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# SEMIDEFINITE PROGRAMMING RELAXATIONS OF NON-CONVEX PROBLEMS IN CONTROL AND COMBINATORIAL OPTIMIZATION

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*In celebration of Tom Kailath's contributions, on his 60th birthday.*

*Dedicated to Tom Kailath: mentor, model, colleague, teacher, friend.*

## ABSTRACT

We point out some connections between applications of semidefinite programming in control and in combinatorial optimization. In both fields semidefinite programs arise as convex relaxations of NP-hard quadratic optimization problems. We also show that these relaxations are readily extended to optimization problems over bilinear matrix inequalities.

## 1 SEMIDEFINITE PROGRAMMING

In a *semidefinite program* (SDP) we minimize a linear function of a variable $x \in \mathbf{R}^m$ subject to a matrix inequality:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) \geq 0 \end{array} \tag{1.1}$$

where

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i.$$

The problem data are the vector $c \in \mathbf{R}^m$ and $m+1$ symmetric matrices $F_0, \ldots, F_m \in \mathbf{R}^{n \times n}$. The inequality sign in $F(x) \geq 0$ means that $F(x)$ is positive semidefinite, *i.e.*, $z^T F(x) z \geq 0$ for all $z \in \mathbf{R}^n$. We call the inequality $F(x) \geq 0$ a *linear matrix inequality* (LMI).

Semidefinite programs can be regarded as an extension of linear programming where the componentwise inequalities between vectors are replaced by matrix inequalities, or, equivalently, the first orthant is replaced by the cone of positive semidefinite matrices. Semidefinite programming unifies several standard problems (*e.g.*, linear and quadratic programming), and finds many applications in engineering and combinatorial optimization (see [Ali95], [BEFB94], [VB96]). Although semidefinite programs are much more general than linear programs, they are not much harder to solve. Most interior-point methods for linear programming have been generalized to semidefinite programs. As in linear programming, these methods have polynomial worst-case complexity, and perform very well in practice.

## 2 SEMIDEFINITE PROGRAMMING AND COMBINATORIAL OPTIMIZATION

Semidefinite programs play a very useful role in non-convex or combinatorial optimization. Consider, for example, the quadratic optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \; i = 1, \ldots, L \end{array} \tag{1.2}$$

where $f_i(x) = x^T A_i x + 2b_i^T x + c_i$, $i = 0, 1, \ldots, L$. The matrices $A_i$ can be indefinite, and therefore problem (1.2) is a very hard, non-convex optimization problem. For example, it includes all optimization problems with polynomial objective function and polynomial constraints (see [NN94, §6.4.4], [Sho87]).

For practical purposes, *e.g.*, in branch-and-bound algorithms, it is important to have good and cheaply computable lower bounds on the optimal value of (1.2). Shor and others have proposed to compute such lower bounds by solving the semidefinite program (with variables $t$ and $\tau_i$)

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots + \tau_L \begin{bmatrix} A_L & b_L \\ b_L^T & c_L \end{bmatrix} \geq 0 \\ & \tau_i \geq 0, \quad i = 1, \ldots, L. \end{array} \tag{1.3}$$

One can easily verify that this semidefinite program yields lower bounds for (1.2). Suppose $x$ satisfies the constraints in the nonconvex problem (1.2), *i.e.*,

$$f_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0$$

for $i = 1, \ldots, L$, and $t$, $\tau_1$, $\ldots$, $\tau_L$ satisfy the constraints in the semidefinite program (1.3). Then

$$\begin{aligned} 0 &\leq \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left( \begin{bmatrix} A_0 & b_0 \\ b_0^T & c_0 - t \end{bmatrix} + \tau_1 \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \cdots + \tau_L \begin{bmatrix} A_L & b_L \\ b_L^T & c_L \end{bmatrix} \right) \begin{bmatrix} x \\ 1 \end{bmatrix} \\ &= f_0(x) - t + \tau_1 f_1(x) + \cdots + \tau_L f_L(x) \\ &\leq f_0(x) - t. \end{aligned}$$

Therefore $t \leq f_0(x)$ for every feasible $x$ in (1.2), as desired. Problem (1.3) can also be derived via Lagrangian duality; for a deeper discussion, see Shor [Sho87], or Poljak, Rendl, and Wolkowicz [PRW94].

Most semidefinite relaxations of NP-hard combinatorial problems seem to be related to the semidefinite program (1.3), or the related one,

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr} X A_0 + 2 b_0^T x + c_0 \\ \text{subject to} & \mathbf{Tr} X A_i + 2 b_i^T x + c_i \leq 0, \quad i = 1, \ldots, L \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \geq 0, \end{array} \tag{1.4}$$

where the variables are $X = X^T \in \mathbf{R}^{k \times k}$ and $x \in \mathbf{R}^k$. It can be shown that (1.4) is the semidefinite programming *dual* of Shor's relaxation (1.3); the two problems (1.3) and (1.4) yield the same bound.

Note that the constraint

$$\begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \geq 0 \tag{1.5}$$

is equivalent to $X \geq xx^T$. The semidefinite program (1.4) can therefore be directly interpreted as a relaxation of the original problem (1.2), which can be written as

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr} X A_0 + 2b_0^T x + c_0 \\ \text{subject to} & \mathbf{Tr} X A_i + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, L \\ & X = xx^T. \end{array} \tag{1.6}$$

The only difference between (1.6) and (1.4) is the replacement of the (nonconvex) constraint $X = xx^T$ with the convex relaxation $X \geq xx^T$. It is also interesting to note that the relaxation (1.4) becomes the original problem (1.6) if we add the (nonconvex) constraint that the matrix on the left hand side of (1.5) is rank one.

As an example, consider the $(-1,1)$-quadratic program

$$\begin{array}{ll} \text{minimize} & x^T A x + 2b^T x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, k, \end{array} \tag{1.7}$$

which is NP-hard. The constraint $x_i \in \{-1, 1\}$ can be written as the quadratic equality constraint $x_i^2 = 1$, or, equivalently, as two quadratic inequalities $x_i^2 \leq 1$ and $x_i^2 \geq 1$. Applying (1.4) we find that the semidefinite program in $X = X^T$ and $x$

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr} X A + 2b^T x \\ \text{subject to} & X_{ii} = 1, \quad i = 1, \ldots, k \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \geq 0 \end{array} \tag{1.8}$$

yields a lower bound for (1.7). In a recent paper on the MAX-CUT problem, which is a specific case of (1.7) where $b = 0$ and the diagonal of $A$ is zero, Goemans and Williamson have *proved* that the lower bound from (1.8) is at most 14% suboptimal (see [GW94] and [GW95]). This is much better than any previously known bound. Similar strong results on semidefinite programming relaxations of NP-hard problems have been obtained by Karger, Motwani, and Sudan [KMS94].

The usefulness of semidefinite programming in combinatorial optimization was recognized more than twenty years ago (see, *e.g.*, Donath and Hoffman [DH73]). Many people seem to have developed similar ideas independently. We should however stress the importance of the work by Grötschel, Lovász, and Schrijver [GLS88, Chapter 9], [LS91] who have demonstrated the power of semidefinite relaxations on some very hard combinatorial problems. The recent development of efficient interior-point methods has turned these techniques into powerful practical tools; see Alizadeh [Ali92b, Ali91, Ali92a], Kamath and Karmarkar [KK92, KK93], Helmberg, Rendl, Vanderbei and Wolkowicz [HRVW94].

For a more detailed survey of semidefinite programming in combinatorial optimization, we refer the reader to the recent paper by Alizadeh [Ali95].

# 3 SEMIDEFINITE PROGRAMMING AND CONTROL THEORY

Semidefinite programming problems arise frequently in control and system theory; Boyd, El Ghaoui, Feron and Balakrishnan catalog many examples in [BEFB94]. We will describe one simple example here.

Consider the *differential inclusion*

$$\frac{dx}{dt} = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \qquad |u_i(t)| \leq |y_i(t)|, \; i = 1, \ldots, p \tag{1.9}$$

where $x(t) \in \mathbf{R}^l$, $u(t) \in \mathbf{R}^p$, and $y(t) \in \mathbf{R}^p$. In the terminology of control theory, this is described as a linear system with uncertain, time-varying, unity-bounded, diagonal feedback.

We seek an invariant ellipsoid, *i.e.*, an ellipsoid $\mathcal{E}$ such that for any $x$ and $u$ that satisfy (1.9), $x(T) \in \mathcal{E}$ implies $x(t) \in \mathcal{E}$ for all $t \geq T$. The existence of such an ellipsoid implies, for example, that all solutions of the differential inclusion (1.9) are bounded.

The ellipsoid $\mathcal{E} = \{x \mid x^T P x \leq 1\}$, where $P = P^T > 0$, is invariant if and only if the function $V(t) = x(t)^T P x(t)$ is nonincreasing for any $x$ and $u$ that satisfy (1.9). In this case we say that $V$ is a quadratic Lyapunov function that proves stability of the differential inclusion (1.9).

We can express the derivative of $V$ as a quadratic form in $x(t)$ and $u(t)$:

$$\frac{d}{dt}V(x(t)) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{1.10}$$

We can express the conditions $|u_i(t)| \leq |y_i(t)|$ as the quadratic inequalities

$$u_i^2(t) - y_i^2(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} -c_i^T c_i & 0 \\ 0 & E_{ii} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \leq 0, \qquad i = 1, \ldots, p, \tag{1.11}$$

where $c_i$ is the $i$th row of $C$, and $E_{ii}$ is the matrix with all entries zero except the $ii$ entry, which is 1.

Putting it all together we find that $\mathcal{E}$ is invariant if and only if (1.10) holds whenever (1.11) holds. Thus the condition is that one quadratic inequality should hold whenever some other quadratic inequalities hold, *i.e.*:

$$\text{for all } z \in \mathbf{R}^{l+p}, \quad z^T T_i z \leq 0, \ i = 1, \ldots, p \quad \Longrightarrow \quad z^T T_0 z \leq 0 \tag{1.12}$$

where

$$T_0 = \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix}, \qquad T_i = \begin{bmatrix} -c_i^T c_i & 0 \\ 0 & E_{ii} \end{bmatrix}, \ i = 1, \ldots, p.$$

In the general case, simply verifying that (1.12) holds for a given $P$ is very difficult. But an obvious sufficient condition is

$$\text{there exist} \quad \tau_1 \geq 0, \ldots, \tau_p \geq 0 \ \text{ such that } \ T_0 \leq \tau_i T_1 + \cdots + \tau_p T_p. \tag{1.13}$$

Replacing the condition (1.12) with the stronger condition (1.13) is called the $\mathcal{S}$-procedure in the Soviet literature on control theory, and dates back at least to 1944 (see [BEFB94, p.33], [FY79], [LP44]). Note the similarity between Shor's bound (see (1.2) and (1.3)) and the $\mathcal{S}$-procedure ((1.12) and (1.13)). Indeed Shor's bound is readily derived from the $\mathcal{S}$-procedure, and vice versa.

Returning to our example, we apply the $\mathcal{S}$-procedure to obtain a sufficient condition for invariance of the ellipsoid $\mathcal{E}$: for some $D = \mathbf{diag}(\tau_1, \ldots, \tau_p)$,

$$\begin{bmatrix} A^T P + PA + C^T DC & PB \\ B^T P & -D \end{bmatrix} \leq 0. \tag{1.14}$$

This is a linear matrix inequality in the (matrix) variables $P = P^T$ and (diagonal) $D$. Hence, by solving a semidefinite feasibility problem we can find an invariant ellipsoid (if the problem is feasible). One can also optimize various quantities over the feasible set; see [BEFB94]. Note that (1.14) is really a convex relaxation of the condition that $\mathcal{E}$ be invariant, obtained via the $\mathcal{S}$-procedure.

The close connections between the $\mathcal{S}$-procedure, used in control theory to form semidefinite programming relaxations of hard control problems, and the various semidefinite relaxations used in combinatorial optimization, do not appear to be well known.

# 4 EXTENSION TO BILINEAR MATRIX INEQUALITIES

We now consider an extension of the SDP (1.1), obtained by replacing the linear matrix inequality constraints by *bilinear* (or bi-affine) matrix inequalities (BMIs),

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{j,k=1}^{m} x_j x_k G_{jk} \geq 0. \end{array} \tag{1.15}$$

The problem data are the vector $c \in \mathbf{R}^m$ and the symmetric matrices $F_i$, $G_{jk} \in \mathbf{R}^{n\times n}$.

Bilinear matrix inequality problems are NP-hard, and include a wide variety of control problems (see, *e.g.*, [GLTS94], [GSP94], [SGL94], [GTS+94], [GSL95]). They also include all quadratic problems (when the matrices in (1.15) are diagonal), all polynomial problems, all $\{0,1\}$ and integer programs, etc.

Several heuristic methods for BMI problems have been presented in the literature cited above and reported to be useful in practice. Our purpose here is to point out that the semidefinite relaxations for quadratic problems can be easily extended to bilinear matrix inequalities.

We first express the BMI problem as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{j,k=1}^{m} w_{jk} G_{jk} \geq 0 \\ & w_{jk} = x_j x_k, \quad j,k = 1,\ldots,m, \end{array}$$

and then relax the second constraint as an LMI

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{j,k=1}^{m} w_{jk} G_{jk} \geq 0 \\ & \begin{bmatrix} W & x \\ x^T & 1 \end{bmatrix} \geq 0. \end{array}$$

This is an SDP in the variables $W$, $x$. Its optimal value is a lower bound for the optimal value of problem (1.15). As in Section 2, this SDP can also be interpreted as the dual of the Lagrangian relaxation of problem (1.15).

We should point out that the SDP relaxation may require some manipulation (*e.g.*, when some of the matrices $G_{ii}$ are zero), just as in the general indefinite quadratic programming case.

We do not yet have any numerical experience with this SDP relaxation of the BMI problem.

## 5 CONCLUSION

The simultaneous discovery of semidefinite programming applications in control and combinatorial optimization is remarkable and raises several interesting questions. For example,

- can we obtain Goemans and Williamson-type results in control theory, *i.e.*, solve a (polynomial-time) SDP and get a *guaranteed* bound on suboptimality?
- what is the practical performance of semidefinite relaxations in nonconvex quadratic or BMI problems?

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