STABILITY ROBUSTNESS OF LINEAR SYSTEMS TO REAL PARAMETRIC PERTURBATIONS

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Abstract

We consider linear time-invariant systems subject to real, parametric variations. The problem of computing the half-sidelength $1/\mu_\infty$ of the largest stability hypercube in the parameter space is formulated in a frequency-independent way. The frequency-dependent approach developed in μ analysis is impracticable, because μ_∞ is a discontinuous function of frequency. We derive an accurate upper bound for μ_∞, using block-diagonal scaling of the largest singular value of a real, frequency-independent matrix M. The optimal scaling is found using quasi-convex optimization. A numerical example illustrates the method.

1 Introduction

In this paper we address the problem of finding the minimum $\mathbf{l}_\infty$ distance to the stability boundary of a parameter-dependent plant in the parameter space, or $\mathbf{l}_\infty$-*parameter margin*. This quantity is nothing else than the inverse of the "real μ" of Doyle [1]. Unfortunately, the only fool-proof algorithms available now are exponential in CPU time [2, 3], because a global minimum has to be found. The two-parameter case is solved in [4].

μ analysis [1] provides a framework for robustness analysis of systems subject to various types of perturbations (neglected high-order dynamics, complex parametric perturbations, ...). The analysis is done at each frequency, and then a line search is done over all frequencies. This approach is impracticable here since the real μ is a *discontinuous* function of frequency [5, 9].

For the very general case when the parameters appear rationally in the coefficients of the plant's dynamic matrix, a new, frequency-independent framework is introduced in § 2. The scaling ideas developed for μ analysis [1] are then used in § 3, to get an accurate upper bound ν for the real μ, involving the largest singular value of a scaled matrix. The new feature is that the scaling matrices are now block-diagonal instead of diagonal, yielding a quasi-convex problem instead of convex. The ellipsoid algorithm with constraints [6] is used to compute ν. A numerical example, taken from [7], illustrates the method in § 4.

2 Parameter Margins

Consider a linear, time-invariant system of order n:

$$\dot{x} = \mathcal{F}(a)\ x,$$

where x is the state and $\mathcal{F}(a)$ is the $n \times n$ dynamic matrix of the plant. $\mathcal{F}(a)$ is a function of a *parameter vector* a of length p, whose nominal value can always be reset to zero. The $\mathbf{l}_\infty$-*Parameter Margin* of $\mathcal{F}(a)$ is [9]:

$$pm_\infty(\mathcal{F}) \stackrel{\Delta}{=} \min_{a \in \mathbf{R}^p} \{ \parallel a \parallel_\infty \mid \ \mathcal{F}(a) \text{ is unstable } \}$$

$\mu_\infty \stackrel{\Delta}{=} 1/pm_\infty$ is nothing else than the structured singular value with repeated real scalar blocks found in μ analysis [1]. We assume without great loss of generality that $\mathcal{F}(a)$ is a rational function of a, and that the nominal system $(a = 0)$ is stable.

We can form a real matrix $\mathcal{B}(a)$, of size $n(n+1)/2$, such that the problem reads [8, 9]:

$$pm(\mathcal{F}) = \min_{a \in \mathbf{R}^p} \{ \parallel a \parallel \mid \det[\mathcal{B}(a)] = 0 \}$$

$\mathcal{B}(a)$ is called the *Lyapunov matrix* of $\mathcal{F}(a)$ [9]; its elements are linear combinations of those of $\mathcal{F}(a)$. Exploiting the rational dependence of $\mathcal{F}(a)$, one can write the problem in the same form as in μ analysis:

$$pm = \min_{a \in \mathbf{R}^p} \{ \parallel a \parallel \ \mid \det[I + \Delta(a) M] = 0 \}$$

M is now a *real, frequency-independent* matrix, and Δ has the structured form:

$$\Delta(a) \stackrel{\Delta}{=} \text{block-diag}[a_1 I_{r_1}\ \ldots\ a_p I_{r_p}]\ ,$$

where each I_{r_p} is an identity matrix of size r_i; $r = [r_1 \dots r_p]$ is called the structure vector.

Using transformations of the form UMU^T, where U is a block-diagonal rotation, we can reduce the size N of M, so that each "block-row" and "block-column" is full rank [4].

3 An Upper Bound for μ_∞

For a given structure vector r, define:

$$\begin{aligned}
\mathcal{B} &= \{\text{block-diag}[B_i]_{i=1}^p\ ,\ B_i\ \in\ \mathbf{R}^{r_i \times r_i}\} \\
\mathcal{D} &= \{D \in \mathcal{B},\ \det D \neq 0\} \\
\mathcal{P} &= \{P \in \mathcal{B},\ P = P^T > 0\} \\
\mathcal{E} &= \{\text{block-diag}\ [\epsilon_i I_i]_{i=1}^p\ ,\ \epsilon_i = \pm 1, \\
&\qquad I_i = \text{identity matrix of size } r_i\}
\end{aligned}$$

We have the well-known bounds:

$$\max_{E \in \mathcal{E}} \rho_R(ME) \leq \mu_\infty(M, r) \leq \nu(M, r) \triangleq \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1}) \tag{1}$$

where $\rho_R(M)$ is the largest real eigenvalue of M (or zero if none exist). It turns out that the upper bound is an excellent approximation for μ_∞. This can be understood from the following new characterization of μ_∞ [9]:

$$\begin{aligned}
\mu_\infty(M, r) &= \max_{v \neq 0} \inf_{D \in \mathcal{D}} \frac{\| DMv \|}{\| Dv \|} \\
&\leq \inf_{D \in \mathcal{D}} \max_{v \neq 0} \frac{\| DMD^{-1}v \|}{\| v \|} = \nu(M, r)
\end{aligned}$$

The above characterization generalizes to the block-diagonal case the one found in [10].

The lower bound in (1) is attained if and only if the largest stability box touches the stability boundary on one of its corners. This is the case if the largest singular value ν is simple, or if $r = [1 \dots 1]$ [11].

Given a real matrix M and a structure vector r, we now address the block-diagonal scaling problem:

$$\nu(M, r) = \inf_{D \in \mathcal{D}} \overline{\sigma}(DMD^{-1}) \tag{2}$$

Problem (2) can be reduced to the following (non-differentiable) quasi-convex optimization program:

$$\nu(M, r) = \min_{P \in \mathcal{P}} \overline{\sigma}(P^{\frac{1}{2}} M P^{-\frac{1}{2}}) \ \text{ subject to: } \lambda_{min}(P) \geq 1\ ,$$

where $\lambda_{min}(P)$ is the smallest eigenvalue of P. For this problem, the ellipsoid algorithm with convex constraints [6] can be used.

At each step, we compute one singular value decomposition, one symmetric square root, a few matrix multiplications and take one inverse. This is of the order of N^3 flops, where N is the size of M.

We were not able to derive a satisfactory stopping criterion: we took a criterion which works for convex functions only. This problem is linked to the conditions for which the infimum in (2) is attained.

4 Helicopter Example

This example is taken from [7, page 109]. The linearized closed-loop dynamic matrix of a VTOL helicopter in the vertical plane, at an airspeed of 135 knots and for typical load and flight conditions, is:

$$\mathcal{F}(a) = \begin{bmatrix} -0.0366 & -0.4174 & 0.0188 & -0.4555 \\ 0.0482 & -18.8189 - 1.6352 a_3 & 0.0024 & -4.0208 \\ 0.1002 & 16.4988 + a_1 & -0.707 & 1.42 + a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The parameters a_i have zero nominal value, and are subject to the bounds:

$$\mid a_1 \mid \leq 0.01 \ , \ \mid a_2 \mid \leq 0.05 \ , \ \mid a_3 \mid \leq 0.04 \ . \qquad (3)$$

The algorithm converges to a precision of $\epsilon = 10^{-5}$ after 450 steps. The largest singular value of the scaled matrix is simple, so it is actually equal to μ_∞:

$$\mu_\infty = \nu = 0.7021 \ , \quad \Longrightarrow \quad pm_\infty = 1.4243 \ .$$

Thus, the system is stable inside the rectangular box given by inequalities (3): no robustification is needed, contrarily to what is claimed in [7].

5 Conclusion

In this paper we addressed the issue of minimizing the largest singular value of a real matrix using block-diagonal scalings. the problem is formulated as a quasi-convex problem with a convex constraint, and was solved using the ellipsoid algorithm. The resulting algorithm yields an excellent approximation for the real parametric μ, for which a frequency-independent formulation was derived. Further research should concentrate on the conditions for existence and unicity of an optimal scaling, and also the conditions for which μ_∞ equals its bounds.

References

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