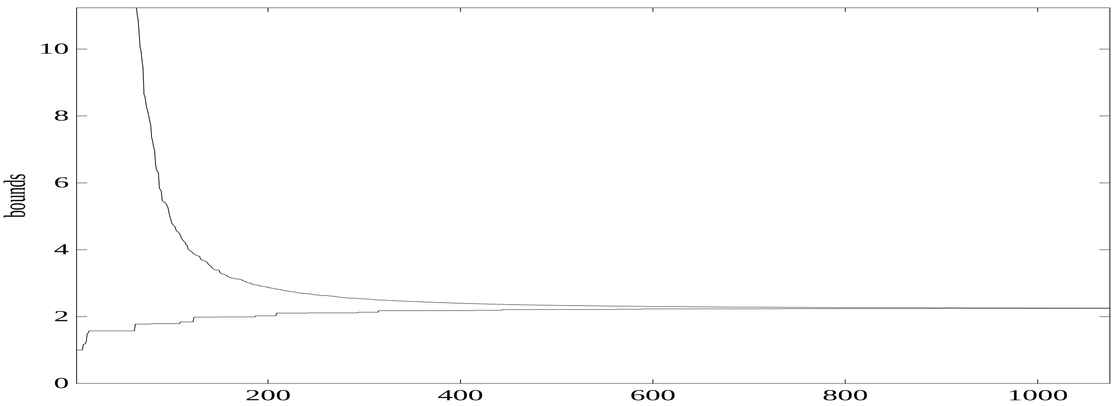

Figure 8: Bounds for the maximum $\mathbf{H}_\infty$-norm of the sensitivity transfer function for the spring mass system.

Figure 8 shows the convergence of the upper and lower bounds as a function of iterations. We observe that by about 50 iterations, the upper bound on $\mathcal{H}_{\max}$ is finite, indicating that the system is robustly stable. At the end of 500 iterations, the algorithm guarantees that $2.21 \leq \mathcal{H}_{\max} \leq 2.34$. The algorithm takes about 1100 iterations to return $\mathcal{H}_{\max} = 2.25$ to within an absolute accuracy of 0.01; thus relatively large parameter variations, even in the worst case, only degrade system performance a little bit. Of course, control theory folklore holds that LQR state feedback is quite "robust". But the folklore does not help us with this specific problem—for example, if the LQR optimal regulator for weights $Q = I$, $\rho = 10$ is used instead, not much can be concluded from conventional LQR wisdom, while our algorithm rapidly determines that $\mathcal{H}_{\max} = \infty$, *i.e.* , the system is not robustly stable.

The algorithm returns the worst-case parameters $m_1 = 3/2$, $m_2 = 2/3$ and $k = 3/2$, which happen to lie on a vertex of the parameter box. Needless to say, this is not the case in general. It is likely that local optimization methods would find this set of parameters fairly quickly. However, unlike our algorithm, local methods have no way of guaranteeing that the maximum they find is the global maximum.

5 Conclusions

We have presented two simple branch and algorithms, based on which we may optimize the $\mathbf{H}_\infty$-norm of systems with parametric uncertainties. The algorithms enjoy the following advantages:

- The algorithms maintain *guaranteed* upper and lower bounds for the quantities they compute, so that they can be terminated at any stage yielding valuable information.
- They attack problems for which no conventional methods exist.
- Improvements in the computation of the bounds may be readily incorporated into the algorithms, thereby improving the overall performance, sometimes significantly (see [**?**]).

However, it is quite easy to construct examples where the algorithms perform poorly.

The basic branch and bound algorithm itself is very simple. It can be easily implemented, and the only problem-specific task is the computation of the upper and lower bounds over a given parameter region for the function whose maximum, minimum or minimax is to be found. Thus they can be applied to a wider class of problems than those addressed in this report.

A Proof of convergence

We prove that the branch and bound algorithm applied to the computation of $\mathcal{H}_{\min}$ and $\mathcal{H}_{\max}$ converges in a finite number of steps. The proof of convergence of $\mathcal{H}_{\mathrm{minmax}}$ computation is under preparation [**?**].

In order to prove the convergence for $\mathcal{H}_{\max}$ and $\mathcal{H}_{\min}$, we must first show that the bounds for $\mathcal{H}_{\max}$ and $\mathcal{H}_{\min}$ satisfy condition (R2) in section 2.

Then it is shown in [**?**] that we can establish an upper bound for the number of iterations of the branch and bound algorithms described in section 2 in order to compute $\Phi_{\max}$ resp. $\Phi_{\min}$ within a desired precision.

Our last task is therefore to prove that the bounds for $\mathcal{H}_{\min}$ and $\mathcal{H}_{\max}$ satisfy condition (R2).

A.1 $\mathcal{H}_{\max}$ convergence

Proposition A.1 *If $\mathcal{H}_{\max}(\mathcal{Q}) < \infty$, there exist positive real numbers M_1 and δ such that for every rectangle $\mathcal{Q}$ with $\|\mathcal{Q}\| < \delta$,*

$$\Phi_{ub}(\mathcal{Q}) - \Phi_{lb}(\mathcal{Q}) < M_1 \|\mathcal{Q}\|^{\frac{1}{2}}, \tag{A.1}$$

where Φ_{lb} an d Φ_{ub} are bounds on the maximum $\mathbf{H}_\infty$-norm, as given in equations (3.2) *and* (3.4) .

Proof: We denote by $\tilde{P}$ the loop-transformed system. We the observe that we may write $\tilde{P}_\beta$ as

$$\tilde{P}_\beta = \begin{bmatrix} \beta^{-1/2} & 0 \\ 0 & F^{\frac{1}{2}} \end{bmatrix} \hat{P} \begin{bmatrix} \beta^{-1/2} & 0 \\ 0 & F^{\frac{1}{2}} \end{bmatrix}, \tag{A.2}$$

where

$$\hat{P} = \begin{bmatrix} \hat{P}_{zw} & \hat{P}_{zu} \\ \hat{P}_{yw} & \hat{P}_{yu} \end{bmatrix}$$

is given by the state space realization

$$\begin{aligned} \hat{A} =& \quad [A + B_u T^{-1} K C_y]\,; \\ \hat{B} =& \quad \begin{bmatrix} B_w + B_u T^{-1} K D_{yw}, & B_u T^{-1} \end{bmatrix}; \\ \hat{C} =& \quad \begin{bmatrix} C_z + D_{zu} T^{-1} K C_y \\ C_y + D_{yu} T^{-1} K C_y \end{bmatrix}; \\ \hat{D} =& \quad \begin{bmatrix} D_{zw} + D_{zu} T^{-1} K D_{yw}, & D_{zu} T^{-1} K \\ D_{yw} + D_{yu} T^{-1} K D_{yw}, & D_{yu} T^{-1} \end{bmatrix}. \end{aligned} \tag{A.3}$$

Note that $\hat{P}_{zw} = P_{zw}$ which we will use in the sequel. Note also that $\hat{P}$ depends on K; we will not show this dependence explicitly in what follows.

$$\Phi_{ub} - \Phi_{lb} \leq \lim_{\epsilon \to 0} \left\{ \beta : \left\| \begin{bmatrix} P_{zw}/\beta & F^{\frac{1}{2}} \hat{P}_{zu}/\beta^{\frac{1}{2}} \\ \hat{P}_{yw} F^{\frac{1}{2}}/\beta^{\frac{1}{2}} & F^{\frac{1}{2}} \hat{P}_{yu} F^{\frac{1}{2}} \end{bmatrix} \right\|_\infty = 1 - \epsilon \right\} - \|P_{zw}\|_\infty,$$

By noting that $\|F\|_\infty = \|\mathcal{Q}\|$ and using a simple norm inequality, we derive the larger bound

$$\leq \lim_{\epsilon \to 0} \left\{ \beta : \left(\|P_{zw}\|_\infty \frac{1}{\beta} + \left(\|\hat{P}_{zu}\|_\infty + \|\hat{P}_{yw}\|_\infty \right) \sqrt{\frac{\|\mathcal{Q}\|}{\beta}} + \|\hat{P}_{yu}\|_\infty \|\mathcal{Q}\| \right) = 1 - \epsilon \right\} - \|P_{zw}\|_\infty,$$

Now let us introduce for simplicity of notation

$$\begin{aligned} a &= \|P_{zw}\|_\infty^{\frac{1}{2}}; \\ b &= \max \left\{ \frac{\|\hat{P}_{zu}\|_\infty + |\hat{P}_{yw}\|_\infty}{2\|P_{zw}\|_\infty^{\frac{1}{2}}}, \quad \|\hat{P}_{yu}\|_\infty^{\frac{1}{2}} \right\}. \end{aligned} \tag{A.4}$$

Then from the previous expression we derive this looser inequality:

$$\leq \lim_{\epsilon \to 0} \left\{ \beta : \left(a^2 \frac{1}{\beta} + 2ab \sqrt{\frac{\|\mathcal{Q}\|}{\beta}} + b^2 \|\mathcal{Q}\| \right) = 1 - \epsilon \right\} - a^2.$$

Equivalently:

$$= \lim_{\epsilon \to 0} \left\{ \beta : \left(\frac{a}{\beta^{\frac{1}{2}}} + b \|\mathcal{Q}\|^{\frac{1}{2}} \right)^2 = 1 - \epsilon \right\} - a^2 = \lim_{\epsilon \to 0} \frac{a^2}{(\sqrt{1 - \epsilon} - b\|\mathcal{Q}\|^{\frac{1}{2}})^2} - a^2$$

Thus, for $\|\mathcal{Q}\| \leq \delta < 1/4b^2$ and letting ϵ go to zero

$$\leq \frac{a^2}{1-2b\|\mathcal{Q}\|^{\frac{1}{2}}} - a^2 \leq \frac{2a^2 b}{1-2b\sqrt{\delta}}\|\mathcal{Q}\|^{\frac{1}{2}}$$

Therefore by defining with equation (A.4)

$$b_{max} = \sup_{q \in \mathcal{Q}_{\text{init}}} \{b\}, \quad a_{max} = \sup_{q \in \mathcal{Q}_{\text{init}}} \{a\}$$

we can state that for any δ and $\mathcal{Q} \subseteq \mathcal{Q}_{\text{init}}$ such that $\|\mathcal{Q}\| \leq \delta < 1/4b_{max}^2$ there exists

$$M_1 = \frac{2a_{max}^2 b_{max}}{1-2b_{max}\sqrt{\delta}}$$

such that

$$\Phi_{ub}(\mathcal{Q}) - \Phi_{lb}(\mathcal{Q}) < M_1 \|\mathcal{Q}\|^{\frac{1}{2}}.$$

Note that $\mathcal{H}_{\max}(\mathcal{Q}_{\text{init}}) < \infty$ implies that $M_1 < \infty$. ■

A.2 $\mathcal{H}_{\min}$ convergence

Proposition A.2 *If the system is stable for all $q \in \mathcal{Q}_{\text{init}}$, there exist positive real numbers M_2 and δ every rectangle $\mathcal{Q}$ with $\|\mathcal{Q}\| < \delta$,*

$$\Phi_{ub}(\mathcal{Q}) - \Phi_{lb}(\mathcal{Q}) < M_2 \|\mathcal{Q}\|, \tag{A.5}$$

where Φ_{lb} and Φ_{ub} are bounds on the maximum $\mathbf{H}_\infty$-norm, as given in equations (3.8) *and* (3.7) .

Proof: We observe that the loop-transformed transfer matrix $\tilde{P}$ can be written as

$$\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & F^{\frac{1}{2}} \end{bmatrix} \hat{P} \begin{bmatrix} I & 0 \\ 0 & F^{\frac{1}{2}} \end{bmatrix}, \tag{A.6}$$

where $\hat{P}$ is given by the state space realization (A.3). Again, we will not show, as in the computation of the convergence for the bounds on $\mathcal{H}_{\max}$, the explicit dependence on K. Then, from (3.8)

$$\Phi_{lb}(\mathcal{Q}) = \begin{cases} \max\left\{\|\hat{P}_{zw}\|_\infty - \dfrac{\|\hat{P}_{zu}F^{\frac{1}{2}}\|_\infty \|F^{\frac{1}{2}}\hat{P}_{yw}\|_\infty}{1-\|F^{\frac{1}{2}}\hat{P}_{yu}F^{\frac{1}{2}}\|_\infty},\ 0\right\} & \text{if } \|F^{\frac{1}{2}}\hat{P}_{yu}F^{\frac{1}{2}}\|_\infty \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

By noting that $\|F\|_\infty = \|\mathcal{Q}\|$ and taking only regions such that $\|\mathcal{Q}\| \leq 1/2\|\hat{P}_{yu}\|_\infty$ we have $\|F^{\frac{1}{2}}\hat{P}_{yu}F^{\frac{1}{2}}\|_\infty \leq \|F\|_\infty \|\hat{P}_{yu}\|_\infty \leq 1/2$ and therefore

$$\begin{aligned} \Phi_{ub}(\mathcal{Q}) - \Phi_{lb}(\mathcal{Q}) &\leq \|\hat{P}_{zw}\|_\infty - \left\{\|\hat{P}_{zw}\|_\infty - \frac{\|\hat{P}_{zu}\|_\infty \|\hat{P}_{yw}\|_\infty \|\mathcal{Q}\|}{1-\|\hat{P}_{yu}\|_\infty \|\mathcal{Q}\|/2}\right\} \\ &\leq 2\|\hat{P}_{zu}\|_\infty \|\hat{P}_{yw}\|_\infty \|\mathcal{Q}\| \end{aligned}$$

By choosing $M_2 = 2\max_{q\in\mathcal{Q}} \|\hat{P}_{yw}\|_\infty \max_{q\in\mathcal{Q}} \|\hat{P}_{zu}\|_\infty$ and having a number of iterations so large that $2\|\mathcal{Q}\|\|\hat{P}_{yu}\|_\infty \leq 1$ we can conclude that

$$\Phi_{ub}(\mathcal{Q}) - \Phi_{lb}(\mathcal{Q}) \leq M_2\|\mathcal{Q}\|.$$

■