Quadratic Stabilization and Control of Piecewise-Linear Systems [1]

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Abstract

We consider analysis and controller synthesis of piecewise-linear systems. The method is based on constructing quadratic and piecewise-quadratic Lyapunov functions that prove stability and performance for the system. It is shown that proving stability and performance, or designing (state-feedback) controllers, can be cast as convex optimization problems involving linear matrix inequalities that can be solved very efficiently. A couple of simple examples are included to demonstrate applications of the methods described.

Key words: Piecewise-linear systems, quadratic stabilization, linear matrix inequality (LMI).

1 Introduction

Promising new methods for the analysis and design of controllers for linear and nonlinear uncertain systems have emerged over the last few years. The basic idea of these methods is to reformulate the control analysis or synthesis problem in terms of certain optimization problems that involve matrix inequalities (LMIs), which are then solved numerically by new interior-point algorithms. The theory (up to 1994) is covered in the monograph [1] and the many references cited there. Since then, many researchers have applied LMI methods in a variety of settings, such as synthesis of gain-scheduled (parameter-varying) controllers [2, 3], mixed-norm and multi-objective control design [4], analysis and synthesis of systems with integral quadratic constraints [5, 6], fuzzy control [7, 8], and hybrid dynamical systems [9, 10].

In this paper, using approaches that are standard in the LMI context, we address the question of stability and control of piecewise-linear time-invariant systems. Such systems can model, for example, a wide range of nonlinear systems, including linear systems with memoryless nonlinearities such as saturators. Using Lyapunov theory, we will derive *sufficient* conditions for stability and performance that can be checked by solving convex optimization problems with LMI constraints. The method is to search among special classes of Lyapunov functions for a Lyapunov function that proves stability or performance for the piecewise-linear system. We will consider two different classes of Lyapunov functions:

- *Quadratic Lyapunov functions.* In this case the Lyapunov function is simply $V(x) = x^TPx$ for some $P = P^T \succ 0$. It is shown that by searching over such Lyapunov functions, both analysis and (state-feedback) synthesis can be formulated as convex optimization problems with LMI constraints.

- *Continuous piecewise-quadratic Lyapunov functions.* This class of Lyapunov functions is more general than the previous one and therefore gives less conservative results in the analysis. Specially, such Lyapunov functions can also deal with piecewise-linear systems with multiple equilibrium points. However, in this case, it doesn't seem that (state-feedback) synthesis can be expressed as a convex optimization problem.

We will also demonstrate applications of the methods described by considering controller synthesis of a simple mechanical system subject to input saturation, and stability analysis of an electrical circuit with multiple equilibrium points.

2 Problem statement

Consider the piecewise-linear (PL) system

$$\begin{aligned} \dot{x} &= A_{\alpha(x)}x + b_{\alpha(x)} + B^{(1)}_{\alpha(x)}w + B^{(2)}_{\alpha(x)}u, \\ z &= C^{(1)}_{\alpha(x)}x + D^{(1)}_{\alpha(x)}w + D^{(2)}_{\alpha(x)}u \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^{n_u}$ is the control input, $w(t) \in \mathbf{R}^{n_w}$ is the exogenous input, and $z(t) \in \mathbf{R}^{n_z}$ is the output. The PL system (1) can be in any of M linear operation modes depending on where the state x is, and this is determined by the function $\alpha : \mathbf{R}^n \to \{1, \dots, M\}$. The set of all x satisfying $\alpha(x) = i$ is called the *ith operating region* of the system and is denoted by $\mathcal{R}_i$. We assume that given any initial condition $x(0) = x_0$, and input signals u and w, the differential equation (1) has a unique solution for $t > 0$.

The goal is to find control inputs u that provide *stability* and *performance* for the PL system (1). In particular, we are interested in finding a control input u that *regulates* the output z in terms of bounding the $\mathbf{L}_2$ gain from w to z, *i.e.*, for a given $\gamma > 0$

$$\frac{\|z\|_2}{\|w\|_2} \le \gamma \quad \text{with (1) and } x(0) = 0,$$

in which the 2-norm is defined as $\|\xi\|_2^2 = \int_0^\infty \xi^T\xi\, dt$.

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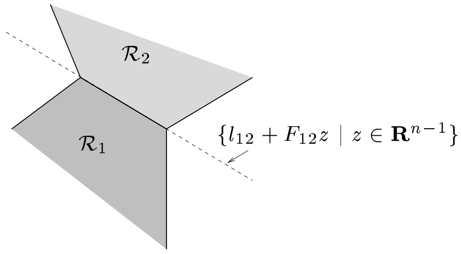


Figure 1: $\mathcal{R}_i$ is polytopic and the boundary of $\mathcal{R}_i$ and $\mathcal{R}_j$ is characterized as being a subset of $\{l_{ij} + F_{ij}z \mid z \in \mathbf{R}^{n-1}\}$ (here $n = 2$).

3 Description for operating regions

We assume that the operating regions $\mathcal{R}_i$ are polytopic, *i.e.*,

$$\begin{aligned}\mathcal{R}_i &= \{ x \mid \alpha(x) = i \} \\ &= \{ x \mid h_{ij}^T x < g_{ij},\ j = 1, \ldots, p_i \}.\end{aligned} \tag{2}$$

Moreover, we assume that if $\bar{\mathcal{R}}_i \bigcap \bar{\mathcal{R}}_j \neq \emptyset$ then $F_{ij} \in \mathbf{R}^{n\times(n-1)}$ (full rank) and $l_{ij} \in \mathbf{R}^n$ exist such that

$$\bar{\mathcal{R}}_i \bigcap \bar{\mathcal{R}}_j \subseteq \{ l_{ij} + F_{ij}z \mid z \in \mathbf{R}^{n-1} \},$$

for $i = 1, \ldots, M$ and $j = i+1, \ldots, M$ (see Figure 3). Also, whenever necessary, we suppose that each $\mathcal{R}_i$ can be *outer approximated* by a union of (possibly degenerate) ellipsoids $\mathcal{E}_{ij}$ for $j = 1, \ldots, m_i$. In other words, matrices E_{ij} and f_{ij} exist such that

$$\mathcal{R}_i \subseteq \bigcup_{j=1}^{m_i} \mathcal{E}_{ij} \quad \text{where} \quad \mathcal{E}_{ij} = \{x \mid \|E_{ij}x + f_{ij}\| \leq 1\}. \tag{3}$$

(This may require a bounded $\mathcal{R}_i$.) In §8 we will briefly discuss how this ellipsoidal outer approximation can be done.

4 Analysis using a single quadratic Lyapunov function

In this section we analyze the PL system (1) using a single *quadratic* Lyapunov function, *i.e.*,

$$V(x) = x^T P x, \quad P = P^T \succ 0, \tag{4}$$

where $P = P^T \in \mathbf{R}^{n\times n}$. In other words, we search over all Lyapunov functions of the form (4) to prove stability or performance for the PL system (1).

4.1 Stability

We first study stability of the PL system (1) for $w = 0$ and $u = 0$. A sufficient condition for this is that V decreases (or equivalently $dV(x)/dt < 0$) along every nonzero trajectory of the system. If such a V exists, the system is said to be *quadratically stable.*

For all $x \in \mathcal{R}_i$ we have

$$\begin{aligned}\tfrac{d}{dt}V(x) &= (A_ix + b_i)^T P x + x^T P (A_i x + b_i) \\ &= \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i & Pb_i \\ b_i^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix},\end{aligned}$$

and therefore the condition for quadratic stability becomes the existence of a $P \succ 0$ such that for $i = 1, \ldots, M$

$$x \in \mathcal{R}_i \Rightarrow \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i & Pb_i \\ b_i^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \prec 0. \tag{5}$$

Now according to §3, suppose that each region $\mathcal{R}_i$ can be covered by a union of ellipsoids $\mathcal{E}_{ij}$ as defined in (3). Relaxing the condition $x \in \mathcal{R}_i$ in (5) by $x \in \mathcal{E}_{ij}$ for $j = 1, \ldots, m_i$ gives

$$x \in \mathcal{E}_{ij} \Rightarrow \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i & Pb_i \\ b_i^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \prec 0,$$

or, for all x satisfying

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} E_{ij}^T E_{ij} & E_{ij}^T f_{ij} \\ f_{ij}^T E_{ij} & -(1 - f_{ij}^T f_{ij}) \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \prec 0$$

there should exist a $P \succ 0$ such that

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i & Pb_i \\ b_i^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \prec 0.$$

Now using the $\mathcal{S}$-procedure (see, *e.g.*, [1]), this is equivalent to the existence of P and λ_{ij} satisfying

$$P \succ 0, \quad \lambda_{ij} < 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, m_i,$$

$$\begin{bmatrix} A_i^T P + PA_i + \lambda_{ij} E_{ij}^T E_{ij} & Pb_i + \lambda_{ij} E_{ij}^T f_{ij} \\ (Pb_i + \lambda_{ij} E_{ij}^T f_{ij})^T & -\lambda_{ij}(1 - f_{ij}^T f_{ij}) \end{bmatrix} \prec 0. \tag{6}$$

Clearly, (6) is an LMI in P and λ_{ij}, and gives a sufficient condition for quadratic stability. This sufficient condition will not be too conservative when the union of the covering ellipsoids $\mathcal{E}_{ij}$ is a good outer approximation to $\mathcal{R}_i$.

With the new variables $Q = P^{-1}$, $\mu_{ij} = 1/\lambda_{ij}$, condition (6) is also equivalent to the existence of μ_{ij} and Q satisfying the LMI

$$Q \succ 0, \quad \mu_{ij} < 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, m_i,$$

$$\begin{bmatrix} A_i Q + QA_i^T + \mu_{ij} b_i b_i^T & \mu_{ij} b_i f_{ij}^T + QE_{ij}^T \\ (\mu_{ij} b_i f_{ij}^T + QE_{ij}^T)^T & -\mu_{ij}(I - f_{ij} f_{ij}^T) \end{bmatrix} \prec 0. \tag{7}$$

This equivalent form is crucial for the controller synthesis problem in §7.

Remark. Note that instead of using the ellipsoidal approximation description of $\mathcal{R}_i$, we could have used its polytopic description (2) to obtain a stability condition similar to (6) by using the $\mathcal{S}$-procedure. One problem with this alternative condition, however, is that it can be very conservative because of the conservativeness of the $\mathcal{S}$-procedure for $p_i > 1$ (as noted in §6, one such case in which conservativeness hurts is when A_i is unstable). Another problem with this alternative is that no such equivalent condition for stability as in (7) with $Q = P^{-1}$ exists, and therefore, it does not seem that the controller synthesis problem can be formulated as an LMI (see §7).

4.2 $\mathbf{L}_2$ gain and other performance measures

Using standard Lyapunov arguments (see, *e.g.*, [1]) and a method similar to the previous subsection, the $\mathbf{L}_2$ gain from input w to output z of (1) is bounded by $\gamma > 0$ if Q and μ_{ij} exist such that

$$Q \succ 0, \quad \mu_{ij} < 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, m_i,$$

$$\begin{bmatrix} \begin{pmatrix} A_i Q + QA_i^T \\ +B_i^{(1)} B_i^{(1)T} \\ +\mu_{ij} b_i b_i^T \end{pmatrix} & \begin{pmatrix} \mu_{ij} b_i f_{ij}^T \\ +QE_{ij}^T \end{pmatrix} & \begin{pmatrix} B_i^{(1)} D_i^{(1)T} \\ +QC_i^{(1)T} \end{pmatrix} \\ \begin{pmatrix} \mu_{ij} f_{ij} b_i^T \\ +E_{ij} Q \end{pmatrix} & \begin{pmatrix} -\mu_{ij} I + \\ \mu_{ij} f_{ij} f_{ij}^T \end{pmatrix} & 0 \\ \begin{pmatrix} D_i^{(1)} B_i^{(1)T} \\ +C_i^{(1)} Q \end{pmatrix} & 0 & \begin{pmatrix} -\gamma^2 I + \\ D_i^{(1)} D_i^{(1)T} \end{pmatrix} \end{bmatrix} \prec 0, \tag{8}$$

The best provable bound on the $\mathbf{L}_2$ gain can be found by minimizing γ^2 subject to (8).

Note that many other performance measures for (1), such as decay rate, output energy, output peak, *etc.*, can also be cast as LMIs.

5 Analysis using a continuous piecewise-quadratic Lyapunov function

In this section we consider piecewise-quadratic Lyapunov functions of the form

$$V(x) = x^T P_{\alpha(x)} x + 2q_{\alpha(x)}^T x + r_{\alpha(x)},$$
$$V(x) > 0, \quad V \text{ is continuous},$$

where $P_i = P_i^T \in \mathbf{R}^{n\times n}$, $q_i \in \mathbf{R}^n$ and $r_i \in \mathbf{R}$ for $i = 1, \ldots, M$. Note that since V is piecewise-quadratic, $V(x) > 0$ also implies that $V(x) \to +\infty$ as $\|x\| \to \infty$. Clearly, this choice of Lyapunov function is more general than that of §4, and for example, it can also deal with PL systems with multiple equilibrium points.

5.1 Stability

The stability we refer to in this section is that, as $t \to \infty$, the state converges to one or more of the points in the set

$$\mathcal{Q} = \left\{-P_1^{-1}q_1, -P_2^{-1}q_2, \ldots, -P_M^{-1}q_M\right\}.$$

Clearly, all local minima of the Lyapunov function V are in $\mathcal{Q}$.

Using standard Lyapunov arguments it can be shown that the PL system (1) with $u = 0$ and $w = 0$ is stable if for $i = 1, \ldots, M$ and $j = i+1, \ldots, M$,

$$\begin{aligned} &F_{ij}^T(P_i - P_j)F_{ij} = 0,\\ &F_{ij}^T(P_i - P_j)l_{ij} + F_{ij}^T(q_i - q_j) = 0,\\ &l_{ij}^T(P_i - P_j)l_{ij} + 2(q_i - q_j)^T l_{ij} + (r_i - r_j) = 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\begin{bmatrix} P_i & q_i - H_i^T\lambda \\ q_i^T - \lambda^T H_i & r_i - 2g_i^T\lambda \end{bmatrix} \succ 0, \quad \lambda > 0, \quad \tau_i > 0,\\ &\begin{bmatrix} A_i^T P_i + P_i A_i & P_i b_i + A_i^T q_i - H_i \tau_i \\ b_i^T P_i + q_i^T A_i - \tau_i^T H_i^T & 2(g_i^T \tau_i + b_i^T q_i) \end{bmatrix} \prec 0, \end{aligned} \tag{10}$$

in which $\lambda, \tau_i \in \mathbf{R}^{p_i}$, and

$$H_i = [h_{1j}\ h_{2j} \cdots h_{p_ij}], \quad g_i = [g_{1j}\ g_{2j}\ \cdots g_{p_ij}]^T.$$

Equality constraints (9) guarantee that V is continuous, the first LMI in (10) guarantees that V is positive, and the second LMI in (10) guarantees that V decreases along all state trajectories.

Alternatively, if an outer ellipsoidal approximation to $\mathcal{R}_i$ as in (3) is given, condition (10) can be replaced by

$$\begin{aligned} &\lambda_{ij} > 0, \quad \tau_{ij} > 0, \quad j = 1, \ldots p_i,\\ &\begin{bmatrix} P_i + \lambda_{ij}E_{ij}^T E_{ij} & q_i + \lambda_{ij}E_{ij}^T f_{ij} \\ q_i^T + \lambda_{ij} f_{ij}^T E_{ij} & r_i + \lambda_{ij}(f_{ij}^T f_{ij} - 1) \end{bmatrix} \succ 0,\\ &\begin{bmatrix} A_i^T P_i + P_i A_i - \tau_{ij}E_{ij}^T E_{ij} & P_i b_i + A_i^T q_i - \tau_{ij}E_{ij}^T f_{ij} \\ b_i^T P_i + q_i^T A_i - \tau_{ij} f_{ij}^T E_{ij} & 2b_i^T q_i - \tau_{ij}(f_{ij}^T f_{ij} - 1) \end{bmatrix} \prec 0. \end{aligned} \tag{11}$$

Therefore, stability of the PL system is guaranteed if conditions (9) and (10), or, (9) and (11) hold. In §6 we discuss why the second set of conditions is relevant.

Remark. The equilibrium points of the PL system (1) should be the local minima of any Lyapunov function candidate. Therefore, if $x_{\mathrm{eq},i} \in \mathcal{R}_i$ is an equilibrium point we must have $x_{\mathrm{eq},i} = -P_i^{-1}q_i$. In other words, for $x \in \mathcal{R}_i$

$$V(x) = (x - x_{\mathrm{eq},i})^T P_i (x - x_{\mathrm{eq},i}) + r_i.$$

Replacing q_i by $-P_i x_{\mathrm{eq},i}$ and r_i by $r_i + x_{\mathrm{eq},i}^T P_i x_{\mathrm{eq},i}$ in (9), (10), and (11), gives a new set of conditions that are more favorable from a numerical point of view because the LMIs are not *strictly infeasible* anymore. Refer to §9.2 for an example.

5.2 Other performance measures

Using standard Lyapunov arguments, many other performance measures can be explored for the PL system (1). These include, $\mathbf{L}_2$ gain, decay rate, output energy, output peak, reachable sets, *etc.* Refer to [1].

6 Polytopic vs. ellipsoidal outer approximation description for operation regions $\mathcal{R}_i$

In most practical cases, a polytopic description of the regions $\mathcal{R}_i$ as in (2) is naturally available, so conditions (9) and (10) can be used to prove stability of the PL system. However, these conditions can be very conservative because of the conservativeness of the $\mathcal{S}$-procedure for $p_i > 1$. For example, in order for (10) to hold, the $(1,1)$ block entry of the first LMI, P_i, should be positive definite and the $(1,1)$ block entry of the second LMI, $A_i^T P_i + P_i A_i$, should be negative definite. Therefore, by Lyapunov's theorem for linear systems A_i must be stable. This means that we will never be able to prove stability of a PL system if one of the A_i's is unstable. Of course, there are many cases for which one or more of the A_i's are unstable but the overall system is stable (see §9.2).

If an ellipsoidal outer approximation for $\mathcal{R}_i$ is known, the LMIs in (11) can be used instead of those in (10). The underlying $\mathcal{S}$-procedure is a necessary and sufficient condition in this case, and potentially, using (9) and (11), we can prove stability for systems with one or more unstable A_i's. For example, note that we are subtracting a negative semidefinite term from the $(1,1)$ block entry of the second matrix in (11), and therefore, we can still be feasible without $A_i^T P_i + P_i A_i$ being negative definite.

As shown in the next section, having an ellipsoidal outer approximation for $\mathcal{R}_i$ has another advantage: The state-feedback synthesis problem using a single quadratic Lyapunov function can be cast as an LMI.

7 State-feedback synthesis using a single quadratic Lyapunov function

In this section we consider the PL system (1) and seek PL state-feedback control signals of the form $u = K_{\alpha(x)}x$. Therefore, the *closed-loop* state equations become

$$\begin{aligned} \dot{x} &= (A_{\alpha(x)} + B_{\alpha(x)}^{(2)} K_{\alpha(x)})x + b_{\alpha(x)} + B_{\alpha(x)}^{(1)} w,\\ z &= (C_{\alpha(x)}^{(1)} + D_{\alpha(x)}^{(2)} K_{\alpha(x)})x + D_{\alpha(x)}^{(1)} w. \end{aligned} \tag{12}$$

7.1 Quadratic stabilizability

Using (7), and by introducing the new variables $Y_i = K_i Q$ for $i = 1, \ldots, M$ we get the following LMI in the variables Q, Y_i and μ_{ij}

$$Q \succ 0, \quad \mu_{ij} < 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, m_i$$

$$\begin{bmatrix} \begin{pmatrix} A_i Q + Q A_i^T + \mu_{ij} b_i b_i^T \\ + B_i^{(2)} Y_i + Y_i^T B_i^{(2)T} \end{pmatrix} & \mu_{ij} b_i f_{ij}^T + Q E_{ij}^T \\ (\mu_{ij} b_i f_{ij}^T + Q E_{ij}^T)^T & -\mu_{ij}(I - f_{ij} f_{ij}^T) \end{bmatrix} \prec 0. \tag{13}$$

When Q and Y_i's satisfying (13) exist, the PL state-feedback control command $u = K_{\alpha(x)} x$ stabilizes (1) where $K_i = Y_i Q^{-1}$ for $i = 1, \ldots, M$.

> **Remark.** Another natural choice of input command would be one that is *affine* in the state x, *i.e.*, $u = K_{\alpha(x)} x + l_{\alpha(x)}$. However, it doesn't seem that the condition for stabilizability using this type of input command can be cast as an LMI.

7.2 $\mathbf{L_2}$ gain synthesis

Using (8) and (12), the state-feedback control $u = K_{\alpha(x)} x$ gives an $\mathbf{L}_2$ gain of less than γ from w to z if there exists Q, Y_i and μ_{ij} such that

$$Q \succ 0, \quad \mu_{ij} \leq 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, m_i,$$

$$\begin{bmatrix} \begin{pmatrix} A_i Q + Q A_i^T \\ + \mu_{ij} b_i b_i^T \\ + B_i^{(2)} Y_i \\ + Y_i^T B_i^{(2)T} \\ + B_i^{(1)} B_i^{(1)T} \end{pmatrix} & \begin{pmatrix} \mu_{ij} b_i f_{ij}^T \\ + Q E_{ij}^T \end{pmatrix} & \begin{pmatrix} B_i^{(1)} D_i^{(1)T} \\ + Q C_i^{(1)T} \\ + Y_i^T D_i^{(2)T} \end{pmatrix} \\ \begin{pmatrix} \mu_{ij} f_{ij} b_i^T \\ + E_{ij} Q \end{pmatrix} & \begin{pmatrix} -\mu_{ij} I + \\ \mu_{ij} f_{ij} f_{ij}^T \end{pmatrix} & 0 \\ \begin{pmatrix} D_i^{(1)} B_i^{(1)T} \\ + C_i^{(1)} Q \\ + D_i^{(2)} Y_i \end{pmatrix} & 0 & \begin{pmatrix} -\gamma^2 I + \\ D_i^{(1)} D_i^{(1)T} \end{pmatrix} \end{bmatrix} \prec 0, \tag{14}$$

where $K_i = Y_i Q^{-1}$. Clearly, (14) is an LMI in Q, Y_i and μ_{ij}.

7.3 A few practical notes

In this section, we assumed that the state-feedback gains are indexed by $\alpha(x)$, *i.e.*, $u = K_{\alpha(x)} x$. In other words, the controller itself is piecewise-linear with the same operating region function α. However, this assumption is not necessary for controller synthesis using LMIs such as (14). For example, as an extreme case, we can consider a *constant* state-feedback gain K which corresponds to a *linear* controller. In this case, the condition for a γ-level of performance in $\mathbf{L}_2$ gain is obtained by replacing $Y_i = K_i Q$ in (14) by $Y = KQ$. As another alternative, each $\mathcal{R}_i$ can be partitioned into smaller regions in which different state-feedback gains are used. The hope is that by introducing more state-feedback gains (or extra free variables in the optimization problem) we will get better performance for the closed loop system.

Computing the covering ellipsoids $\mathcal{E}_{ij}$ as defined in (3) is crucial for the synthesis method described in this paper. For an unbounded region $\mathcal{R}_i$, covering ellipsoids do not generally exist, and therefore, we need to bound the state-space beforehand. We will come back to this in §8.

A general assumption throughout this paper is that the operating mode of (1), through the function $\alpha(\cdot)$, depends on the state x *only*. Hence, we have ruled out the possibility that the control input u directly affect the operating mode of (1), and therefore, modelings in which there is a feed-through from u to a nonlinearity should be avoided. We can always add states to the system to overcome such unwanted feed-throughs, for example, by adding a first-order system before the nonlinearity with a "large" enough bandwidth (see §9.1).

Conditions for quadratic stability and $\mathbf{L}_2$ gain performance are given in (6) and (8) respectively. Clearly, when $1 - f_{ij}^T f_{ij} < 0$, we can only have $\lambda_{ij} = 0$ for the LMIs to hold, and the LMIs are no longer strictly feasible. $1 - f_{ij}^T f_{ij} < 0$ means that the origin lies inside the ellipsoid $\mathcal{E}_{ij}$. In this case $\frac{d}{dt} V(x) < 0$ for $x \in \mathcal{E}_{ij}$ is equivalent to $A_i^T P + P A_i \prec 0$ (assuming $b_i = 0$ so that the origin is an equilibrium point of the system) which is the condition on *global* stability of the linear system $\dot{x} = A_i x$. Therefore, for example, the (modified) state-feedback synthesis formulation for quadratic stabilizability (that avoids strict infeasibility) becomes the existence of Q, Y_i ($K_i = Y_i Q^{-1}$) and μ_{ij} such that for $i = 1, \ldots, M$ and $j = 1, \ldots, m_i$

$$\begin{array}{ll} A_i Q + Q A_i^T + B_i^{(2)} Y_i + Y_i^T B_i^{(2)T} \prec 0 & \text{when } 0 \in \mathcal{R}_i \\ (13) & \text{when } 0 \notin \mathcal{R}_i. \end{array} \tag{15}$$

Similarly, condition (14) for a γ-level of $\mathbf{L_2}$ gain should be modified (to avoid strict infeasibility) when $0 \in \mathcal{R}_i$. This can be done by removing the second row and column of the 3×3 block matrix in (14), which is then equivalent to having a γ-level of $\mathbf{L_2}$ gain from w to z in the linear system

$$\dot{x} = (A_i + B_i^{(2)} K_i) x + B_i^{(1)} w, \quad z = (C_i^{(1)} + D_i^{(2)} K_i) x + D_i^{(1)} w.$$

8 Computing ellipsoidal outer approximations for operating regions $\mathcal{R}_i$

In theory, any region $\mathcal{R}_i$ can be (outer) approximated arbitrarily well by a union of ellipsoids $\mathcal{E}_{ij}$ for $j = 1, \ldots, m_i$. However, as far as we know, there is no general and numerically efficient method to approximate an arbitrary region $\mathcal{R}_i$ to any desired accuracy.

When the $\mathcal{R}_i$'s are polytopic as in (2), there are many well-known methods to compute ellipsoidal outer approximations. A discussion of these methods is out of the scope of this paper and we only refer the interested reader to the references [1, 11, 12, 13].

Let us note that if $\alpha(x)$ does not explicitly depend on all state variables, the regions $\mathcal{R}_i$ are elongated to $\pm\infty$ in directions that correspond to state variables that do not appear in $\alpha(x)$. If $\bar{x}$ is the vector of state variables that explicitly appear in $\alpha(x)$, once a covering ellipsoid $\|E_{ij} \bar{x} + f_{ij}\| \leq 1$ is computed for a *cross section* of $\mathcal{R}_i$ in which all state variables other than $\bar{x}$ are constant, a degenerate ellipsoid that covers $\mathcal{R}_i$ is simply found by adding zeros to E_{ij} and f_{ij} at positions that correspond to the missing state variables.

In order to be able to compute covering ellipsoids that have finite volume in the directions $\bar{x}$, the state variables that explicitly appear in $\alpha(x)$ should be bounded, say, by adding the component-wise inequality constraint $-a \leq \bar{x} \leq b$. This is not a practical problem, however, as we can always take a and b larger than the physical limitations of the system. Therefore, each $\mathcal{R}_i$ should be redefined as $\mathcal{R}_i \leftarrow \mathcal{R}_i \bigcap \{x \in \mathbf{R}^n \mid -a \leq \bar{x} \leq b\}$.

When $\mathcal{R}_i$ is a slab, a (degenerate) ellipsoid of the form $\|E_{i1} x + f_{i1}\| \leq 1$ can be found that approximates

$\mathcal{R}_i$ *exactly.* Suppose that $\mathcal{R}_i = \{ \ x \mid d_1 \le c^T x \le d_2 \ \}$, then it is easy to see that we can take $E_{i1} = 2c/(d_2 - d_1)$ and $f_{i1} = (d_2 + d_1)/(d_2 - d_1)$.

Finally, note that if we are using the single quadratic Lyapunov function approach to analyze or design controllers for (1), according to the discussion in §7.3, and condition (15), we do not need to compute an ellipsoidal outer approximation for regions $\mathcal{R}_i$ that contain the origin.

9 Examples

9.1 Mechanical system with saturating actuator

In this example[1] we consider the simple mechanical system in Figure 2(a). The goal is to design a (state-feedback) controller that makes the $\mathbf{L_2}$ gain from the exogenous input w to the displacement x_1 small. Without any control input, the $\mathbf{L}_2$ norm from w to x_1 is approximately equal to $\gamma_{\text{OL}} = 11.8$.

The actuator is subject to a saturation nonlinearity as shown in Figure 2(b). Note that the first-order system $1/(\tau s + 1)$ is introduced before the nonlinearity so that there is no feed-through from the control input u to the nonlinearity (see §7.3). In this case, α becomes only a function of x_5, and $M = 3$. It is straightforward to compute the system matrices in each operation region, as well as ellipsoidal outer approximations which are exact because the $\mathcal{R}_i$'s are slabs. (Note that the state-space should be bounded in the x_5 direction, say, by adding the constraint $-10^3 \le x_5 \le 10^3$.)

Assuming that $k_1 = 1$, $k_2 = 1$, $b_1 = 0.1$, and $b_2 = 0.1$, the modes of the mechanical system become

$$p_{1,2} = -0.1309 \pm j1.6127, \quad p_{3,4} = -0.0191 \pm j0.6177.$$

We let $1/\tau = 10$ which is a couple of orders of magnitude larger than the decay rate of the mechanical system.

The regulating output is chosen to be $z = [x_1 \ 0.1u]^T$. Note that the input command u is also included in the regulating output to avoid getting large state-feedback gains. Using the results of §7.2 we design the state-feedback gains for a level of $\mathbf{L}_2$ gain of $\gamma = 7$ from w to z. It turns out that the state-feedback gains in each region $\mathcal{R}_i$ become the same and are equal to

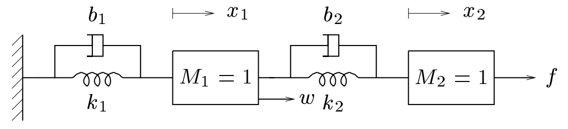
$$K_{1,2,3} = [\ 2.45 \quad -12.50 \quad -401.3 \quad -645.3 \quad -172.0\ ]$$

where the state was chosen as $x = [x_1 \ x_2 \ \dot{x}_1 \ \dot{x}_2 \ x_5]^T$. Now if we recompute a bound on the $\mathbf{L}_2$ gain from input w to output x_1 of the *closed loop* system using the results of §4.2 we get the value of $\gamma_{\text{CL}} = 0.09$ which is a significant improvement over the open loop $\mathbf{L}_2$ gain of $\gamma_{\text{OL}} = 11.8$.

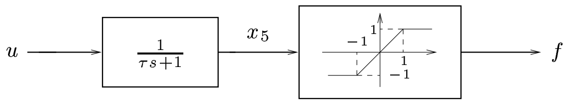
9.2 Circuit with multiple equilibrium points

Here we consider the simple electric circuit of Figure 3(a). The nonlinear resistor has a tunnel-diode type (i, v)-characteristic as shown in Figure 3(b). The state of the system is $x = [i_L \ v_c]^T$ and there are three equilibrium points $x_{\text{eq},1} = [0.14 \ 0.71]^T$, $x_{\text{eq},2} = [0.45 \ 0.50]^T$, and $x_{\text{eq},3} = [0.64 \ 0.37]^T$. Clearly, a single quadratic Lyapunov function cannot handle this system as there are three equilibrium points. Also, because of the negative slope of the nonlinear resistor, A_2 is unstable and

[1]The example in this section and the next were carried out using the semidefinite program solver package `sdpsol` [14].



(a) Simple mechanical system with two degrees of freedom.



(b) Actuator input/output behavior.

Figure 2: Controller design for a simple mechanical system subject to input saturation nonlinearity.

as a result, we need an ellipsoidal outer approximation for $\mathcal{R}_2$. Since $\mathcal{R}_2$ is just a slab, an exact ellipsoidal approximation exists (see §8). For regions $\mathcal{R}_1$ and $\mathcal{R}_3$ we can just use the polytopic description. Note that $p_1 = p_3 = 1$ and therefore conditions in (10) are necessary and sufficient.

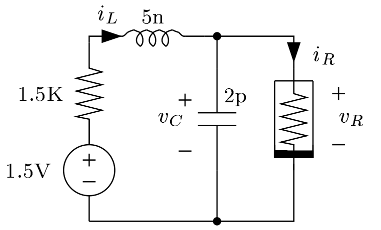
Using the results of §5 it is possible to prove stability for this system in the sense of §5.1. Figures 3(c) and 3(d) show one of the piecewise-quadratic Lyapunov functions that achieve this. It can be seen that $x_{\text{eq},2}$ is a saddle point of the Lyapunov function which means that $x_{\text{eq},2}$ is unstable. $x_{\text{eq},2}$ and $x_{\text{eq},3}$ are local minima of the Lyapunov function and are therefore stable. In fact, this is a bistable circuit.

10 Conclusions and further research

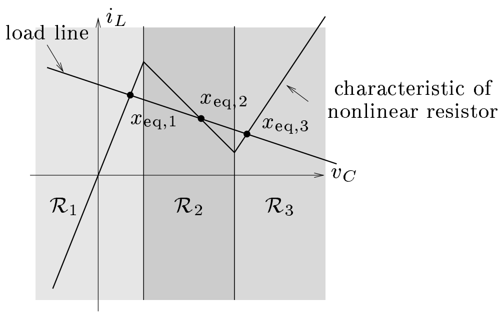
In this paper we have given a method for analysis of PL systems by Lyapunov methods. The analysis involves solving convex optimization problems involving LMIs that can be done very efficiently. If a single quadratic Lyapunov function is used, state-feedback synthesis of PL systems can also be formulated as LMIs. (If the full state is not available for feedback, observer-based controllers can be designed by solving LMIs, although this was not mentioned in this paper.) On the other hand, piecewise-quadratic Lyapunov functions are specially useful for dealing with PL systems with multiple equilibrium points. A central idea in this paper was to use an ellipsoidal outer approximation to the operating regions $\mathcal{R}_i$. This enabled us to reduce the conservatism of the methods and to derive an LMI formulation for the synthesis problem.

A very interesting problem to be explored in the controller synthesis problem is the partitioning of the operating regions $\mathcal{R}_i$ into smaller cells in which different state-feedback gains are used. The hope is that by introducing more state-feedback gains (or extra free variables in the optimization problem) we will get better performance for the closed loop system.

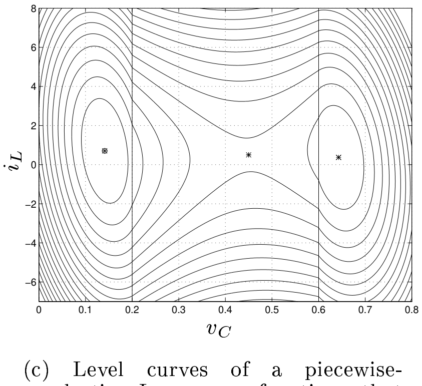
Finally, it should be noted that the same ideas in this paper can be extended to the analysis of *hybrid dynamical systems.* Hybrid dynamical systems are systems that incorporate both discrete and continuous dynamics, with the discrete dynamics governed by finite automata and the continuous dynamics usually repre-



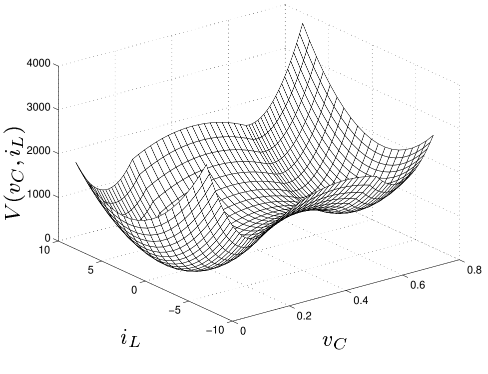
(a) Electrical circuit with nonlinear resistor.



(b) Different operating regions and equilibrium points of the electrical circuit.



(c) Level curves of a piecewise-quadratic Lyapunov function that proves stability for the electrical circuit.



(d) A piecewise-quadratic Lyapunov function that proves stability for the electrical circuit.

Figure 3: Lyapunov function construction for an electrical circuit having multiple equilibria.

sented by ordinary differential equations. The two interact at "event times" determined by the continuous state hitting certain event sets in the continuous state space. Hybrid dynamical systems can model a vast array of important practical systems for which piecewise-linear systems is just one of the simplest. Some examples are: systems with hysteresis, multi-modal systems, systems with logic, timing circuits, automated highway systems [15], computer disk drives [16], transmissions and stepper motors [17], and systems with both digital and analog components. Even hybrid systems with very simple continuous dynamics, *e.g.*, only integrators, can have many practical applications and very complex behavior.

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