EXISTENCE AND UNIQUENESS OF OPTIMAL MATRIX SCALINGS*

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Abstract. The problem of finding a diagonal similarity scaling to minimize the scaled singular value of a matrix arises frequently in robustness analysis of control systems. We show that the set of optimal diagonal scalings is nonempty and bounded if and only if the matrix that is being scaled is irreducible. For an irreducible matrix, we derive a sufficient condition for the uniqueness of the optimal scaling.

Key words. Diagonal similarity scalings, scaled singular value minimization, irreducible matrices.

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Notation. $\mathbf{R}$ ($\mathbf{C}$) denotes the set of real (complex) numbers. $\mathbf{R}_+$ stands for the set of positive real numbers. For $z \in \mathbf{C}$, $\mathrm{Re}\, z$ is the real part of z. The set of $m \times n$ matrices with real (complex) entries is denoted $\mathbf{R}^{m\times n}$ ($\mathbf{C}^{m\times n}$). I stands for the identity matrix, with size determined from context. For a matrix $P \in \mathbf{C}^{m\times n}$, P^T stands for the transpose and P^* stands for the complex conjugate of P^T. $\|P\|$ is the spectral norm (maximum singular value) of P, given by the square root of the maximum eigenvalue of P^*P. (For a vector $v \in \mathbf{C}^n$, $\|v\|$ is just the Euclidean norm.) For $P \in \mathbf{C}^{n\times n}$, $\mathrm{Tr}\, P$ stands for the trace, that is, the sum of the diagonal entries of P.

1. Introduction. Given a complex matrix $M \in \mathbf{C}^{n\times n}$ and a nonsingular diagonal matrix $D \in \mathbf{C}^{n\times n}$, the *similarity-scaled singular value* of M corresponding to scaling D

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is defined as

$$f(M, D) = \left\| DMD^{-1} \right\|.$$

The optimal diagonal scaling problem is to minimize $f(M, D)$ over all diagonal nonsingular matrices D:

$$f_{\min}(M) = \inf \left\{ \left\| DMD^{-1} \right\| \;\middle|\; D \in \mathbf{C}^{n\times n},\ D \text{ is diagonal and nonsingular} \right\}. \tag{1}$$

We will refer to $f_{\min}(M)$ as the *optimally scaled singular value* of M.

Problem (1) arises in the robustness analysis of control systems with structured uncertainties. For further details, see references [6, 7, 2].

Reformulation as a convex optimization problem. We note that $f(M, |D|) = f(M, D)$; we also observe that $f(M, D)$ is homogeneous of degree zero in D, that is, $f(M, \alpha D) = f(M, D)$ for all nonzero $\alpha \in \mathbf{C}$. Therefore, we may rewrite (1) as

$$f_{\min}(M) = \inf \left\{ \left\| e^{D} M e^{-D} \right\| \;\middle|\; D \in \mathbf{R}^{n\times n},\ D \text{ is diagonal}, \mathrm{Tr}\ D = 0 \right\}. \tag{2}$$

The reason for rewriting (1) as (2) is that $\left\| e^{D} M e^{-D} \right\|$ is a *convex* function of D — this fact will prove important in the sequel — while $\| DMD^{-1} \|$ is not [8, 9]. For convenience, we let

$$\mathcal{D} = \left\{ D \;\middle|\; D \in \mathbf{R}^{n\times n},\ D \text{ is diagonal}, \mathrm{Tr}\ D = 0 \right\}.$$

In this paper, we will not concern ourselves with the solution of (2). We will instead investigate the set of *minimizers* for (2), that is, the set of *optimal* scalings $\mathcal{D}_{\mathrm{opt}}$ defined by

$$\mathcal{D}_{\mathrm{opt}} \triangleq \left\{ D \;\middle|\; D \in \mathcal{D}, \left\| e^{D} M e^{-D} \right\| = f_{\min}(M) \right\}. \tag{3}$$

In the process, we will provide a sufficient condition for $\mathcal{D}_{\mathrm{opt}}$ to be nonempty (which means the infimum in (2) is *achieved*) and a sufficient condition for $\mathcal{D}_{\mathrm{opt}}$ to be a singleton (which means that there is a *unique* optimal scaling).

2. Boundedness of $\mathcal{D}_{\text{opt}}$. We start with a few definitions:

DEFINITION. A *permutation* matrix P is a real, orthogonal $n \times n$ matrix (*i.e.*, $PP^T = P^TP = I$) with entries that are either one or zero. We will let $\mathcal{P}$ denote the set of $n \times n$ permutation matrices.

DEFINITION. A complex matrix M is said to be *reducible* if there exists some $P \in \mathcal{P}$ such that PMP^T is block upper-triangular, that is,

$$PMP^T = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix},$$

where M_{11}, M_{22} are square matrices of appropriate sizes [3, 1]. A matrix that is not reducible is termed *irreducible*.

Remark. For any permutation matrix P,

$$\left\| e^D M e^{-D} \right\| = \left\| Pe^D P^T PMP^T Pe^{-D} P^T \right\|.$$

Note that Pe^DP^T is diagonal, and corresponds to just a reordering of the diagonal entries of e^D. Therefore, as far as the scaling problem is concerned, if a matrix M is reducible, we may assume without loss of generality that

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix},$$

bearing in mind that a reordering of the entries of the scaling D might be necessary. In the sequel, the phrase "within a permutation" will refer to such a reordering of the entries of D and the corresponding permutation similarity transformation on M.

Let $\mathcal{D}_\gamma$ denote the *sub-level* set

$$\left\{ D \in \mathcal{D} \;\middle|\; \left\| e^D M e^{-D} \right\| < \gamma \right\}.$$

The following theorem relates the irreducibility of M to the boundedness of the sub-level sets.

THEOREM 2.1. *For any $\gamma > f_{\min}(M)$, the sublevel set $\mathcal{D}_\gamma$ is bounded if and only if M is irreducible.*

Proof. We first note

LEMMA 2.2.

$$\left\| \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \right\|,$$

where M_{11}, M_{12} and M_{22} are matrices of appropriate sizes. (The proof is left to the reader.)

We first assume that M is reducible. Then to within a permutation,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}.$$

Then given any $\gamma > f_{\min}(M)$ and $D \in \mathcal{D}_\gamma$ (note that $\mathcal{D}_\gamma$ is nonempty), partition D conformally with the block upper-triangular structure of M above as

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}.$$

Consider now a sequence of scaling matrices $D^{(i)}$ of the form

$$D^{(i)} = \begin{bmatrix} D_1 - i\,(n-r) & 0 \\ 0 & D_2 + i\,r \end{bmatrix}, \; i = 1, 2, \ldots$$

where D_1 is of size $r \times r$ (note that $D^{(i)} \in \mathcal{D}$ for $i = 1, 2, \ldots$).

For such a sequence of scalings, $\left\| e^{D^{(i)}} M e^{-D^{(i)}} \right\|$ converges to

$$\max\left(\left\| e^{D_1} M_{11} e^{-D_1} \right\|, \left\| e^{D_2} M_{22} e^{-D_2} \right\| \right),$$

which is less than or equal to $\left\| e^{D^{(i)}} M e^{-D^{(i)}} \right\|$ for every i, from Lemma 1. Thus for every $\gamma > f_{\min}(M)$, the set $\mathcal{D}_\gamma$ is unbounded.

To prove the converse, let us assume that for some $\gamma > f_{\min}(M)$, $\mathcal{D}_\gamma$ is not bounded. Then there is a sequence of scalings $D^{(i)}$ in $\mathcal{D}_\gamma$ with some of the elements of the diagonal scaling matrix $D^{(i)}$ with absolute value tending to infinity. Then, there exists a subsequence $D^{(n_i)}$, which can be partitioned to within a permutation as

$$D^{(n_i)} = \begin{bmatrix} D_{1,n_i} & 0 \\ 0 & D_{2,n_i} \end{bmatrix},$$

where every element of D_{1,n_i} diverges to $-\infty$ with i, while every element of D_{2,n_i} is bounded below. (In fact, at least one of the elements of D_{2,n_i} must diverge to ∞, but we will not use this fact.)

Thus the maximum singular value of

$$M = \begin{bmatrix} e^{D_{1,n_i}} M_{11} e^{-D_{1,n_i}} & e^{D_{1,n_i}} M_{12} e^{-D_{2,n_i}} \\ e^{D_{2,n_i}} M_{21} e^{-D_{1,n_i}} & e^{D_{2,n_i}} M_{22} e^{-D_{2,n_i}} \end{bmatrix},$$

remains bounded with every element of D_{1,n_i} diverging to $-\infty$ while the elements of D_{2,n_i} are bounded below. This immediately means that $M_{21} = 0$, which shows that M must be reducible. $\square$

COROLLARY 2.3. *$\mathcal{D}_{\mathrm{opt}}$ is nonempty and bounded if M is irreducible.*

Proof. If M is irreducible, the sublevel set $\mathcal{D}_\gamma$ is bounded for every $\gamma > f_{\min}(M)$; since $\left\| e^D M e^{-D} \right\|$ is a continuous function of D over $\mathcal{D}_\gamma$, the infimum in (2) is achieved. Thus $\mathcal{D}_{\mathrm{opt}}$ is nonempty if M is irreducible. Boundedness of $\mathcal{D}_{\mathrm{opt}}$ follows from an argument similar to the one in the proof of theorem 2.1. $\square$

Remark. Thus, irreducibility of M is a sufficient condition for the existence of optimal matrix scalings. If M is reducible, two cases are possible: $\mathcal{D}_{\mathrm{opt}}$ may be empty or it may be nonempty and unbounded. The following examples illustrate this.

EXAMPLE 1 ($\mathcal{D}_{\mathrm{opt}}$ EMPTY).

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

It can be verified that $\mathcal{D}_{\mathrm{opt}}$ is empty. The optimally scaled singular value is the limit of the sequence of scaled singular values corresponding to scalings $D(\epsilon)$ with $\epsilon \downarrow 0$:

$$D(\epsilon) = \begin{bmatrix} \log(\epsilon) & 0 & 0 \\ 0 & \log(\epsilon) & 0 \\ 0 & 0 & -2\log(\epsilon) \end{bmatrix}.$$

EXAMPLE 2 ($\mathcal{D}_{\mathrm{opt}}$ NONEMPTY AND UNBOUNDED).

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

It can be verified that

$$\mathcal{D}_{\mathrm{opt}} = \left\{ D \;\middle|\; D = \begin{bmatrix} \log(\epsilon) & 0 & 0 \\ 0 & \log(\epsilon) & 0 \\ 0 & 0 & -2\log(\epsilon) \end{bmatrix},\ \epsilon \in (\,0, (3/2)^{1/6}\,] \right\}.$$

3. $\mathcal{D}_{\mathrm{opt}}$ for irreducible matrices. We next derive a sufficient condition for $\mathcal{D}_{\mathrm{opt}}$ to be a singleton. Our derivation depends on the analyticity properties of the singular values and singular vectors of a complex matrix that depends on a real parameter. We summarize these in the following theorem, taken from [5, 4].

THEOREM 3.1. *Let $A(x)$ be a (complex) $m \times n$ matrix, the entries of which are analytic functions of a real parameter x. There are real analytic functions $f_i : \mathbf{R} \to$*

$\mathbf{R}, i = 1, \ldots, \min(m,n)$ *such that, for all* $x \in \mathbf{R}$,

$$\{\sigma_i(A(x)), i = 1, \ldots, \min(m,n)\} = \{|f_i(x)|, i = 1, \ldots, \min(m,n)\}, \tag{4}$$

where $\sigma_i(A(x))$ *stands for the ith singular value of* $A(x)$. *(Thus, the* f_i*'s are the unordered and unsigned singular value functions of* $A(x)$.*)*

Moreover, for $x \in \mathbf{R}$, *there exist analytic matrix-valued functions* $U(x)$ *and* $V(x)$ *whose columns are the left and right singular vectors respectively of* $A(x)$.

For convenience, we let $\gamma = f_{\min}(M)$. Assuming that $\mathcal{D}_{\text{opt}}$ is not a singleton, let $D_1, D_2 \in \mathcal{D}_{\text{opt}}$, with $D_1 \neq D_2$. Then, from the convexity of $\mathcal{D}_{\text{opt}}$, $D(\lambda) = \lambda D_1 + (1-\lambda) D_2 \in \mathcal{D}_{\text{opt}}$ for every $\lambda \in [0,1]$. Thus for every $\lambda \in [0,1]$, there must exist vectors $u(\lambda)$ and $v(\lambda)$, with $\|u(\lambda)\| = \|v(\lambda)\| = 1$, such that

$$\begin{aligned} e^{D(\lambda)} M\, e^{-D(\lambda)}\, v(\lambda) &= \gamma\, u(\lambda) \\ e^{-D(\lambda)} M^*\, e^{D(\lambda)}\, u(\lambda) &= \gamma\, v(\lambda). \end{aligned} \tag{5}$$

Since a singular value of $e^{D(\lambda)}\, M\, e^{-D(\lambda)}$ is constant over an interval, from Theorem 3.1, it is constant for *all* $\lambda \in \mathbf{R}$. Moreover, $u(\lambda)$ and $v(\lambda)$ can be chosen to be analytic functions of λ.

Since $D_1 \neq D_2$, to within a permutation,

$$D_1 - D_2 = \begin{bmatrix} D_+ & 0 \\ 0 & D_- \end{bmatrix},$$

where D_+ and $-D_-$ are diagonal matrices with positive entries, and are of sizes p and $n-p$ respectively.

Partitioning M, D_2, $u(\lambda)$ and $v(\lambda)$ as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad D_2 = \begin{bmatrix} D_{2,1} & 0 \\ 0 & D_{2,2} \end{bmatrix}, \quad u = \begin{bmatrix} u_1(\lambda) \\ u_2(\lambda) \end{bmatrix}, \quad v = \begin{bmatrix} v_1(\lambda) \\ v_2(\lambda) \end{bmatrix},$$

where $M_{11} \in \mathbf{C}^{p\times p}$, $D_{2,1} \in \mathbf{R}^{p\times p}$ and diagonal, $u_1(\lambda)$, and $v_1(\lambda) \in \mathbf{C}^p$, we may rewrite equation (5) as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} e^{-\lambda D_+} e^{-D_{2,1}} v_1(\lambda) \\ e^{-\lambda D_-} e^{-D_{2,2}} v_2(\lambda) \end{bmatrix} = \gamma \begin{bmatrix} e^{-\lambda D_+} e^{-D_{2,1}} u_1(\lambda) \\ e^{-\lambda D_-} e^{-D_{2,2}} u_2(\lambda) \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} M_{11}^* & M_{21}^* \\ M_{12}^* & M_{22}^* \end{bmatrix} \begin{bmatrix} e^{\lambda D_+} e^{D_{2,1}} u_1(\lambda) \\ e^{\lambda D_-} e^{D_{2,2}} u_2(\lambda) \end{bmatrix} = \gamma \begin{bmatrix} e^{\lambda D_+} e^{D_{2,1}} v_1(\lambda) \\ e^{\lambda D_-} e^{D_{2,2}} v_2(\lambda) \end{bmatrix} \tag{7}$$

Letting λ tend to $+\infty$ and $-\infty$ in equations (6) and (7), and noting that the entries of $u(\lambda)$ and $v(\lambda)$ are bounded for all $\lambda \in \mathbf{R}$, we conclude the existence of vectors $\tilde{u}_1, \tilde{v}_1 \in \mathbf{C}^p$ and $\tilde{u}_2, \tilde{v}_2 \in \mathbf{C}^{(n-p)}$ with $[\tilde{u}_1^T \;\; \tilde{u}_2^T] \neq 0$ and $[\tilde{v}_1^T \;\; \tilde{v}_2^T] \neq 0$ such that

$$\begin{aligned} M_{11}\, \tilde{v}_1 &= \gamma\, \tilde{u}_1 \quad \text{and} \quad M_{21}\, \tilde{v}_1 = 0, \\ M_{22}\, \tilde{v}_2 &= \gamma\, \tilde{u}_2 \quad \text{and} \quad M_{12}\, \tilde{v}_2 = 0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} M_{11}^*\, \tilde{u}_1 &= \gamma\, \tilde{v}_1 \quad \text{and} \quad M_{12}^*\, \tilde{u}_1 = 0, \\ M_{22}^*\, \tilde{u}_2 &= \gamma\, \tilde{v}_2 \quad \text{and} \quad M_{21}^*\, \tilde{u}_2 = 0. \end{aligned} \tag{9}$$

Two cases are possible:

- $\tilde{u}_1 \neq 0 \neq \tilde{v}_1$. Then equations (8) and (9) mean that $[\tilde{u}_1^T \;\; 0]^T$ and $[\tilde{v}_1^T \;\; 0]^T$ are left and right singular vectors of $e^{D_2} M e^{-D_2}$ corresponding to the singular value γ.
- $\tilde{u}_1 = 0 = \tilde{v}_1$. Then $[0 \;\; \tilde{u}_2^T]^T$ and $[0 \;\; \tilde{v}_2^T]^T$ are left and right singular vectors of $e^{D_2} M e^{-D_2}$ corresponding to the singular value γ.

Thus, either case means that there exist right and left singular vectors of $e^{D_2} M e^{-D_2}$ corresponding to the maximum singular value belonging to the same *coördinate subspace*, *i.e.*, a subspace of the form $\bigcup_{i\in\mathbf{I}} \operatorname{span}\{e_i\}$, where $\mathbf{I}$ is a proper subset of the set of indices $\{1,\ldots,n\}$ and $\{e_i,\ i=1,\ldots,n\}$ are the unit vectors in $\mathbf{R}^n$ in the standard basis. Thus, we have the following sufficient condition for the uniqueness of optimal scaling:

THEOREM 3.2. *For an irreducible matrix M, the optimal scaling D is unique if no pair of vectors v and u, with $\|u\| = \|v\| = 1$ satisfying*

$$\begin{aligned} e^D M e^{-D}\, v &= \gamma\, u \\ e^{-D} M^* e^D\, u &= \gamma\, v, \end{aligned}$$

belong to the same coördinate subspace.

Remark. Theorem 3.2 does not provide a necessary condition for the uniqueness of the optimal scaling. Consider the following two examples in which at an optimal scaling D, there exist left and right singular vectors u and v corresponding to the maximum singular value that belong to the same coordinate subspace:

EXAMPLE 3 ($\mathcal{D}_{\rm opt}$ IS A SINGLETON).

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

It can be verified that the unique optimal scaling is the 0, *i.e.*, the 'identity' scaling, though $[0\ \ 1/\sqrt{2}\ \ 1/\sqrt{2}]^T$ is both a left and right singular vector corresponding to the maximum singular value of two.

EXAMPLE 4 ($\mathcal{D}_{\rm opt}$ IS NOT A SINGLETON).

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

It can be verified that $\mathcal{D}_{\rm opt}$ is given by

$$\mathcal{D}_{\rm opt} = \left\{ D \;\middle|\; D = \begin{bmatrix} \frac{2}{3}\log(\epsilon) & 0 & 0 \\ 0 & -\frac{1}{3}\log(\epsilon) & 0 \\ 0 & 0 & -\frac{1}{3}\log(\epsilon) \end{bmatrix},\ \epsilon \in [\epsilon_* \,, 1/\epsilon_* \,] \right\}$$

where $\epsilon_* = \sqrt{\dfrac{4.5-\sqrt{4.25}}{4}}$. For every $D \in \mathcal{D}_{\rm opt}$, $[0 \;\; 1/\sqrt{2} \;\; 1/\sqrt{2}]^T$ is both a left and right singular vector of $e^D M e^{-D}$ corresponding to the maximum singular value of two.

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