Convex Optimization in Electrical Engineering

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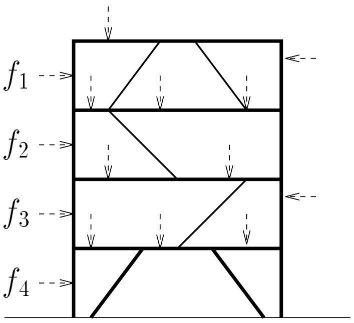
Main idea

- many electrical engineering design problems can be cast as **convex optimization problems**
- such problems can appear very difficult, but can be solved **very efficiently** by recently developed methods
- (unfortunately) convexity is often not recognized, hence not exploited

Example 1

- linear elastic structure; forces $f_1, \dots, f_{100}$ induce deflections $d_1, \dots, d_{300}$
- $0 \le f_i \le F_i^{\max}$, several hundred other constraints: max load per floor, max wind load per side, etc.



Problem 1a: find worst-case deflection, *i.e.*, $\max_i |d_i|$

Problem 1b: find worst-case deflection, with each force "on" or "off", *i.e.*, $f_i = 0$ or $F_i^{\max}$

Example 1

problem 1a is **very easy**

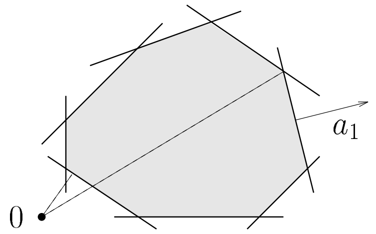
- readily solved in a few minutes on small workstation
- general problem has polynomial complexity

problem 1b is **very difficult**

- could take weeks to solve ...
- general problem NP-complete

Example 2

polytope described by linear inequalities, $a_i^T x \leq b_i$, $i = 1, \ldots, L$



Problem 2a: find point **closest** to origin, $i.e.$, $\min \|x\|$

Problem 2b: find point **farthest** from origin, $i.e.$, $\max \|x\|$

Example 2

problem 2a is **very easy**

- readily solved on small workstation
- polynomial complexity

problem 2b is **very difficult**

- difficult even with supercomputer
- NP-complete

moral:

very difficult and **very easy** problems can look quite similar

Outline

- **Convex optimization**
- Examples
- Interior-point methods

Convex optimization

minimize $f_0(x)$

subject to $f_1(x) \leq 0, \ldots f_L(x) \leq 0$

$f_i : \mathbf{R}^n \to \mathbf{R}$ are convex, $i.e.$, for all x, y, $0 \leq \lambda \leq 1$,

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f_i(y)$$

- can have linear equality constraints
- differentiability not needed
- examples: linear programs (LPs), problems 1a, 2a
- other formulations possible (feasibility, multicriterion)

(roughly speaking,)

Convex optimization problems are fundamentally tractable

- computation time is small, grows gracefully with problem size and required accuracy
- large problems solved quickly in practice
- what "solve" means:
  - find **global** optimum within a given tolerance, or,
  - find **proof** (certificate) of infeasibility
- not widely enough appreciated

Outline

- Convex optimization
- **Examples**
- Interior-point methods

Well-known example: FIR filter design

transfer function: $H(z) \triangleq \sum_{i=0}^{n} h_i z^{-i}$

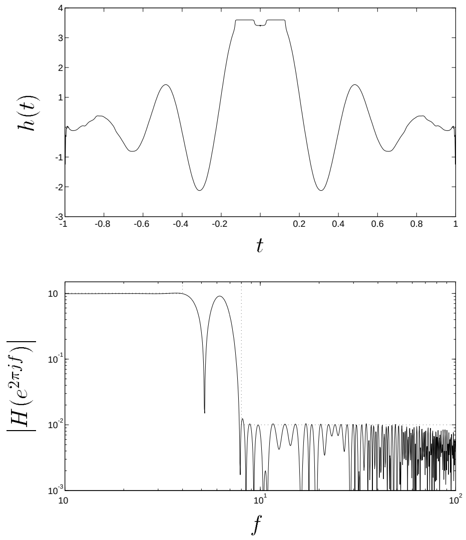
design variables: $x \triangleq [h_0\ h_1\ \ldots\ h_n]^T$

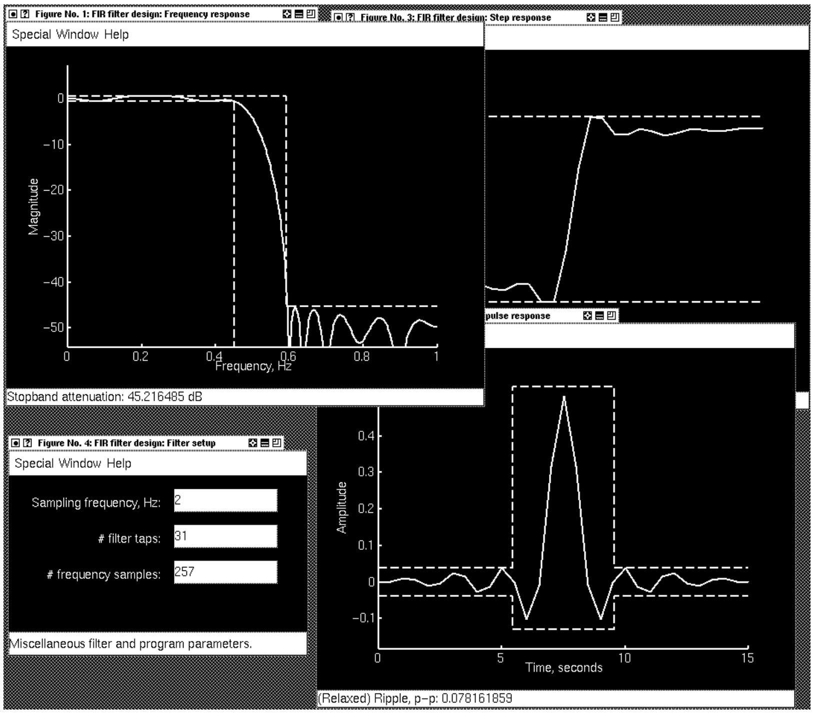
sample convex constraints:

- $H(e^{j0}) = 1$ (unity DC gain)
- $H(e^{j\omega_0}) = 0$ (notch at ω_0)
- $|H(e^{j\omega})| \leq 0.01$ for $\omega_s \leq \omega \leq \pi$
  (min. 40dB atten. in stop band)
- $|H(e^{j\omega})| \leq 1.12$ for $0 \leq \omega \leq \omega_b$
  (max. 1dB upper ripple in pass band)
- $h_i = h_{n-i}$ (linear phase constraint)
- $s(t) \triangleq \sum_{i=0}^{t} h_i \leq 1.1 H(e^{j0})$ (max. 10% step response overshoot)

FIR filter design example (M. Grant)

- sample rate $2/n$ sec^{-1}
- linear phase
- max ± 1dB ripple up to 0.4Hz
- min 40dB atten above 0.8Hz
- minimize $\max_i |h_i|$
- some solution times:
  $n = 255$: 5 sec
  $n = 2047$: 4 min





Stopband attenuation: 45.216485 dB

Sampling frequency, Hz: 2

filter taps: 31

frequency samples: 257

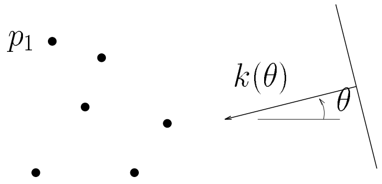
Miscellaneous filter and program parameters.

(Relaxed) Ripple, p−p: 0.078161859

Beamforming

omnidirectional antenna elements at positions $p_1, \ldots, p_n \in \mathbf{R}^2$
plane wave incident from angle θ:

$$\exp j(k(\theta)^T p - \omega t), \qquad k(\theta) = -[\cos\theta \ \ \sin\theta]^T$$



demodulate to get $y_i = \exp(jk(\theta)^T p_i)$
form weighted sum $y(\theta) = \sum_{i=1}^{n} w_i y_i$

design variables: $x = [\textbf{Re}\, w^T \ \textbf{Im}\, w^T]^T$
(antenna array weights or shading coefficients)

$G(\theta) \triangleq |y(\theta)|$ antenna gain pattern

Sample convex constraints:

- $y(\theta_t) = 1$ (target direction normalization)
- $G(\theta_0) = 0$ (null in direction θ_0)
- w is real (amplitude only shading)
- $|w_i| \leq 1$ (attenuation only shading)

Sample convex objectives:

- $\max\{G(\theta) \mid |\theta - \theta_t| \geq 5°\}$
  (sidelobe level with $10°$ beamwidth)
- $\sigma^2 \sum_i |w_i|^2$ (noise power in y)

Open-loop trajectory planning

discrete-time linear system, input $u(t) \in \mathbf{R}^p$, output $y(t) \in \mathbf{R}^q$

sample convex constraints:

- $|u_i(t)| \leq U$ (limit on input amplitude)
- $|u_i(t+1) - u_i(t)| \leq S$ (limit on input slew rate)
- $l_i(t) \leq y_i(t) \leq u_i(t)$ (envelope bounds for output)

sample convex objective:

- $\max_{t,i} |y_i(t) - y_i^{\mathrm{des}}(t)|$ (peak tracking error)

Robust open-loop trajectory planning

input must work well with **multiple plants**

one input u applied to L plants; outputs are $y^{(1)}, \ldots, y^{(L)}$

constraints are to hold for **all** $y^{(i)}$

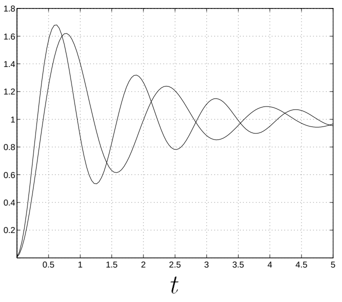
sample objectives:

- weighted sum of objectives for each i (average performance)
- max over objectives for each i (worst-case performance)

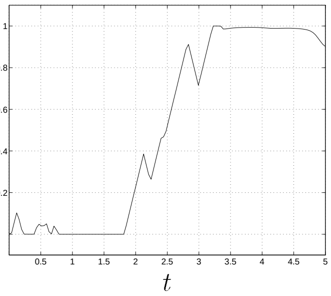
Example

- two plants
- $0 \le u(t) \le 1$
- $|\Delta u(t)| \le 1.25$/sec
- minimize worst-case peak tracking error
- 128 time samples (variables)

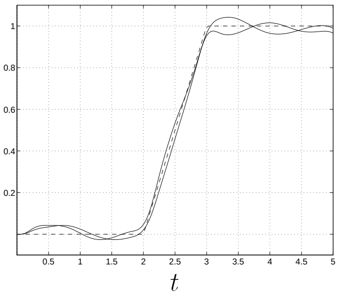
Plant step responses



u^{opt}

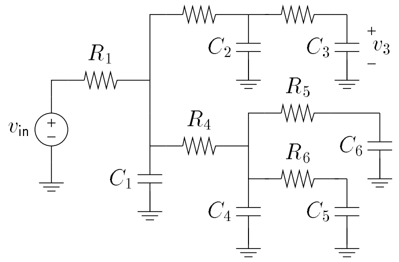


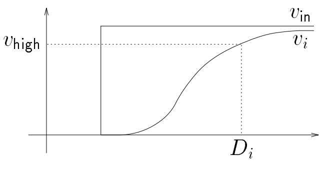
$y_1,\ y_2,\ y^{\text{des}}$



VLSI transistor sizing

subcircuit during transition contains Rs, Cs:
'on' transistor resistances; drain, source to ground capacitances





- x_i: size (width) of transistors, so total area is $A(x) = f^T x + g$
- 'on' conductance of transistor i: $g_i = \alpha x_i$
- capacitance at node j : $c_j = a_j^T x + b_j$
- sample problem: minimize area s.t. max delay, limits on widths

Approximation via dominant time constant

circuit dynamics: $C(x)\frac{dv}{dt} = -G(x)v(t)$

- conductance matrix $G(x)$, capacitance matrix $C(x)$ affine in x
- solutions have form $v(t) = \sum_i \alpha_i e^{\lambda_i t}$
- eigenvalues $0 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ given by

$$\det(\lambda_i C(x) + G(x)) = 0$$

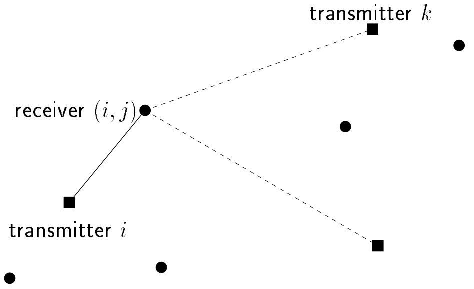
- slowest ("dominant") time constant given by $T_{\mathsf{dom}} = -1/\lambda_1$ (related to delay)
- $T_{\mathsf{dom}} \leq T_{\mathsf{max}} \Longleftrightarrow (-1/T_{\mathsf{max}})C(x) + G(x) \geq 0$

Problem: (convex, in fact, a semidefinite program)

$$\begin{array}{ll} \text{minimize} & A(x) \\ \text{subject to} & (-1/T_{\mathsf{max}})C(x) + G(x) \geq 0, \quad x_i^{\mathsf{min}} \leq x_i \leq x_i^{\mathsf{max}} \end{array}$$

Transmitter power allocation

- m transmitters, mn receivers all at same frequency
- transmitter i wants to transmit to n receivers labeled (i, j), $j = 1, \ldots, n$



- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)
- variables: transmitter powers p_k, $k = 1, \ldots, m$

signal power at receiver (i,j): $S_{ij} = A_{iji}p_i$

noise plus interference power at receiver (i,j): $I_{ij} = \sum_{k\neq i} A_{ijk}p_k + N_{ij}$

signal to interference/noise ratio (SINR) at receiver (i,j): S_{ij}/I_{ij}

Problem: choose p_i to maximize smallest SINR:

$$\begin{array}{ll} \text{maximize} & \displaystyle\min_{i,j} \frac{A_{iji}p_i}{\sum_{k\neq i} A_{ijk}p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\mathsf{max}} \end{array}$$

...a (quasi-) convex problem (same methods work)

Other examples

- control system analysis and design
- array signal processing, *e.g.*, broadband beamforming
- filter/controller realization
- experiment design & system identification
- structural optimization, *e.g.*, truss design
- design centering & yield maximization
- statistical signal processing
- computational geometry

Outline

- Convex optimization
- Examples
- **Interior-point methods**

Interior-point convex programming methods

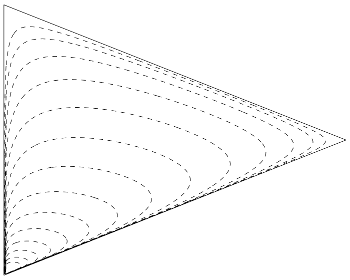
history:

- Dikin; Fiacco & McCormick's SUMT (1960s)
- Karmarkar's LP algorithm (1984); many more since then
- Nesterov & Nemirovsky's general formulation (1989)

general:

- # iterations small, grows slowly with problem size
  (typical number: $5 - -50$)
- each iteration is basically least-squares problem
  (hence can exploit problem structure via conjugate-gradients)

Basic idea (oversimplified)



choose (potential fct.) φ s.t.

- φ smooth
- $\varphi(x) \to +\infty$
  as $x \to$ feasible set boundary
- $\varphi(x) \to -\infty$ as $x \to$ optimal

minimize φ by (modified) Newton method

each Newton step is (mostly) a least-squares problem

if φ is properly chosen:
algorithm is polynomial, extremely efficient in practice

Conclusions

- diverse EE design problems can be cast as convex optimization problems, hence efficiently solved
- applications: numerical engines for CAD tools, embedded systems

as available computing power increases, this observation becomes more relevant

How many practical problems are convex?

many, but certainly not all, or even most

myth # 1: very few practical problems are convex

myth # 2: it's very hard to determine convexity in practice

these are self-propagating

. . . the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.

— R. Rockafellar, SIAM Review 1993

(Pointers to) references

where:

- anonymous ftp to `isl.stanford.edu`, in `pub/boyd`
- URL `http://www-isl.stanford.edu/ boyd`

what:

- survey articles with **many** references
- code
- lecture notes for EE392: *Introduction to Convex Optimization with Engineering Applications*

Course: EE364, Winter 1996-7