Structured and Simultaneous Lyapunov Functions for System Stability Problems

Stephen Boyd and Qinping Yang[1]
Electrical Engineering Department
Stanford University
Stanford CA 94305

January 22, 2001

[1]Research sponsored in part by ONR under N00014-86-K-0112, NSF under ECS-85-52465, an IBM faculty development award, and Bell Communications Research.

Abstract

It is shown that many system stability and robustness problems can be reduced to the question of when there is a quadratic Lyapunov function of a certain structure which establishes stability of $\dot{x} = Ax$ for some appropriate A. The existence of such a Lyapunov function can be determined by solving a convex program. We present several numerical methods for these optimization problems. A simple numerical example is given.

1 Notation and preliminaries

$\mathbb{R}$ ($\mathbb{R}_+$) will denote the set of real numbers (nonnegative real numbers). The set of $m \times n$ matrices will be denoted $\mathbb{R}^{m\times n}$. I_k will denote the $k \times k$ identity matrix (we will sometimes drop the subscript k if it can be determined from context). $\mathbb{R}I_k$ will denote all multiples of I_k. If $G_i \in \mathbb{R}^{k_i\times k_i}$, $i = 1, \ldots, m$, then $\bigoplus_{i=1}^{m} G_i = G_1 \oplus \cdots \oplus G_m$ will denote the block diagonal matrix with diagonal blocks $G_1, \ldots, G_m$. We extend this notation to sets of matrices, so that for example $\bigoplus_{i=1}^{3} \mathbb{R}$ is the set of diagonal 3×3 matrices, and

$$\mathbb{R}^{2\times 2} \oplus \mathbb{R}I_2 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & a_5 \end{bmatrix} \right| a_1, \ldots, a_5 \in \mathbb{R} \right\}.$$

Many of our results will pertain to the basic feedback system (shown in figure 1),

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

$$u = \Delta(y) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t),\ y(t) \in \mathbb{R}^k$, and Δ is a (possibly nonlinear) causal operator mapping $[\mathbf{L}^2_{loc}(\mathbb{R}_+)]^k$ into itself (see Desoer and Vidyasagar [DV75] for a complete definition of causality and more background). Throughout this paper we will assume that the linear system (1) is minimal, that is, controllable and observable. We will say that the system (1-2) is *stable* if for all solutions, $x(t)$ is bounded for $t \geq 0$.

Sometimes the operator Δ can be decomposed into a number of smaller operators in parallel, as shown in figure 2. More precisely, suppose u and y can be partitioned as $u^T = \left[u_1^T \cdots u_m^T\right]$, $y^T = \left[y_1^T \cdots y_m^T\right]$, $u_i(t),\ y_i(t) \in \mathbb{R}^{k_i}$, such that (2) can be expressed as

$$u_i = \Delta_i(y_i), \quad i = 1, \ldots, m. \tag{3}$$

In this case we say the operator Δ has the *block structure* $[k_1, \ldots, k_m]$. If Δ has block structure $[1, \ldots, 1]$, we say Δ is a *diagonal* operator.

The term 'block structure' and the symbol Δ follow Doyle's usage[Doy82].

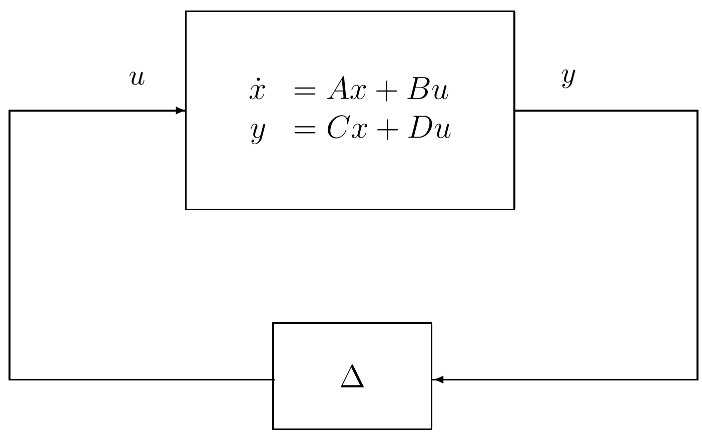

Figure 1: Basic feedback system.

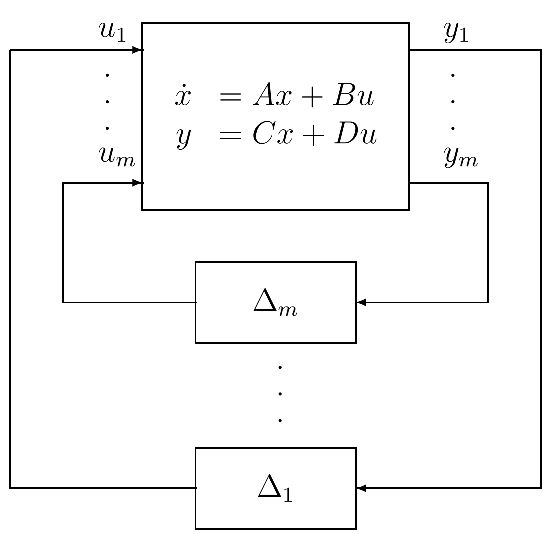

Figure 2: Basic feedback system with block structured feedback.

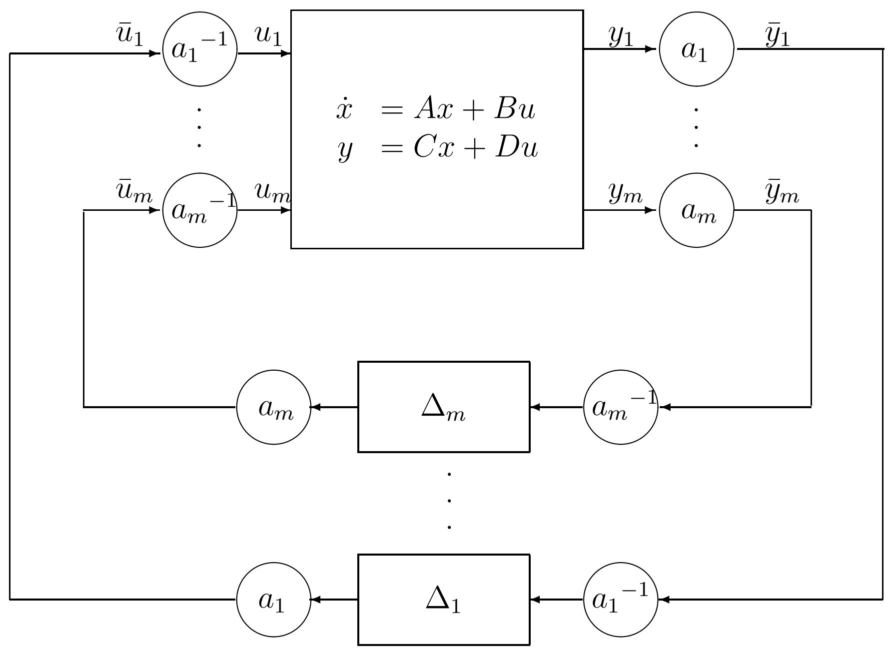

Figure 3: Basic feedback system with block structured feedback, showing the 'block structure preserving scaling transformation'.

If Δ has block structure $[k_1, \ldots, k_m]$ and $a_1, \ldots, a_m$ are nonzero constants, we can perform the block structure preserving scaling transformation $\bar{u}_i = a_i u_i$, $\bar{y}_i = a_i y_i$, so that (1-2) can be expressed

$$\begin{aligned} \dot{x} &= Ax + BF^{-1}\bar{u} \\ \bar{y} &= FCx + FDF^{-1}\bar{u} \end{aligned} \tag{4}$$

$$\bar{u} = \bar{\Delta}(\bar{y}) \tag{5}$$

where $F = \bigoplus_{i=1}^{m} a_i I_{k_i}$ and $\bar{\Delta}$ is the operator defined by $\bar{u}_i = a_i \Delta_i(a_i^{-1}\bar{y}_i)$. A block diagram of this scaling transformation is shown in figure 3. We note for future reference that $\bar{\Delta}$ also has block structure $[k_1, \ldots, k_m]$ and the transfer matrix of the scaled linear system (4) is $FH(s)F^{-1}$, where $H(s) = C(sI - A)^{-1}B + D$ is the transfer matrix of the original linear system (1).

2 Structured Lyapunov functions

We say $A \in \mathbb{R}^{n\times n}$ is *stable* if all solutions of $\dot{x} = Ax$ are bounded for $t \geq 0$, or equivalently, all eigenvalues of A have nonpositive real parts, and the pure imaginary eigenvalues are simple zeros of the minimal polynomial of A.[1] A famous result of Lyapunov theory states that A is stable if and only if there is a $P = P^T > 0$ such that $A^T P + PA \leq 0$. In this case we say the 'Lyapunov function' $V(x) = x^T P x$ establishes stability of the differential equation $\dot{x} = Ax$, since V is positive definite and $\dot{V}(x) = -2x^T PAx$ is negative semidefinite.

The topic of this paper is the following question: given A, is there a P of a certain structure, for example, block diagonal, for which the Lyapunov function $V(x) = x^T P x$ establishes stability of $\dot{x} = Ax$? We call this the *structured Lyapunov problem* for A. We will show that (a) several problems involving stability of the basic feedback system (1-2) can be answered by solving a structured Lyapunov problem for a certain structure and matrix A, and (b) practical (numerical) solution of the structured Lyapunov problem involves a convex minimization problem. The implication of (b) is that there are *effective* algorithms for solving the structured Lyapunov problem.

Definition 1 *Let* $\mathbf{S}$ *be a subspace of* $\mathbb{R}^{n\times n}$*.* $A \in \mathbb{R}^{n\times n}$ *is* $\mathbf{S}$-structured Lyapunov stable *(*$\mathbf{S}$*-SLS or just SLS if* $\mathbf{S}$ *is understood) if there is a* $P \in \mathbf{S}$ *such that* $P = P^T > 0$ *and* $A^T P + PA \leq 0$*.*

We will refer to $\mathbf{S}$ as a *structure,* and V an $\mathbf{S}$-structured Lyapunov function ($\mathbf{S}$-SLF) for A. The structures we will encounter will be very simple, usually consisting of block diagonal matrices, perhaps with some blocks repeated.

Note the distinction between a 'block structure' $[k_1, \ldots, k_m]$ (an attribute of an operator), and a 'structure' $\mathbf{S}$ (a subset of $\mathbb{R}^{n\times n}$).

If $\mathbf{S} = \mathbb{R}^{n\times n}$, then by Lyapunov's theorem, A is SLS if and only if A is stable (this could be called unstructured Lyapunov stability); but in general the condition that A is $\mathbf{S}$-SLS is stronger than mere stability of A. At the other extreme, if $\mathbf{S} = \mathbb{R}I_n$, then A is SLS if and only if $A + A^T \leq 0$, which is sometimes referred to as *dissipative dynamics.* This is precisely the condition under which all solutions of $\dot{x} = Ax$ are not only bounded, but in addition

[1] This is often called *marginal stability* in the linear systems literature.

$\|x\| = \sqrt{x^T x}$ is nonincreasing. For intermediate structures, the condition that A be SLS will fall between these two extremes: stability, and dissipative dynamics.

A very important special case of the structured Lyapunov problem is the following:

$$\mathbf{S} = \left\{ P \oplus \cdots \oplus P \;\middle|\; P \in \mathbb{R}^{n\times n} \right\}, \tag{6}$$

$$A = A_1 \oplus \cdots \oplus A_k, \quad A_i \in \mathbb{R}^{n\times n} \tag{7}$$

In this case, A is $\mathbf{S}$-SLS if and only if there is a *single Lyapunov function* $V(x) = x^T P x$ (no structure requirement on P) which establishes the stability of the matrices $A_1, \ldots, A_k$. If the matrix A in (7) is $\mathbf{S}$-SLS, then we say the set of matrices $\{A_1, \ldots, A_k\}$ is *simultaneously Lyapunov stable* (SILS), and the Lyapunov function V is a *simultaneous Lyapunov function* (SILF) for the set.

Several nontrivial cases of the structured Lyapunov problem have been investigated, notably for the case where P is diagonal. This problem is considered in Araki [Ara76], Barker, Berman, and Plemmons [BBP78], Moylan and Hill [MH78], Khalil and Kokotovic [KK79], Khalil [Kha82], and others; their applications range from stability of large interconnected systems to multi-parameter singular perturbations. Matrices which are diagonal-SLS or satisfy a very similar condition (for example, $A^T P + PA > 0$) are sometimes called D-stable or diagonally stable. In the papers cited above various relations (often, sufficient conditions for D-stability) have been found between such matrices and M-matrices, so called quasi-dominant matrices, and P_0-matrices. We refer the reader to the papers mentioned above and the references therein. We mention that a structured Lyapunov stability problem with a *block-diagonal* structure is briefly mentioned in Khalil and Kokotovic [KK79].

The simultaneous Lyapunov problem has also been investigated, more or less directly in Horisberger and Belanger [HB76], and also in connection with the absolute stability problem (see §4) in, for example, Kamenetskii [Kam83] and Pyatnitskii and Skorodinskii [PS83].

In the next two sections (§3 and §4) we show how the important system theoretic notions of passivity and nonexpansivity are easily characterized in terms of SLS problems. The Kalman Yacubovich Popov lemma establishes the equivalence between these important system theoretic notions and the existence of quadratic Lyapunov functions which establish the stability of the basic feedback system for appropriate classes of Δ's (passive and nonexpansive, respectively).

More importantly, we show that the questions of whether a linear system can be scaled so as to be passive or nonexpansive are also readily cast as SLS problems. These conditions are weaker than passivity or nonexpansivity, and we show that the conditions are related to the existence of a quadratic Lyapunov function establishing the stability of the basic feedback system for appropriate classes of *block diagonal* Δ's.

In §5 we show how the general (multiple nonlinearity, nonzero D) absolute stability problem can be attacked as a SILS problem, extending the results of Kamenetskii [Kam83], Pyatnitskii and Skorodinskii [PS83], and Horisberger and Belanger [HB76].

In §6 we show how the results of the previous sections can be combined to yield SLS problems which can determine the existence of a quadratic Lyapunov function establishing the stability of a complex system containing sector bounded memoryless nonlinearities and nonexpansive Δ_i.

In §7 we discuss numerical methods for determining whether a given A is **S**-SLS for some given structure **S**. We establish that this question can be cast as a nondifferentiable convex programming problem, a fact which has been noted for several special cases by several authors (see §7). We give some basic results for this optimization problem, such as optimality conditions and descriptions of subgradients and descent directions. We describe several algorithms appropriate for these convex programs.

In §8 we present a numerical example which demonstrates some of the results of this paper.

3 Passivity and scaled passivity

Recall that the linear system (1) is *passive* if every solution of (1) with $x(0) = 0$ satisfies

$$\int_0^T u(t)^T y(t) dt \geq 0 \tag{8}$$

for all $T \geq 0$. This implies that A is stable.[2] Passivity is equivalent to the transfer matrix $H(s) = C(sI - A)^{-1}B + D$ being *positive real* (PR) (see e.g.

[2]We remind the reader of the minimality assumption in force, and our use of the term stable.

[DV75]), meaning,

$$H(s) + H(s)^* \geq 0 \quad \text{for} \quad \Re s > 0. \tag{9}$$

Theorem 3.1 *Let* $\mathbf{S} = \mathbb{R}^{n\times n} \oplus \mathbb{R} I_k$*. Then the matrix*

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \tag{10}$$

is $\mathbf{S}$*-stable if and only if the linear system (1) is passive.*

Proof

First suppose that the linear system (1) is passive. By the Kalman Yacubovich Popov (KYP) lemma (see e.g. Anderson [And67, p179]), there is a symmetric positive definite matrix P and matrices L and W such that

$$A^T P + PA = -LL^T \tag{11}$$

$$PB = C^T - LW \tag{12}$$

$$W^T W = D + D^T. \tag{13}$$

The only property of P important for us, and indeed in any application, is that the function $V(x) = x^T P x$ satisfies the inequality (see e.g. [Wil71a, Wil72])

$$\frac{1}{2}\frac{d}{dt} V(x) = u^T y - \frac{1}{2}(L^T x + Wu)^T (L^T x + Wu) \leq u^T y \tag{14}$$

for any solution of (1),[3] or equivalently,

$$x^T P(Ax + Bu) \leq u^T (Cx + Du) \quad \text{for all} \quad x,\ u. \tag{15}$$

We rewrite (15) as

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0 \tag{16}$$

for all x, u. Since the lefthand matrix in (16) is in $\mathbf{S}$, we have shown that the matrix (10) is $\mathbf{S}$-SLS.

[3]Equation (14) has the following simple interpretation: the time rate of increase of the Lyapunov function V does not exceed the power input ($u^T y$) to the system.

Now we prove the converse. Suppose that (10) is **S**-SLS. Without loss of generality we may assume the SLF has the form of the lefthand matrix in (16), so that (16) holds. If we define $V(x) = x^T P x$, then from (16) we may conclude the inequality (15), and hence (14). This implies the system (1) is passive, since (14) implies for $T \geq 0$,

$$\int_0^T u(t)^T y(t) dt \geq -x(0)^T P x(0).$$

■

In the single-input single-output strictly proper case, we can recast the structured Lyapunov stability condition in theorem 3.1 as a simultaneous Lyapunov stability condition.

Corollary 3.2 *The single-input, single-output, strictly proper system $\dot{x} = Ax + bu$, $y = cx$, is passive if and only if the matrices A and $-bc$ are simultaneously Lyapunov stable.*

Proof

First suppose the system is passive. By theorem 3.1, there is a symmetric positive definite matrix P such that

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & b \\ -c & 0 \end{bmatrix} + \begin{bmatrix} A & b \\ -c & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A^T P + PA & Pb - c^T \\ b^T P - c & 0 \end{bmatrix} \leq 0.$$

This implies $Pb = c^T$, $A^T P + PA \leq 0$ (which the reader may recognize as the simple form the KYP lemma takes in this case). P also establishes stability of $-bc$, since

$$(-bc)^T P + P(-bc) = -2c^T c \leq 0$$

and thus $V(x) = x^T P x$ is a SILF for $\{A,\ -bc\}$.

For the converse direction, see the proof of Corollary (3.4). ■

Passivity is an important tool in stability analysis. The *passivity theorem* (e.g. [DV75]) can be used to establish stability of the feedback system (1-2).

It states that if the linear system (1) is passive and $-\Delta$ is a passive operator, meaning for any signal z and $T \geq 0$,

$$\int_0^T z(t)^T(-\Delta(z)(t))dt \geq 0, \tag{17}$$

then the feedback system (1-2) is stable. This conclusion is immediate from (14) and (17), since integration yields

$$V(x(T)) \leq V(x(0)) + \int_0^T u(t)^T y(t)dt \leq V(x(0))$$

and thus x is bounded for $t \geq 0$.

If $-\Delta$ is not only passive but has block structure $[k_1, \ldots, k_m]$, then it can be advantageous to apply a block structure preserving transformation to the system before applying the passivity theorem. Such a transformation does not affect the passivity of the feedback, that is, $-\bar{\Delta}$ is also passive. This results in the following less conservative condition for stability: if there exists an invertible matrix $F \in \bigoplus_{i=1}^m \mathbb{R}I_{k_i}$ such that $FH(s)F^{-1}$ is PR, where H is the transfer matrix of the system (1), then the feedback system (1-2) is stable. This block structure preserving scaled passivity condition is also readily cast as a SLS problem.

Theorem 3.3 *Let* $\mathbf{S} = \mathbb{R}^{n\times n} \oplus \bigoplus_{i=1}^m \mathbb{R}I_{k_i}$. *Then the matrix (10) is* $\mathbf{S}$*-stable if and only if there exists an invertible* $F \in \bigoplus_{i=1}^m \mathbb{R}I_{k_i}$ *such that* $FH(s)F^{-1}$ *is PR. Under this condition, the feedback system (1-2) is stable whenever* $-\Delta$ *is passive and* Δ *has block structure* $[k_1, \ldots, k_m]$.

Proof

First suppose (10) is $\mathbf{S}$-stable. We express the $P \in \mathbf{S}$ which establishes stability of (10) as $P = P_0 \oplus P_1$, where $P_0 \in \mathbb{R}^{n\times n}$ and $P_1 \in \bigoplus_{i=1}^m \mathbb{R}I_{k_i}$. Thus we have

$$0 \geq \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}^T \begin{bmatrix} P_0 & 0 \\ 0 & P_1 \end{bmatrix} + \begin{bmatrix} P_0 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$$

$$= (I \oplus P_1^{1/2}) \left\{ \begin{bmatrix} A & \tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix}^T (P_0 \oplus I) + (P_0 \oplus I) \begin{bmatrix} A & \tilde{B} \\ -\tilde{C} & -\tilde{D} \end{bmatrix} \right\} (I \oplus P_1^{1/2})$$

where

$$\tilde{B} = BP_1^{-1/2}, \tag{18}$$

$$\tilde{C} = P_1^{1/2}C, \tag{19}$$

$$\tilde{D} = P_1^{1/2}DP_1^{-1/2}. \tag{20}$$

It follows from theorem 3.1 that $\tilde{C}(sI - A)^{-1}\tilde{B} + \tilde{D} = P_1^{1/2}HP_1^{-1/2}$ is PR. Thus, there is an invertible (indeed, positive definite) matrix $F = P_1^{1/2} \in \bigoplus_{i=1}^m \mathbb{R}I_{k_i}$ such that $FH(s)F^{-1}$ is PR.

To prove the converse, suppose $F \in \bigoplus_{i=1}^m \mathbb{R}I_{k_i}$, and $FH(s)F^{-1} = (FC)(sI - A)^{-1}(BF^{-1}) + FDF^{-1}$ is PR. By theorem 3.1 there is a symmetric positive definite $P_0 \in \mathbb{R}^{n\times n}$ and positive $p_1 \in \mathbb{R}$ such that $P_0 \oplus p_1 I_k$ establishes stability of

$$\begin{bmatrix} A & BF^{-1} \\ -FC & -FDF^{-1} \end{bmatrix}.$$

By a calculation similar to the one above, it follows that $P = P_0 \oplus p_1 F^2 \in \mathbf{S}$ and establishes stability of the matrix (10). We defer the proof of the last assertion in theorem 3.3. ■

Just as in corollary 3.2, we can recast the structured Lyapunov stability condition appearing in theorem 3.3 as a simultaneous Lyapunov stability condition when the linear system is strictly proper and the structure is diagonal.

Corollary 3.4 *Suppose that $D = 0$, $b_i \neq 0$, and $c_i \neq 0$, $i = 1, \ldots, k$, where b_i (c_i) is the ith column (row) of B (C). Then there exists a diagonal invertible F such that FHF^{-1} is PR if and only if $\{A, -b_1c_1, \ldots, -b_kc_k\}$ is simultaneously Lyapunov stable.*

Proof

First suppose that FHF^{-1} is PR. Let $P_0 \oplus P_1 \in \mathbb{R}^{n\times n} \oplus \bigoplus_{i=1}^k \mathbb{R}$ establish stability of the matrix (10) in theorem 3.3, where $P_0 \in \mathbb{R}^{n\times n}$ and $P_1 \in \bigoplus_{i=1}^k \mathbb{R}$. Then we have

$$\begin{bmatrix} A^TP_0 + P_0A & P_0B - C^TP_1 \\ B^TP_0 - P_1C & 0 \end{bmatrix} \leq 0$$

from which we conclude that $A^T P_0 + P_0 A \leq 0$ and $P_0 B = C^T P_1$. Thus $P_0 b_i = \lambda_i c_i^T$ where $P_1 = \bigoplus_{i=1}^k \lambda_i$. It follows that $V(x) = x^T P_0 x$ is a SILF for $\{A, -b_1 c_1, \ldots, -b_k c_k\}$, since

$$(-b_i c_i)^T P_0 + P_0(-b_i c_i) = -\lambda_i c_i^T c_i \leq 0.$$

Conversely suppose that $x^T P_0 x$ is a SILF for the set $\{A, -b_1 c_1, \ldots, -b_k c_k\}$, so that for each i,

$$(P_0 b) c_i + c_i^T (P_0 b_i)^T \geq 0.$$

Quite generally if $uv^T + vu^T \geq 0$ for two nonzero vectors u and v, then $u = \lambda v$ for some $\lambda > 0$. Thus we conclude that $P_0 b_i = \lambda_i c_i^T$ for some positive constants λ_i (here we use the hypothesis that none of the b_i or c_i are zero). With $F = \bigoplus_{i=1}^k \lambda_i^{1/2}$, we find that FHF^{-1} is PR by reversing the first few steps in the proof of the converse above. ■

We can give a more intuitive statement of theorem 3.3, which moreover provides an interpretation of the SLF of theorem 3.3.

Theorem 3.5 *There exists an invertible $F \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i}$ such that $FH(s)F^{-1}$ is PR if and only if there is a symmetric positive definite $P_0 \in \mathbb{R}^{n \times n}$ and positive constants $\lambda_1, \ldots, \lambda_m$ such that for all solutions of (1), with $V(x) = x^T P_0 x$, we have*

$$\frac{d}{dt} V(x(t)) \leq \sum_{i=1}^m \lambda_i u_i(t)^T y_i(t). \tag{21}$$

Like (14), (21) has a simple and obvious interpretation. We note that this theorem provides an immediate proof of the last assertion of theorem 3.3, since if (21) and (17) hold, integration yields

$$V(x(T)) \leq V(x(0)) + \sum_{i=1}^m \lambda_i \int_0^T u_i(t)^T y_i(t) dt \leq V(x(0)) \tag{22}$$

and hence stability of the feedback system.

We also note that theorem 3.5 shows that the structured Lyapunov condition of theorem 3.3 is very nearly the most general condition under which a quadratic Lyapunov function exists which establishes stability of the feedback system (1-2) for arbitrary Δ with block structure $[k_1, \ldots k_m]$ and $-\Delta$

passive. The gap is simply this: the quadratic Lyapunov function V would still establish stability if it satisfied (22), but with some of the λ_i *zero*, as opposed to positive.

Proof

Suppose $F \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i}$ is invertible and $FH(s)F^{-1}$ is PR. Let the matrix of the SLF of theorem 3.3 be

$$P_0 \oplus \lambda_1 I_{k_1} \oplus \cdots \oplus \lambda_m I_{k_m}. \tag{23}$$

Routine computations establish (21).

Conversely if (21) holds, let us define $F = \bigoplus_{i=1}^m \sqrt{\lambda_i} I_{k_i}$. Then the calculation (22) implies that $FH(s)F^{-1}$ is PR. ■

4 Nonexpansivity and scaled nonexpansivity

We now turn to the important notion of *nonexpansivity.* The linear system (1) is nonexpansive if every solution with $x(0) = 0$ satisfies

$$\int_0^T y(t)^T y(t) dt \leq \int_0^T u(t)^T u(t) dt, \qquad \forall T \geq 0. \tag{24}$$

Nonexpansivity also implies that A is stable. In terms of the transfer matrix $H(s) = C(sI - A)^{-1}B + D$, nonexpansivity is equivalent to

$$\|H\|_\infty = \sup_{\Re s > 0} \sigma_{max}(H(s)) \leq 1 \tag{25}$$

where $\sigma_{max}(\cdot)$ denotes the maximum singular value.

If the linear system (1) is nonexpansive, then the feedback system (1-2) is stable for any nonexpansive Δ, meaning

$$\int_0^T \Delta(z)^T \Delta(z) dt \leq \int_0^T z^T z dt \qquad \forall\ z,\ T \geq 0. \tag{26}$$

A simple proof of this follows from a nonexpansivity form of the KYP lemma which states that the linear system (1) is nonexpansive if and only if there exists a symmetric positive definite $P \in \mathbb{R}^{n \times n}$ such that with $V(x) = x^T P x$, we have

$$\frac{d}{dt} V(x(t)) \leq u(t)^T u(t) - y(t)^T y(t) \tag{27}$$

for any solution of (1).[4] Integration of (27), along with (26) yields $V(x(T)) \leq V(x(0))$ for all $T \geq 0$.

By means of the Cayley transformation, the results on passivity in the previous section can be made to apply to nonexpansivity. If S is a complex $k \times k$ matrix with $\det(I+S) \neq 0$, we define its Cayley transform to be $Z = (I-S)(I+S)^{-1}$. It can be shown that $\sigma_{max}(S) \leq 1$ if and only if $Z + Z^* \geq 0$. Let us now apply the Cayley transform to the transfer matrix $H = C(sI-A)^{-1}B + D$. If $\det(I+D) \neq 0$, the transfer matrix $H = C(sI-A)^{-1}B+D$ satisfies $\|H\|_\infty \leq 1$ if and only if $G = (I-H)(I+H)^{-1}$ is PR. A state space realization of G can be derived: $G = C_c(sI-A_c)^{-1}B_c+D_c$, where

$$\begin{array}{rcl} A_c & = & A - B(I+D)^{-1}C \\ B_c & = & B(I+D)^{-1} \\ C_c & = & -2(I+D)^{-1}C \\ D_c & = & (I-D)(I+D)^{-1}. \end{array} \tag{28}$$

Theorem 4.1 *Let* $\mathbf{S} = \mathbb{R}^{n\times n} \oplus \bigoplus_{i=1}^m \mathbb{R} I_{k_i}$, *and suppose that* $\det(I+D) \neq 0$. *Then the matrix*

$$\begin{bmatrix} A_c & B_c \\ -C_c & -D_c \end{bmatrix} \tag{29}$$

is $\mathbf{S}$*-stable if and only if there exists an invertible* $F \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i}$ *such that* $\|FHF^{-1}\|_\infty \leq 1$. *If this condition holds, then the feedback system (1-2) is stable for any nonexpansive* Δ *with block structure* $[k_1, \ldots, k_m]$.

Proof

The proof follows from theorem 3.3, the observations above, and the facts that $\det(I+D) = \det(I+FDF^{-1})$ and

$$F(I-H)(I+H)^{-1}F^{-1} = (I - FHF^{-1})(I+FHF^{-1})^{-1}.$$

■

Remark

To apply theorem (4.1) when $\det(I+D) = 0$, we simply pick a sign matrix S such that $\det(I+DS) \neq 0$ and apply the theorem to the modified linear system $\{A, BS, C, DS\}$. Let us justify this. The modified linear system

[4] Equation (27) has exactly the same interpretation as (14), if we think of u and y as *scattering variables,* since then $u^Tu - y^Ty$ represents the power input to the system (1).

has transfer matrix HS. Since F above is diagonal, it commutes with any sign matrix, so that $F(HS)F^{-1} = FHF^{-1}S$, and thus $\|F(HS)F^{-1}\|_\infty = \|FHF^{-1}S\|_\infty = \|FHF^{-1}\|_\infty$.

Now let us show that we can always pick a sign matrix S such that $\det(I + DS) \neq 0$. Let $S = s_1 \oplus \cdots \oplus s_k$, where $s_i \in \{-1, 1\}$. By elementary properties of determinants, we have e.g.

$$\det(I + D(1 \oplus s_2 \oplus \cdots \oplus s_k)) + \det(I + D(-1 \oplus s_2 \oplus \cdots \oplus s_k))$$

$$= 2 \det(I + D(0 \oplus s_2 \oplus \cdots \oplus s_k))$$

and thus we have

$$\sum_{s_i \in \{-1,1\}} \det(I + DS) = 2^k.$$

Since the sum of these 2^k numbers is 2^k, at least one of them is nonzero, and that is precisely what we wanted to show.

Block diagonal scaled nonexpansivity can be restated in a 'KYP' form, that is, in terms of the existence of a quadratic Lyapunov function with certain properties:

Theorem 4.2 *There exists an invertible $F \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i}$ such that $\|FHF^{-1}\| \leq 1$ if and only if there is a symmetric positive definite $P \in \mathbb{R}^{n\times n}$ and positive constants $\lambda_1, \ldots, \lambda_m$ such that for all solutions of (1), with $V(x) = x^T P x$, we have*

$$\frac{d}{dt} V(x(t)) \leq \sum_{i=1}^m \lambda_i \left(u_i(t)^T u_i(t) - y_i(t)^T y_i(t) \right). \qquad (30)$$

This can be proved by applying a Cayley transform and theorem 3.5.

Remark

Doyle [Doy82] has studied the feedback system (1-2) for the case when Δ is nonexpansive, has block structure $[k_1, \ldots, k_m]$, and in addition Δ *is a linear time-invariant system.* He shows that necessary and sufficient conditions for stability of the feedback system for all such Δ are that

$$\mu(H(j\omega)) \leq 1 \qquad \forall\, \omega \in \mathbb{R} \qquad (31)$$

where μ denotes the ($[k_1, \ldots, k_m]$–) *structured singular value* of a matrix.

Doyle demonstrates that for any matrix G and any invertible $F \in \bigoplus_{i=1}^{m} \mathbb{R} I_{k_i}$, we have $\mu(G) \leq \sigma_{max}(FGF^{-1})$ (and indeed if the right hand side is minimized over F, the result is thought to be an excellent approximation to $\mu(G)$). Thus the condition $\|FHF^{-1}\|_\infty \leq 1$ appearing in theorem 4.1 immediately implies Doyle's condition (31). Alternatively, we may note that $\|FHF^{-1}\|_\infty \leq 1$ is sufficient to guarantee stability of the feedback system for all (nonlinear, time-varying) nonexpansive Δ with block structure $[k_1, \ldots, k_m]$, and hence in particular for those Δ which are in addition linear and time-invariant. Since Doyle's condition (31) is necessary for stability of the feedback system for all nonexpansive linear time-invariant block structured Δ, it must be implied by $\|FHF^{-1}\|_\infty \leq 1$.

Remark

Theorem 4.1 yields an effective algorithm for computing

$$\bar{\mu}^{-1} = \inf \left\{ \|FHF^{-1}\|_\infty \;\middle|\; F \in \bigoplus_{i=1}^{m} \mathbb{R} I_{k_i}, \;\; \det F \neq 0 \right\}. \tag{32}$$

$\bar{\mu}$ has the following interpretation: the feedback system (1-2) is stable for all Δ with block structure $[k_1, \ldots, k_m]$ and $\mathbf{L}^2$-gain at most $\bar{\mu}$, that is,

$$\int_0^T \Delta(z)^T \Delta(z) dt \leq \bar{\mu}^2 \int_0^T z^T z dt \qquad \forall \, z, \; T \geq 0. \tag{33}$$

Thus $\bar{\mu}$ could be considered an upper bound on a *nonlinear* version of Doyle's structured singular value.

5 The absolute stability problem

We now consider the system (1-2) with Δ diagonal and *memoryless*, meaning

$$u_i(t) = (\Delta_i(y_i))(t) = f_i(y_i(t), t), \tag{34}$$

where the f_i are functions from $\mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}$, in sector $[\alpha_i, \beta_i]$, meaning, for all $a \in \mathbb{R}$ and $t \geq 0$,

$$\alpha_i a^2 \leq a f_i(a, t) \leq \beta_i a^2. \tag{35}$$

The *absolute stability problem* is to find conditions under which all trajectories of the system (1, 34) are bounded for $t \geq 0$, for all f_i satisfying (35).

It is well known that we do not change the absolute stability problem by restricting the f_i to be time-varying linear gains, since the set of trajectories $x(t)$ satisfying (1), (34) for some f_i's satisfying (35) is identical with the set of trajectories satisfying the equations (1) and

$$u_i(t) = k_i(t) y_i(t) \tag{36}$$

for some $k_i(t)$ which satisfy

$$\alpha_i \le k_i(t) \le \beta_i. \tag{37}$$

Since the time-varying linear gains (36) satisfy the sector conditions (35), it is clear that if x satisfies (1) and (36) for some k_i's satisfying (37), then x satisfies (1) and (34) for some f_i satisfying the sector conditions (35). Conversely, suppose x is a trajectory of (1),(34). Then $x(t)$ is also a trajectory of the linear time-varying system (1),(36), where

$$k_i(t) = \begin{cases} \frac{f_i(y_i(t),t)}{y_i(t)} & y_i(t) \neq 0 \\ \alpha_i & y_i(t) = 0 \end{cases} \tag{38}$$

(note that the k_i defined in (38) depends on the particular trajectory $x(t)$). Of course, the k_i defined in (38) satisfy $\alpha_i \le k_i(t) \le \beta_i$.

A Lyapunov method can be used to establish absolute stability of the feedback system. The feedback system is absolutely stable if there is a symmetric positive definite $P \in \mathbb{R}^{n\times n}$ such that for any trajectory $x(t)$ satisfying (1) and (34), or equivalently, (1) and (36), $x(t)^T P x(t)$ is nonincreasing. In this case we say that the (quadratic) Lyapunov function $V(x) = x^T P x$ establishes the absolute stability of the feedback system (1, 34).

In this section we will show that the quadratic Lyapunov function V establishes absolute stability of the feedback system if and only if V is a simultaneous Lyapunov function for the 2^m linear systems resulting when the linear time-varying gains k_i are constant and set to every combination of their extreme values. For the $D = 0$ case, which is considerably simpler, this result appears in Kamenetskii [Kam83] and is implicit in the work of Pyatnitskii and Skorodinskii [PS83]. We note that sufficiency of the simultaneous Lyapunov stability condition for the $D = 0$ case follows from Theorem 1 of Horisberger and Belanger [HB76].

We will prove this main result after examining a more fundamental question.

5.1 Wellposedness

We first consider the question of when the feedback system is *well posed* for any nonlinearities satisfying the sector condition (35). By this we mean simply that equations (34) and

$$y = Cx + Du \tag{39}$$

should determine u as a function of x, for any f_i's satisfying (35). Of course if $D = 0$ the system is well posed, since then $u_i(t) = f_i(c_i x(t), t)$, where c_i is the ith row of C.

In view of the equivalence discussed above, the system will be well posed if and only if equations (36) and (39) determine u as a function of x whenever (37) holds. This is the case if and only if

$$\det \{I - D\,(k_1 \oplus \cdots \oplus k_m)\} \neq 0 \qquad \forall\; k_i \in [\alpha_i, \beta_i]\,. \tag{40}$$

Let $\phi(k_1, \ldots, k_m)$ denote the left hand side of (40).

Theorem 5.1 *Necessary and sufficient conditions for (40) are that the* 2^m *numbers*

$$\phi(k_1, \ldots, k_m), \quad k_i \in \{\alpha_i, \beta_i\}\,, \tag{41}$$

all have the same nonzero sign.

Remark

When the intervals $[\alpha_i, \beta_i]$ are replaced by $(0, \infty)$, the condition (40) is the definition of D being a 'P_0-matrix', and there is a similar necessary and sufficient condition for D to be a 'P_0-matrix' [FP62, FP66].

Proof

It is clear that this condition is necessary, since the image of

$$\mathcal{K} = [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m] \tag{42}$$

under ϕ is connected, and therefore an interval, so if it contains numbers of different signs, it contains zero.

To prove sufficiency we will show that the maximum and minimum of ϕ over $\mathcal{K}$ are achieved at its vertices.[5] Suppose that the maximum is achieved

[5] It is generally *false* that the determinant of a polytope of matrices achieves its maximum or minimum at a vertex.

at $k^* \in \mathcal{K}$ (ϕ is continuous and $\mathcal{K}$ compact). We will find $\tilde{k} \in \mathcal{K}$ which is an extreme point of $\mathcal{K}$ and also achieves the maximum, that is, $\phi(\tilde{k}) = \phi(k^*)$. By elementary properties of determinants,

$$\begin{array}{rcl} \phi(k_1, k_2^*, \ldots, k_m^*) & = & \phi(0, k_2^*, \ldots, k_m^*) \\ & + & k_1 \left[\phi(1, k_2^*, \ldots, k_m^*) - \phi(0, k_2^*, \ldots, k_m^*)\right], \end{array} \tag{43}$$

so ϕ is a polynomial of degree one in k_1. Hence the maximum of (43) over k_1, $\alpha_1 \leq k_1 \leq \beta_1$, must occur at an endpoint unless (43) is in fact independent of k_1. In the first case, k_1^* is extreme (i.e. α_1 or β_1), and we set $\tilde{k}_1 = k_1^*$; in the second case we may set $\tilde{k}_1 = \alpha_1$ without affecting the value of ϕ. We now apply the same argument to k_2, and so on. We have then found a $\tilde{k}$ which achieves the maximum of ϕ on $\mathcal{K}$, and for which each $\tilde{k}_i$ is extreme.

A similar argument establishes that the minimum is achieved at a vertex.

■

5.2 Existence of quadratic Lyapunov function

We suppose now that the wellposedness condition is satisfied, so that $V(x) = x^T P x$, $P = P^T > 0$, establishes absolute stability of the feedback system if and only if every trajectory x of the linear time-varying system

$$\dot{x} = \left(A + BK(t)\,(I - DK(t))^{-1} C\right)\, x \tag{44}$$

has $V(x(t))$ nonincreasing for arbitrary $K(t) \in \bigoplus_{i=1}^{k} [\alpha_i, \beta_i]$. Of course this is equivalent to

$$\left(A + BK(I - DK)^{-1}C\right)^T P + P\left(A + BK(I - DK)^{-1}C\right) \leq 0 \tag{45}$$

for all $K \in \mathcal{K}$.

We note that the set of matrices

$$\mathcal{A} = \left\{A + BK(I - DK)^{-1}C \;\middle|\; K \in \mathcal{K}\right\} \tag{46}$$

is not in general a polytope of matrices, although in fact $\mathcal{A}$ is contained in the convex hull of the images of the vertices of $\mathcal{K}$, that is,

$$\mathcal{A} \subseteq \mathrm{Co}\left\{A + B\tilde{K}(I - D\tilde{K})^{-1}C \;\middle|\; \tilde{K} \text{ a vertex of } \mathcal{K}\right\}. \tag{47}$$

We now state the main result of this section.

Theorem 5.2 *There exists a positive definite quadratic Lyapunov function which establishes absolute stability of the system (1, 34) if and only if the set of* 2^m *matrices*

$$\left\{ A + BK(I - DK)^{-1}C \;\middle|\; K \text{ a vertex of } \mathcal{K} \right\} \tag{48}$$

is simultaneously Lyapunov stable.

Proof

If P satisfies (45) for all $K \in \mathcal{K}$, then in particular it satisfies (45) for K a vertex of $\mathcal{K}$. This means that $V(x) = x^T P x$ is a simultaneous Lyapunov function for the 2^m matrices in (48).

To prove the converse, suppose that $V = x^T P x$ is a simultaneous Lyapunov function for the matrices (48). We must show that (45) holds for all $K \in \mathcal{K}$. This follows from the fact (47) noted above, since if V is a SILF for $A_1, \ldots, A_r$ then V establishes stability of any matrix A in their convex hull. We will give a direct proof instead.

Let $z \in \mathbb{R}^n$ and consider the quadratic form on the left side of (45) evaluated at z, that is,

$$\psi(k_1, \ldots, k_n) = 2z^T P \left(A + BK(I - DK)^{-1}C \right) z.$$

ψ is nonpositive at the vertices of $\mathcal{K}$, and we will show that ψ achieves its maximum at a vertex of $\mathcal{K}$, so that ψ is actually nonpositive for all $K \in \mathcal{K}$. Since this holds for all z, this will show that (45) holds for all $K \in \mathcal{K}$, and hence that V establishes absolute stability of the feedback system (1, 34).

It remains only to show that ψ achieves its maximum on $\mathcal{K}$ at a vertex. Suppose that k^* maximizes ψ over $\mathcal{K}$. As in our proof of the condition for wellposedness, we will show that if k_1^* is not extreme, then in fact ψ does not depend on k_1 at all, and we may then set $k_1^* = \alpha_1$, without affecting the value of ψ. We then apply the same argument to k_2, and so on. Thus we construct a vertex of $\mathcal{K}$ at which the maximum of ψ is achieved.

By elementary properties of determinants, ψ is a linear fractional function of k_1 with denominator ϕ defined in (40):

$$\psi(k_1, k_2^*, \ldots, k_m^*) = \frac{\phi_0(k_2^*, \ldots, k_m^*) + k_1 \phi_1(k_2^*, \ldots, k_m^*)}{\phi(k_1, k_2^*, \ldots, k_m^*)}. \tag{49}$$

(ϕ_0 and ϕ_1 are readily determined, but not important to our argument). By the wellposedness condition, the denominator of (49) does not vanish for $k_1 \in [\alpha_1, \beta_1]$.

Now if we consider a linear fractional function on an interval not containing its pole, then either it achieves its maximum at one of the endpoints only, or else is constant, and hence achieves its maximum at, say, the left endpoint.[6] Thus if k_1^* is not extreme, then in fact ψ does not depend on k_1 at all, and we may set $k_1^* = \alpha_1$ without affecting the value of ψ. ∎

A generalization of this argument may be used to prove (47).

5.3 Brayton-Tong and Safonov results

For the absolute stability problem there are two very nice results available.

Brayton and Tong [BT79, BT80] have derived *necessary and sufficient conditions* for absolute stability: simply, the existence of a *convex* Lyapunov function which (simultaneously) establishes stability of the 2^m matrices (48). They give an effective algorithm for constructing such a Lyapunov function or determining that none exists (in which case the system is not absolutely stable). Note that theorem 5.2 only determines conditions for the existence of a *quadratic* Lyapunov function establishing absolute stability.

Safonov [SW87] has studied a variation on the absolute stability problem: Δ is single-input, single-output, memoryless, time-invariant, and incrementally sector bounded. He has shown that stability of the system for all such Δ can be determined by solving an (infinite dimensional) convex program, and gives a simple algorithm for solving it. Thus for this variation of the absolute stability problem, Safonov has developed an effective algorithm for determining absolute stability.

6 Comparison and hybrid results

Let us compare theorem 4.1, which pertains to the feedback system with Δ diagonal and nonexpansive, with theorem 5.2 with the sector conditions $\alpha_i = -1$, $\beta_i = 1$, which pertains to the feedback system with Δ diagonal, nonexpansive, and memoryless. As mentioned above, theorem 4.1 essentially

[6]The derivative of such a function is either never zero or always zero.

determines the conditions under which a quadratic Lyapunov function establishes stability of the feedback system for all diagonal nonexpansive Δ, whereas theorem 5.2 determines the conditions under which a quadratic Lyapunov function establishes stability of the feedback system for all diagonal nonexpansive *memoryless* Δ, a weaker condition (on A, B, C, D). Doyle's condition (31) is also weaker (as a condition on A, B, C, D) than that of theorem 3.3; it determines the precise conditions under which the feedback system is stable for all diagonal nonexpansive *linear time-invariant* Δ. Doyle's condition (31) and the absolute stability condition of theorem 5.2 are not comparable, that is, neither is a weaker condition on A, B, C, D.

In terms of Lyapunov functions, the difference between theorem 4.1 and theorem 5.2 with sector $[-1, 1]$ nonlinearities can be stated as follows. These theorems determine conditions under which there exists a symmetric positive definite $P \in \mathbb{R}^{n\times n}$ such that $V(x) = x^T P x$ satisfies:
(theorem 4.1; Δ diagonal and nonexpansive):

$$\frac{d}{dt}\, V(x(t)) \leq \sum \lambda_i (u_i(t)^2 - y_i(t)^2), \qquad \lambda_i > 0, \tag{50}$$

(theorem 5.2; Δ diagonal, nonexpansive, and memoryless):

$$\frac{d}{dt}\, V(x(t)) \leq 0 \qquad \text{whenever } \ |u_i| \leq |y_i|. \tag{51}$$

It is clear that (50) implies (51).

This last observation suggests that the two theorems can be combined. Consider the case where Δ is nonexpansive with block structure $[k_1, \ldots, k_m]$, with $k_1 = \cdots = k_s = 1$ and $\Delta_1, \ldots, \Delta_s$ memoryless (figure 4).

We will assume that this system is well posed, meaning that the absolute stability problem resulting by considering only the memoryless operators $\Delta_1, \ldots, \Delta_s$ is well posed.

A quadratic Lyapunov function $V = x^T P x$ would establish stability of the feedback system (1-2) for all such Δ if there are positive $\lambda_{s+1}, \ldots, \lambda_m$ such that

$$\frac{d}{dt} V(x(t)) \leq \sum_{i=s+1}^{m} \lambda_i (u_i^T u_i - y_i^T y_i) \quad \text{whenever } \ |u_i| \leq |y_i|,\ i = 1, \ldots, s. \tag{52}$$

Note that this combines the conditions (50) and (51).

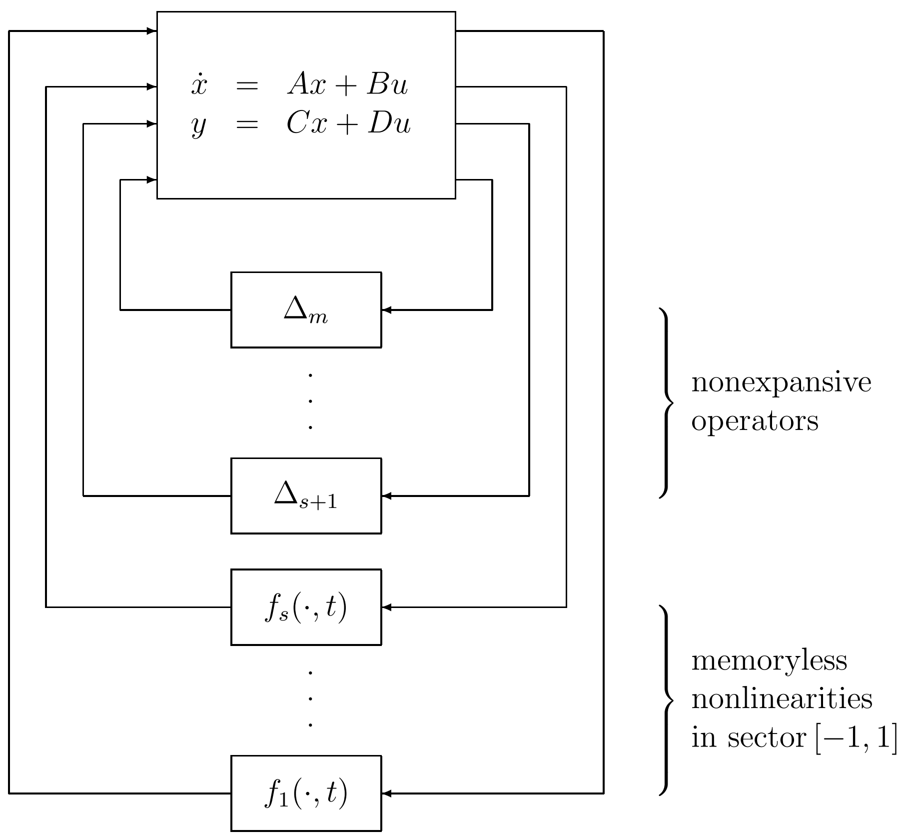

Figure 4: Basic feedback system with block structured nonexpansive Δ, with $\Delta_1, \ldots, \Delta_s$ memoryless.

We will see that the condition (52) can also be cast as a structured Lyapunov stability question. As in the absolute stability problem, we may consider the case where the memoryless nonlinearities are linear time-varying gains, that is, $u_i = k_i(t)y_i$, $i = 1, \ldots, s$. Let us eliminate $u_1, \ldots, u_s$ from (1) to yield:

$$\dot{x} = A^{(k)}x + B^{(k)} \begin{bmatrix} u_{s+1} \\ \vdots \\ u_m \end{bmatrix} \tag{53}$$

$$\begin{bmatrix} y_{s+1} \\ \vdots \\ y_m \end{bmatrix} = C^{(k)}x + D^{(k)} \begin{bmatrix} u_{s+1} \\ \vdots \\ u_m \end{bmatrix} \tag{54}$$

We will spare the reader the formulas for $A^{(k)}, B^{(k)}, C^{(k)}, D^{(k)}$, only noting that they are linear fractional in the k_i's. Let $A_c^{(k)}, B_c^{(k)}, C_c^{(k)}, D_c^{(k)}$ denote the state space Cayley transform of the system (53-54) (formulas (28); we assume that $\det(I + D^{(k)}) \neq 0$). We can now state our result.

Theorem 6.1 *There exists a symmetric positive definite $P \in \mathbb{R}^{n\times n}$ and positive $\lambda_{s+1}, \ldots, \lambda_m$ such that for all solutions of (1), with $V(x) = x^T P x$, (52) holds,* if and only if *there is a symmetric positive definite matrix in $\mathbb{R}^{n\times n} \oplus \bigoplus_{i=s+1}^{m} \mathbb{R} I_{k_i}$ which establishes stability of the 2^s matrices*

$$\begin{bmatrix} A_c^{(k)} & B_c^{(k)} \\ -C_c^{(k)} & -D_c^{(k)} \end{bmatrix} \tag{55}$$

where the linear gains $k_i(t)$ are constant and set to their extreme values, ± 1. In this case, the feedback system (1-2) is stable for all nonexpansive block structured Δ with $\Delta_1, \ldots, \Delta_s$ memoryless.

The proof simply combines several of the arguments used above, and is left to the reader.

7 Numerical methods for the SLS problem

In this section we consider the problem of actually determining whether A is **S**-SLS, given A and **S**. Our first observation is that A is **S**-SLS if and only if the set

$$\mathcal{P} = \left\{ P \,\middle|\, P = P^T > 0,\ P \in \mathbf{S},\ A^T P + PA \le 0 \right\} \tag{56}$$

is nonempty. Since $\mathcal{P}$ is convex, we see that the question of whether A is $\mathbf{S}$-SLS is really a *convex feasibility program,* that is, a (nondifferentiable) convex optimization problem. Similar observations can be found in Horisberger and Belanger [HB76], Khalil [Kha82], Kamenetskii and Pyatnitskii [KP87], and Pyatnitskii and Skorodinskii [PS83]. Of course this means that there are effective algorithms for determining whether a given A is $\mathbf{S}$-SLS.

Although it is possible to use numerical methods to determine whether $\mathcal{P}$ is empty, there are several reasons why in practice we should prefer to check the slightly stronger condition that $\mathcal{P}$ have nonempty interior. In terms of A, this stronger condition, which we will call *strict* SLS or SSLS, is the existence of a symmetric positive definite $P \in \mathbf{S}$ such that $A^T P + PA < 0$ (note the strict inequality here).

First, small perturbations in A (e.g. roundoff error) do not destroy the SSLS property; the same cannot be said of the SLS property. In other words, the set of A which are $\mathbf{S}$-SSLS for some $\mathbf{S}$ is always open, whereas the set of A which are $\mathbf{S}$-SLS need not be. Second, when the strict SLS property is determined, it allows us to conclude *asymptotic stability* ($x(t) \to 0$) of the system under study, as opposed to mere stability ($x(t)$ bounded). In the remainder of this section we will consider numerical methods for determining whether A is SSLS.

Let $Z_1, \ldots, Z_r$ be a basis for the subspace $\left\{ Z \,\middle|\, Z = Z^T \in \mathbf{S} \right\}$. Then A is SSLS if and only if there exist $a_1, \ldots, a_r$ such that $Q = \sum_{i=1}^r a_i Q_i < 0$, where $Q_i = -Z_i \oplus \left(A^T Z_i + Z_i A \right)$. Let us define

$$\Phi(a) = \Phi(a_1, \ldots, a_r) = \lambda_{max} \left(\sum_{i=1}^r a_i Q_i \right).$$

Since Φ is positive homogeneous ($\Phi(\alpha a) = \alpha \Phi(a)$ for all positive α), and thus A is SSLS if and only if $\Phi^* < 0$, where

$$\Phi^* = \min_{|a_i| \le 1} \Phi(a). \tag{57}$$

If A is SSLS, then from the positive homogeneity of Φ we conclude that the optimum a^* always occurs on a boundary of the constraint set, that is, there is at least one i with $|a_i^*| = 1$. Note that A is not SSLS if and only if $a = 0$ is optimal for (57).

Before turning to numerical algorithms appropriate for the convex program (57), let us comment on the significance of the bounds $|a_i| \leq 1$. Suppose the matrices Z_i and A are scaled so that their largest elements are on the order of one, $\Phi^* = \Phi(a^*) < 0$, and $P = \sum_{i=1}^r a_i^* Z_i$. Then not only is $P \oplus -A^T P - PA$ positive definite, but its condition number is at most on the order of $1/|\Phi^*|$. Thus if we test whether the minimum value Φ^* of (57) is less than, say, -10^{-4}, we are really testing whether there is a P such that $P \oplus -A^T P - PA$ is positive definite and has condition number under approximately 10^4.

7.1 Descent directions, subgradients, and optimality conditions

A vector $\delta a \in \mathbb{R}^n$ is said to be a *descent direction* for Φ at a if for small positive h, $\Phi(a + h\delta a) < \Phi(a)$. Note that the existence of a descent direction at $a = 0$ is equivalent to A being SSLS; it will be useful for us to also consider descent directions at other, nonzero a. The conditions for δa to be a descent direction at a are readily determined from perturbation theory for symmetric matrices [Kat84]. Let $\Phi(a) = \lambda$, and let t denote the multiplicity of the eigenvalue λ of $\sum_{i=1}^r a_i Q_i$. Let the columns of $U \in \mathbb{R}^{n \times t}$ be an orthonormal basis for the nullspace of

$$\lambda I - \sum_{i=1}^r a_i Q_i. \tag{58}$$

Then δa is a descent direction if and only if

$$\delta a_1 U^T Q_1 U + \cdots + \delta a_r U^T Q_r U = G < 0 \tag{59}$$

and in fact

$$\lim_{h \searrow 0} \frac{\Phi(a + h\delta a) - \Phi(a)}{h} = \lambda_{max}(G).$$

Thus if the eigenvalue λ has multiplicity one, so that U is a single column, one choice of descent direction is $\delta a_i = -U^T Q_i U$. Indeed this is precisely the condition (i.e. $t = 1$) under which Φ is differentiable at a, and this δa is simply $-\nabla \Phi(a)$.

Whenever $t > 1$, (e.g. when $a = 0$, we have $t = n$) determining a descent direction (or that none exists) is much harder. One general method uses the notion of the *subgradient* $\partial\Phi(a)$ of a convex function Φ at $a \in \mathbb{R}^r$, defined as [Roc72, Cla83]

$$\partial\Phi(a) = \left\{ g \in \mathbb{R}^r \,\middle|\, \Phi(\tilde{a}) - \Phi(a) \geq g^T(\tilde{a} - a),\ \forall\, \tilde{a} \in \mathbb{R}^r \right\}. \tag{60}$$

$\partial\Phi(a)$ can be shown to be nonempty, compact, and convex, and moreover δa is a descent direction at a if and only

$$\delta a^T g < 0 \quad \forall\, g \in \partial\Phi(a), \tag{61}$$

so that descent directions correspond precisely to hyperplanes through the origin with the subgradient in the negative half-space. Thus we have the standard conclusion that there exists a descent direction at a if and only if $0 \notin \partial\Phi(a)$, and indeed in this case we may take as 'explicit' descent direction the negative of the element of $\partial\Phi(a)$ of least norm. In particular we have: A is SSLS if and only if $0 \notin \partial\Phi(0)$.

Polak and Wardi [PW82] have shown that for our particular Φ,

$$\partial\Phi(a) = \mathrm{Co} \left\{ g \in \mathbb{R}^r \,\middle|\, g_i = z^T U^T Q_i U z,\ \ z \in \mathbb{R}^t,\ z^T z = 1 \right\}. \tag{62}$$

In particular if u is *any* unit eigenvector of (58) corresponding to the maximum eigenvalue $\Phi(a)$ of the matrix (58), then $g_i = u^T Q_i u$ yields $g \in \partial\Phi(a)$. So it is very easy to find elements of the set $\partial\Phi(a)$.

From Polak and Wardi's characterization of the subgradient we can readily derive conditions for $a = 0$ to be optimal. These conditions can be found in Overton [Ove87, OW87], but with a completely different proof.

Theorem 7.1 *A is not SSLS, or equivalently,* 0 *is a global minimizer of* Φ*,* if and only if *there is a nonzero* $R = R^T \geq 0$ *such that* $\mathrm{Tr} Q_i R = 0$*,* $i = 1, \ldots, r$*.*

Proof
First suppose that A is not SSLS, so that that $0 \in \partial\Phi(0)$. By Polak and Wardi's characterization of $\partial\Phi(0)$, there are $\lambda_1, \ldots, \lambda_d$, with $\lambda_j > 0$, $\sum_{j=1}^d \lambda_j = 1$, and unit vectors $z_1, \ldots, z_d$, such that

$$\sum_{j=1}^{d} \lambda_j z_j^T Q_i z_j = 0, \qquad i = 1, \ldots, r.$$

We rewrite this as $\mathrm{Tr} Q_i R = 0$, $i = 1, \ldots, r$, where we define $R = \sum_{j=1}^d \lambda_j z_j z_j^T$. Of course $R = R^T \geq 0$; R is nonzero since $\mathrm{Tr} R = 1$. This establishes one direction of theorem (7.1).

To prove the converse, suppose that A is SSLS, say, $Q = \sum a_i^* Q_i < 0$. We must show that there is no nonzero R such that $R = R^T \geq 0$ and $\mathrm{Tr} Q_i R = 0$, $i = 1, \ldots, r$. Suppose that $R = R^T \geq 0$ and $\mathrm{Tr} Q_i R = 0$, $i = 1, \ldots, r$; we will show that $R = 0$. $Q < 0$, so we may write it as $Q = -GG^T$ for some nonsingular G. Thus

$$0 = \mathrm{Tr}\left(\sum_{i=1}^{r} a_i^* Q_i\right) R = \mathrm{Tr} Q R = -\mathrm{Tr} G^T R G.$$

Since $G^T R G$ is positive semidefinite and has trace zero, it must be the zero matrix, and thus $R = 0$ and we are done. ■

7.2 Cutting-Plane method

The algorithm we have found most effective for solving (57) is Kelley's cutting-plane algorithm [Kel60]. The algorithm requires only the ability to evaluate the function (i.e. compute $\Phi(a)$) and find an element in the subgradient at a point (i.e. compute a $g \in \partial\Phi(a)$), which we have already explained how to do. Suppose that $a^{(1)}, \ldots, a^{(s)}$ are the first s iterates with $g^{(i)} \in \partial\Phi(a^{(i)})$. Then from the definition of subgradient we have

$$\Phi(z) \geq \max_{i=1,\ldots,s} \Phi(a^{(i)}) + g^{(i)T}(z - a^{(i)})$$

for all z and thus

$$\Phi^* \geq = \Phi_{LB}^{(s)} = \min_{|z_i| \leq 1} \max_{i=1,\ldots,s} \Phi(a^{(i)}) + g^{(i)T}(z - a^{(i)}). \qquad (63)$$

The right hand side of (63) is readily solved via linear programming, and we take $a^{(s+1)}$ to be the argument which minimizes the right hand side of (63), that is, $a^{(s+1)}$ is chosen such that

$$\Phi_{LB}^{(s)} = \max_{i=1,\ldots,s} \Phi(a^{(i)}) + g^{(i)T}(a^{(s+1)} - a^{(i)}).$$

Of course $\Phi_{LB}^{(s)}$ is a lower bound for Φ^*, which is extremely useful in devising stopping criteria—for example, we may stop when $\Phi_{LB}^{(s)}$ exceeds some threshold, say, -10^{-4}, or when the difference $\Phi(a^{(s)}) - \Phi_{LB}^{(s)}$ is smaller than some specified tolerance.

Although the number of constraints in the linear program which must be solved at each iteration grows with iteration number, if these linear programs are initialized at the last iterate it usually takes only a very few iterations to converge.

The great advantage of the cutting-plane algorithm is that at all times a lower bound $\Phi_{LB}^{(s)}$ and upper bound $\Phi_{UB}^{(s)}$ $(= \min_{i \leq s} \Phi(a^{(i)}))$ on Φ^* are maintained. Of course it is readily shown that $\Phi_{UB}^{(s)} - \Phi_{LB}^{(s)} \to 0$ as $s \to \infty$, so the cutting-plane algorithm is therefore completely effective—it cannot fail to unambiguously determine in a finite number of steps whether or not $\Phi^* < -\epsilon$.[7] The disadvantage is that the computation per iteration can be prohibitive for very large systems. In the next two subsections we describe two other algorithms for (57) which involve less computation per iteration, and thus may be appropriate for large systems.

7.3 Subgradient methods

Shor [Sho85] has introduced a method for solving nondifferentiable convex programs such as (57), the *subgradient algorithm.* In appearance it is quite similar to a descent method for a differentiable convex function. Shor's algorithm generates $a^{(s+1)}$ as

$$a^{(s+1)} = a^{(s)} + h_s \delta a^{(s)} \tag{64}$$

where $\delta a^{(s)}$ is the *direction* and h_s the *step-size* of the sth iteration. In a descent method, $\delta a^{(s)}$ would be a descent direction for Φ at $a^{(s)}$, and then h_s

[7] As mentioned above, $1/\epsilon$ can be interpreted as a maximum allowable condition number for $P \oplus -A^T P - PA$.

might be chosen to minimize or approximately minimize $\Phi(a^{(s)} + h_s \delta a^{(s)})$. In Shor's subgradient methods, the direction $\delta a^{(s)}$ is allowed to be the negative of *any* element of the subgradient $\partial\Phi(a^{(s)})$, and usually the step size h_s depends only on the iteration number s. One possible choice is:

$$-\delta a^{(s)} \in \partial\Phi(a^{(s)}), \qquad h_s = \frac{\alpha}{s\|\delta a^{(s)}\|} \tag{65}$$

where α is the largest number under one which ensures $|a_i^{(s+1)}| \leq 1$.

Thus the subgradient method requires at each iteration the computation of *any* element of the subgradient, as opposed to a descent direction. As we have already noted, finding an element of the subgradient $\partial\Phi(a)$ is straightforward, essentially involving the computation of the largest eigenvalue of the symmetric matrix Q and a vector in its associated eigenspace. This computation can be very efficiently done, even for large systems [Par80].

If the subgradient $\partial\Phi(a^{(s)})$ subtends an angle exceeding $\pi/2$ from the origin, then it is possible that $\delta a^{(s)}$ is *not* a descent direction, and indeed it (often) occurs that $\Phi(a^{(s+1)}) > \Phi(a^{(s)})$. Nevertheless it can be proved that the algorithm (64-65) has guaranteed global convergence, that is,

$$\lim_{s\to\infty} \Phi(a^{(s)}) = \Phi^*. \tag{66}$$

Thus if A is SSLS, so that $\Phi^* < 0$, then the subgradient algorithm (64-65) will find an a with $\Phi(a) < 0$ in a finite number of iterations. These assertions follow immediately from the results in Shor [Sho85] or Demyanov and Vasilev [DV85].

This algorithm involves much less computation per iteration than the cutting-plane method described above, especially for large systems. It has two disadvantages: First, if A is SSLS, it may take a large number of subgradient iterations to produce a SLF. Second, and more important, if A is *not* SSLS, there is no good method to know when to stop—no good lower bounds on Φ^* are available. In other words, the subgradient method cannot unambiguously determine that A is not SSLS—it will simply fail to produce a SLF in a large number of iterations. Even if it appears that the a's are converging to zero, as they must if A is not SSLS, there is no way to be certain of this after only a finite number of iterations.

7.4 Kamenetskii-Pyatnitskii saddle point method

Kamenetskii and Pyatnitskii [KP87] have developed an algorithm for (57) which involves even less computation per iteration than the subgradient method, and thus may be useful for very large systems. Kamenetskii and Pyatnitskii consider the function

$$F(a,x) = x^T(\sum_{i=1}^{r} a_i Q_i)x. \tag{67}$$

Recall that $\tilde{a}$, $\tilde{x}$ is said to be a *saddle point* of F [AHU58, Roc72] if

$$F(\tilde{a},x) \leq F(\tilde{a},\tilde{x}) \leq F(a,\tilde{x}) \qquad \forall\ a,\ x.$$

It is easy to see that $\tilde{a}$, $\tilde{x}$ is a saddle point of F if and only if $\Phi(a) \leq 0$ and $Q_i\tilde{x} = 0$, $i = 0,\ldots,r$, which we assume without loss of generality occurs only if $\tilde{x} = 0$ (otherwise A is clearly not SSLS). This is theorem 2 of [KP87].

The Kamenetskii-Pyatnitskii algorithm is just the gradient method for finding saddle points of functions, most simply expressed as a differential equation for a and x:

$$\begin{array}{rcrcl} \dot{x} & = & \partial F/\partial x & = & 2\sum_{i=1}^{r} a_i Q_i x, \\ \dot{a}_i & = & -\partial F/\partial a_i & = & -x^T Q_i x. \end{array} \tag{68}$$

It can be shown that if F were strictly concave in x for each a (which is not true for our F (67)) and convex in a for each x, then all solutions of the differential equation (68) would converge to saddle points of F [AHU58]. Despite the fact that (67) is not concave in x for each a, Kamenetskii and Pyatnitskii prove the remarkable fact that if A is SSLS, then for arbitrary initial conditions the solutions of DE (68) converge to saddle points of F as $t \to \infty$. Thus $x \to 0$ and $a \to \tilde{a}$, where $\Phi(\tilde{a}) \leq 0$. They show moreover that for almost all initial conditions (zero is one of the exceptions), $\Phi(\tilde{a}) < 0$. Thus if A is SSLS, then the gradient method will find a SLF (for almost all initial x and a).

Of course in practice a suitable discretization of the differential equation (68) is solved (see [KP87]).

Compared to the cutting-plane or subgradient method, this algorithm is extremely simple, requiring no maximum eigenvalue/eigenvector computations. On the other hand, we have found extremely slow convergence of a to $\tilde{a}$, and of course this method shares with the subgradient method the disadvantage of not being able to establish that A is *not* SSLS.

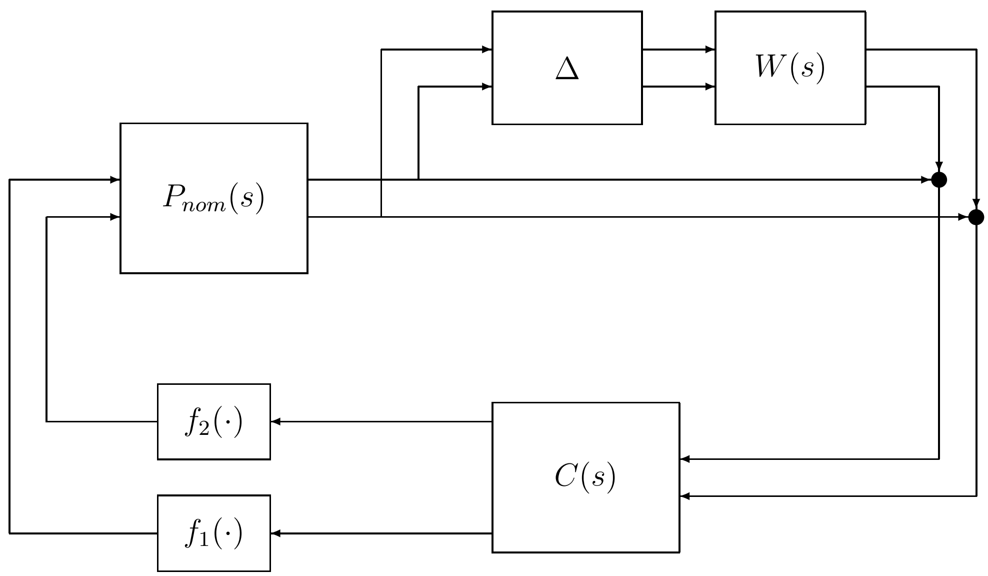

Figure 5: A simple control system. Δ is a nonexpansive operator representing unknown nonlinear dynamic plant perturbation; the memoryless nonlinearities $f_1(\cdot)$ and $f_2(\cdot)$ represent actuator nonlinearity.

8 An example

In this section we demonstrate some of the results of this paper on the simple two-input two-output control system shown in figure 5. The plant is modeled as a nominal LTI plant with transfer matrix

$$P_{nom}(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 2 \\ -.2 & 1 \end{bmatrix}$$

and an *output 'multiplicative' perturbation* which consists of a nonexpansive but unknown two-input, two-output, (possibly nonlinear) operator Δ

followed by a LTI weighting filter with transfer matrix

$$W(s) = 1.5\,\frac{s+1}{s+10}\,I_2.$$

Very roughly speaking, this means that our nominal plant is moderately accurate (about 15%) at low frequencies (say, $\omega < 0.5$), less accurate in the range $0.5 < \omega < 3$, and quite inaccurate for $\omega > 3$.

The two memoryless nonlinearities f_1 and f_2 represent actuator nonlinearities, and are assumed to be in sector $[0.7, 1.3]$.

The controller is a simple proportional plus integral (PI) controller with transfer matrix

$$C(s) = -\left(K_P + \frac{K_I}{s}\right)\left[\begin{array}{cc} .7 & -1.4 \\ 0 & .7 \end{array}\right].$$

We set $K_P = 2\alpha - 1$ and $K_I = 2\alpha^2$, which with the nominal plant without actuator nonlinearity would yield closed loop eigenvalues at approximately $-\alpha \pm j\alpha$. Thus the parameter α approximately determines the closed-loop system bandwidth.

The system can be put into the hybrid system form of figure 4 and thus the question of whether there exists a quadratic Lyapunov function which establishes stability of this control system can be formulated as a SLS problem (see theorem 6.1). For this example, the appropriate structure is $\mathbf{S} = \mathbb{R}^{6\times 6} \oplus \mathbb{R} I_2$. When formulated as a convex optimization problem as described in section 7, we find $r = 22$ (i.e. there are 22 variables in the optimization problem), and the matrices Q_i are 40×40, in fact block diagonal, with five 8×8 blocks. Thus to generate a subgradient involves roughly five 8×8 symmetric matrix eigenvalue computations.

Taking $\epsilon = 10^{-4}$ (see §7), which we remind the reader corresponds to an approximate limit of 10^4 on the condition number of acceptable $P \oplus -A^T P - PA$, we find that when $.5 \le \alpha < 1.2$, A is $\mathbf{S}$-SSLS, and when $\alpha > 1.2$, A is not $\mathbf{S}$-SSLS. Thus for $\alpha < 1.2$, we can find a quadratic Lyapunov function which establishes (asymptotic) stability of the control system, and for $\alpha \ge 1.2$, there exists no quadratic Lyapunov function establishing stability of our control system.

To give some idea of the performance of the algorithms discussed in §7 we will consider two cases: $\alpha = 1$ (A is $\mathbf{S}$-SSLS in this case), and $\alpha = 2$ (A is not $\mathbf{S}$-SSLS in this case).

For $\alpha = 1$, the cutting-plane method takes 109 iterations to find out that the system is **S**-SSLS ($\Phi_{UB} < -\epsilon$), and another 133 iterations to determine that $\Phi^* = -.0031$ within at most 10% (stopping criterion: $\Phi_{UB} - \Phi_{LB} < .1|\Phi_{LB}|$). The condition number of the corresponding $P \oplus -A^T P - PA$ is 382.

For $\alpha = 1$, the subgradient method takes 5507 iterations to determine that the system is **S**-SSLS ($\Phi_{UB} < 0$).

When $\alpha = 2$, the cutting-plane method takes 113 iterations to determine that A is not **S**-SSLS ($\Phi_{LB} > -\epsilon$). Of course the subgradient method simply fails to find a SLF for A; after a very large number of iterations we may suspect that A is not **S**-SSLS, but we cannot be sure.

9 Conclusion

We have introduced the simple notion of structured Lyapunov stability, and shown how several important system theoretic problems involving block diagonally scaled passivity and nonexpansivity can be recast as SLS problems. We have shown (theorem 6.1) how it can be used to determine conditions which guarantee stability of the feedback system (1-2) for all Δ of a specified class, for example of a certain block structure, with some blocks memoryless with sector constraints and other blocks nonexpansive (but possibly nonlinear and dynamic).

A very important fact is that the structured Lyapunov stability problem is equivalent to a convex optimization problem, and can therefore be effectively solved, for example using the methods described in §7.

10 Acknowledgment

The authors would like to thank Craig Barratt, who caught many errors and made several suggestions for clarification.

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