Optimal Linear Static Control with Moment and Yield Objectives

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Abstract

This report describes how to optimize linear static control problems involving first and second moments and yield objectives for Gaussian distributions. These problems can be cast as second-order cone programs, which is a class of convex optimization problems that can be solved very efficiently.

1 Introduction

Yield is of importance in many manufacturing processes. The profit can be directly proportional to the yield, or the yield can be related to quality levels in a nonlinear way, i.e. only as long as a certain percentage of the production meets the desired quality, the customer is paying full price for the product. In the former case it is desirable to maximize the yield, in the latter case it is most likely desirable to minimize the production cost subject to constraints on the yield. Both these problems will be addressed in this paper.

In Section 2 the model to be controlled will be defined. For simplicity only a static model will be considered, but the results presented can easily be extended to the dynamic case using ideas like in [BB91]. Control of static systems in the H_∞ framework has been described in [SP96], and static performance robustness of thermal processes has been described in [KKB94]. In Section 3 the control problems will be defined. Then in Section 4 the solution procedure will be outlined. It will be seen that the problems can be cast as Second-order Cone Programs (SOCP):s, which is a class of convex optimization problems that can be solved very efficiently, see [LVB97]. In sections 5 and 6 some extensions and examples will be investigated. Finally, in Section 7, a few concluding remarks are given.

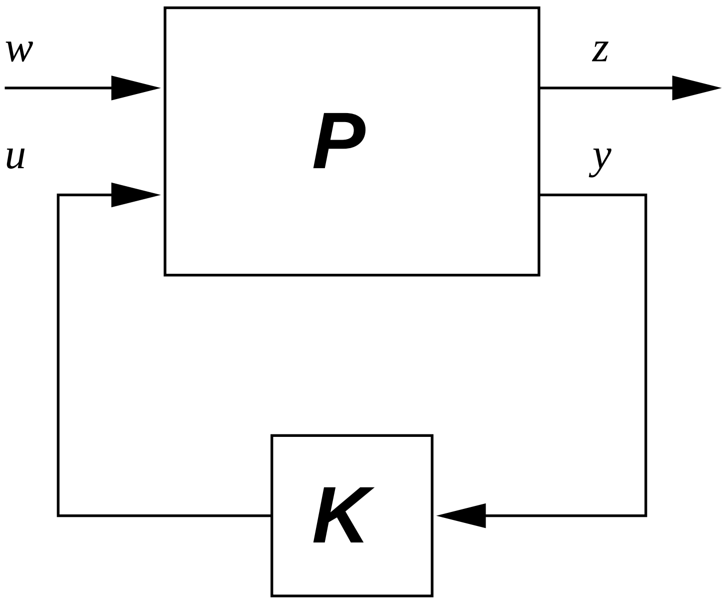

Figure 1: Model of the closed loop control system

2 Model

In this section the model for the control problems will be given. It is static, multi-variable, and depicted in Figure 1. The equations relating the different variables are

$$\begin{aligned}\begin{bmatrix} z \\ y \end{bmatrix} &= P \begin{bmatrix} u \\ w \end{bmatrix} \\ u &= Ky\end{aligned}$$

where $w \in \mathbb{R}^l$ is the exogenous input, $z \in \mathbb{R}^q$ is the output to be controlled, $y \in \mathbb{R}^p$ is the sensed output, $u \in \mathbb{R}^m$ is the actuator input, and where K and

$$P \triangleq \begin{bmatrix} P_{zu} & P_{zw} \\ P_{yu} & P_{yw} \end{bmatrix}$$

are matrices of compatible dimensions. The matrix P is given and describes the plant, whereas the matrix K, called the feedback gain, describes the controller; it is to be determined to optimize some objective.

For K such that $I - P_{yu}K$ is invertible the control problem is said to be well-posed, and simple algebra shows that $z = Hw$, where

$$H \triangleq P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}$$

Most objectives are convex in H, but unfortunately not in K, which is the optimization-variable. However, this problem can be circumvented by the following linear-fractional transformation:

$$Q \triangleq K(I - P_{yu}K)^{-1}$$

which yields

$$H(Q) = P_{zw} + P_{zu}QP_{yw}$$

where H is now affine in Q. This transformation can also be extended to the dynamic case, see e.g. [BB91]. In case $I+QP_{yu}$ is invertible K can be computed as

$$K = (I + QP_{yu})^{-1}Q$$

The case when $I + QP_{yu}$ is not invertible is discussed in Appendix A.

For the purpose of this report it will be assumed that w is a Gaussian random vector with mean $\bar{w}$ and covariances Σ. The covariance may be degenerate, i.e. Σ need not be positive definite, only positive semidefinite. Notice that z is also Gaussian with mean and covariance

$$\begin{aligned} m(Q) &\triangleq \mathrm{E}\left\{z(Q)\right\} = H(Q)\bar{w} \\ \sigma^2(Q) &\triangleq \mathrm{E}\left\{[z(Q) - m(Q)]\,[z(Q) - m(Q)]^T\right\} = H(Q)\Sigma H^T(Q) \end{aligned}$$

respectively.

3 Control Problems

In this section control problems involving yield and first and second moments will be defined. First it will be noticed that the second moment of z is given by

$$M(Q) \triangleq \mathrm{E}\left\{z(Q)z^T(Q)\right\} = H(Q)\left(\Sigma + \bar{w}\bar{w}^T\right)H^T(Q)$$

Furthermore for any entry $z_i(Q) \triangleq e_i^T z(Q)$ of $z(Q)$ it holds that the corresponding means, variances, and second moments are given by

$$\begin{aligned} m_i(Q) &= e_i^T m(Q) \\ \sigma_i^2(Q) &= e_i^T \sigma(Q) e_i \\ M_i(Q) &= e_i^T M(Q) e_i \end{aligned}$$

Notice that they are linear and quadratic in Q, respectively. Finally the yield of the signal z_i with respect to the level $\bar{z}_i \in \mathbb{R}$ is defined as

$$Y_i(Q) \triangleq \mathrm{Prob}\left\{z_i(Q) \leq \bar{z}_i\right\}$$

It can be expressed in the standard Gaussian distribution function $\Phi(\cdot)$ as

$$Y_i(Q) = \Phi\left(\frac{\bar{z}_i - m_i(Q)}{\sigma_i(Q)}\right)$$

Furthermore it holds that $Y_i(Q) \geq \alpha_i$ if and only if

$$\frac{\bar{z}_i - m_i(Q)}{\sigma_i(Q)} \geq \Phi^{-1}(\alpha_i)$$

This follows from the fact that $\Phi(\cdot)$ is an increasing function. Notice that $\Phi^{-1}(\alpha_i) > 0$ if and only if $\alpha_i > 1/2$.

It is now clear that the sets

$$\begin{array}{cc} \{(Q,\beta_i) : m_i(Q) \leq \beta_i\}; & \{(Q,\gamma_i^2) : \sigma_i(Q) \leq \gamma_i^2\} \\ \{(Q,\delta_i^2) : M_i(Q) \leq \delta_i^2\}; & \{Q : Y_i(Q) \geq \alpha_i > 1/2\} \end{array}$$

are convex. Hence natural optimization problems to formulate are:

$$\begin{array}{ll} \text{minimize } m_1(Q), & \text{minimize } \sigma_1^2(Q) \\ \text{minimize } M_1(Q), & \text{maximize } Y_1(Q) \end{array}$$

respectively, subject to

$$\begin{array}{rcl} m_i(Q) & \leq & \beta_i, \quad i \in I_m \\ \sigma_i^2(Q) & \leq & \gamma_i^2, \quad i \in I_\sigma \\ M_i(Q) & \leq & \delta_i^2, \quad i \in I_M \\ Y_i(Q) & \geq & \alpha_i > 1/2, \quad i \in I_Y \end{array}$$

where I_m, I_σ, I_M, and I_Y are subsets of $\{1, 2, \ldots, q\}$. Notice that more general problems can be obtained by considering linear combinations of m_i:s and positive linear combinations of σ_i:s and M_i:s, respectively. For notational simplicity the details are not given.

4 Solution

In this section the control problems of the previous section will be recast as SOCP problems. It is fairly straight forward to see that the constraints common to all optimization problems can be expressed as

$$\begin{array}{rcl} 0 & \leq & \beta_i - H_i(Q)\bar{w}, \quad i \in I_m \\ \left\| \sqrt{\Sigma} H_1^T(Q) \right\|_2 & \leq & \gamma_i, \quad i \in I_\sigma \end{array}$$

$$
\begin{aligned}
\left|\left|\sqrt{\Sigma + \bar{w}\bar{w}^T} H_i^T(Q)\right|\right|_2 &\leq \delta_i, \quad i \in I_M \\
\Phi^{-1}(\alpha_i) \left|\left|\sqrt{\Sigma} H_i^T(Q)\right|\right|_2 &\leq \bar{z}_i - H_i(Q)\bar{w}, \quad i \in I_Y
\end{aligned}
$$

The additional expressions for the different optimization problems are
Problem 1:

$$\text{minimize } t \quad \text{s.t.} \quad 0 \leq t - H_1(Q)\bar{w}$$

Problem 2:

$$\text{minimize } t \quad \text{s.t.} \quad \left|\left|\sqrt{\Sigma} H_1^T(Q)\right|\right|_2 \leq t$$

Problem 3:

$$\text{minimize } t \quad \text{s.t.} \quad \left|\left|\sqrt{\Sigma + \bar{w}\bar{w}^T} H_1^T(Q)\right|\right|_2 \leq t$$

Problem 4:

$$\text{maximize } t \quad \text{s.t.} \quad t \left|\left|\sqrt{\Sigma} H_1^T(Q)\right|\right|_2 \leq \bar{z}_1 - H_1(Q)\bar{w}$$

By the results in Appendix B problems 1–3 are all SOCP problems in Q and t, since $\Phi^{-1}(\alpha_i) > 0$ for $\alpha_i > 1/2$. Furthermore Problem 4 is an SOCP feasibility problem in Q for any fixed value of $t > 0$. Hence it can be solved by bisectioning over $t > 0$. Notice that there exists a Q satisfying the inequality for $t > 0$ if and only if there exist a Q such that $Y_1(Q) > 1/2$.

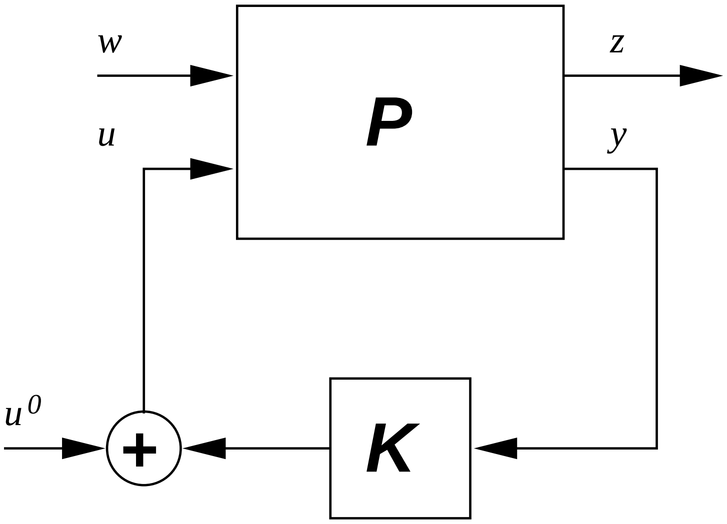

Figure 2: Model of the closed loop control system with an affine controller.

5 Affine Controllers

In this section it will be shown how the model of Section 2 can be used to accommodate not only linear feedback but also affine feedback, see Figure 2. The equations for this are:

$$\begin{aligned} \begin{bmatrix} z \\ y \end{bmatrix} &= P \begin{bmatrix} u \\ w \end{bmatrix} \\ u &= Ky + u^0 \end{aligned}$$

Define $\bar{K}$ and $\bar{y}$ via

$$u = \begin{bmatrix} K & u^0 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} \triangleq \bar{K}\bar{y}$$

and $\bar{P}_{zw}$, $\bar{P}_{yu}$, $\bar{P}_{yw}$, and $\bar{w}$ via

$$\begin{bmatrix} z \\ \bar{y} \end{bmatrix} = \begin{bmatrix} P_{zu} & P_{zw} & 0 \\ P_{yu} & P_{yw} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ 1 \end{bmatrix} \triangleq \begin{bmatrix} P_{zu} & \bar{P}_{zw} \\ \bar{P}_{yu} & \bar{P}_{yw} \end{bmatrix} \begin{bmatrix} u \\ \bar{w} \end{bmatrix}$$

from which it follows that the affine controller can be cast in the linear framework by augmenting y and w appropriately. Notice how this formulation also covers the case $u = u^0$ as a special case. The different strategies will in the sequel be called

- $u = u^0$: feed-forward
- $u = Ky$: feedback
- $u = Ky + u^0$: feedback/feed-forward

Trade-Off between Yield and Production Cost

This example is designed to demonstrate the trade-off between yield and production cost when the latter is proportional to the energy of the control variable. The plant is depicted in Figure 3, and it can be described with the following set of equations:

$$\begin{aligned} x &= Gu + v \\ y &= x + e \end{aligned}$$

where $G \in \mathbb{R}^{p \times m}$ and where v and e are independent random vectors of appropriate dimensions with means $\bar{v}$ and $\bar{e}$ and covariances Σ_v and Σ_e, respectively. The objective to minimize is given by

$$M = \sum_{i=1}^{m} M_i, \quad M_i = \mathrm{E}\left\{u_i^2\right\}, \; i = 1, 2, \ldots, m$$

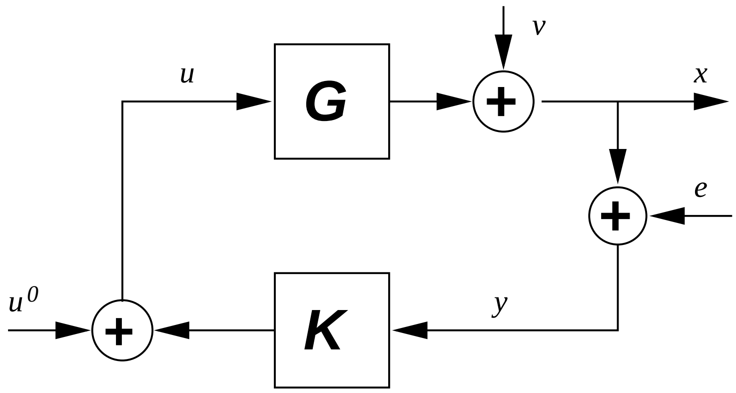

Figure 3: Model of the closed loop control system for trade-off between yield and production cost.

and the constraints are given by $Y_{m+i} = \text{Prob}\{x_i \le \bar{z}_i\} \ge \alpha > 1/2,\ i = 1, 2, \ldots, p$. This can be cast in the framework of sections 2 and 3 by defining

$$\begin{aligned} P_{zu} &= \begin{bmatrix} I \\ G \end{bmatrix}; \quad P_{zw} = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \\ P_{yu} &= G; \quad P_{yw} = [\, I \quad I \,] \end{aligned}$$

and $\bar{w}^T = [\, \bar{v}^T \quad \bar{e}^T \,]$, and $\Sigma = \text{diag}(\Sigma_v, \Sigma_e)$. The matrix G has been taken to have uniformly distributed random entries on $[0, 1]$. In the same way the mean $\bar{v}$ has uniformly distributed random entries on $[0, 1]$, and the covariances Σ_v and Σ_e are given by $\Sigma_v = D^T D$ and $\Sigma_e = E^T E$, where D and E are $p \times p$ matrices with uniformly distributed random entries on $[0, 1]$. The mean of e is given by $\bar{e} = 0$, and $\bar{z}_i = \bar{v}_i + \kappa\sqrt{e_i^T \Sigma_v e_i}$. The number of variables in this example is m for the feed-forward case, mp for the feedback case, and $m(p+1)$ for the feedback/feed-forward case. The number of constraints are $p + 1$. The optimal values of M for $m = 10$, $p = 20$, $\kappa = 1.0$, and values of $\alpha_1 = 0.51, 0.52, \ldots, 0.98$ are shown in Figure 4. It is seen that the pure feed-forward design is the most costly design and that the combined feedback/feed-forward design is the cheapest one. Notice that the trade-off curves not necessarily are concave, which, however, is the case in this example. The computational time was about 8 hours on a reasonably fast computer.

6 Worst Case Exogenous Inputs

In this section it will be shown how to consider worst case exogenous inputs. To this end let w_k define a set of Gaussian random vectors with means $\bar{w}_k$

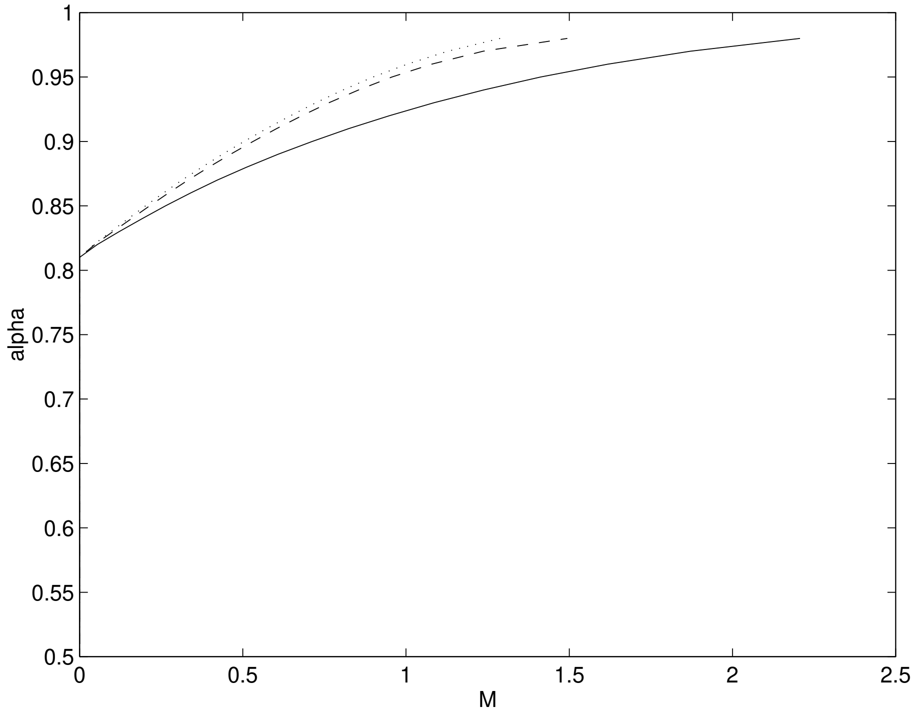

Figure 4: Maximal yield versus cost for feed-forward—solid line, feedback—dashed line, and feedback/feed-forward— dotted line

and covariances $\Sigma_k,\ k \in I_P$, where I_P is a finite index set. Problem 1 is then generalized to

$$\text{minimize}_Q\ \text{maximize}_k\ m_1(Q,k)$$

subject to

$$\begin{aligned} m_i(Q,k) &\leq \beta_i, \quad i \in I_m, k \in I_P \\ \sigma_i^2(Q,k) &\leq \gamma_i^2, \quad i \in I_\sigma, k \in I_P \\ M_i(Q,k) &\leq \delta_i^2, \quad i \in I_M, k \in I_P \\ Y_i(Q,k) &\geq \alpha_i > 1/2, \quad i \in I_Y, k \in I_P \end{aligned}$$

which can be rewritten as

$$\text{minimize } t \quad \text{s.t.} \quad 0 \leq t - H_1(Q)\bar{w}_k,\ k \in I_P$$

and

$$0 \leq \beta_i - H_i(Q)\bar{w}_k, \quad i \in I_m, k \in I_P$$

$$
\begin{aligned}
\left\|\sqrt{\Sigma_k} H_1^T(Q)\right\|_2 &\le \gamma_i, \quad i \in I_\sigma, k \in I_P \\
\left\|\sqrt{\Sigma_k + \bar{w}_k \bar{w}_k^T} H_i^T(Q)\right\|_2 &\le \delta_i, \quad i \in I_M, k \in I_P \\
\Phi^{-1}(\alpha_i)\left\|\sqrt{\Sigma_k} H_i^T(Q)\right\|_2 &\le \bar{z}_i - H_i(Q)\bar{w}_k, \quad i \in I_Y, k \in I_P
\end{aligned}
$$

The other problems can be generalized in a similar fashion.

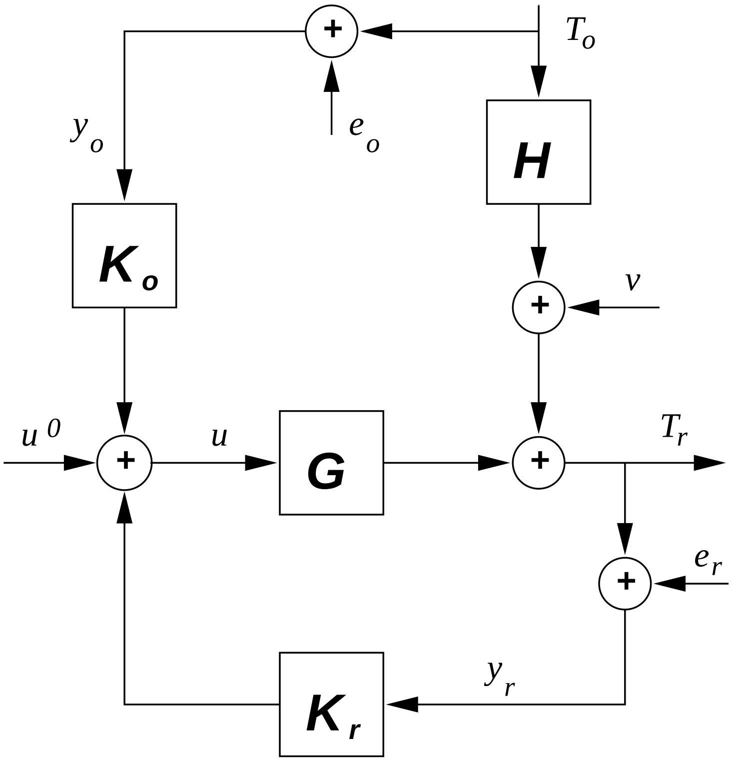

Figure 5: Model of the closed loop control system for the heating example.

Heating of a Room

Consider the model describing heating of a room depicted in Figure 5, which can be described with the equations:

$$
\begin{aligned}
T_r &= Gu + HT_{ok} + v \\
y_r &= T_r + e_r \\
y_o &= T_o + e_o
\end{aligned}
$$

where $T_r \in \mathbb{R}$ is the room temperature, $T_{ok} \in \mathbb{R},\ k \in I_P$ defines a set of out-door temperatures $\{T_{ok} : k \in I_P\}$, $v \in \mathbb{R}$ is a random variable with zero mean and variance σ_v^2, $y_r \in \mathbb{R}$ is the measurement of the room temperature, $e_r \in \mathbb{R}$ is a random variable with zero mean and variance $\sigma_{e_r}^2$, $y_o \in \mathbb{R}$ is the measurement of the out-door temperature, and where $e_o \in \mathbb{R}$ is a random variable with zero mean and variance $\sigma_{e_o}^2$. It is assumed that the random variables are independent of one another. The control signal $u \in \mathbb{R}$ is given by the feedback/feed-forward structure

$$u = K_r y_r + K_o y_o + u^0$$

The objective is to choose K_r, K_o and u^0 as to minimize the maximal control

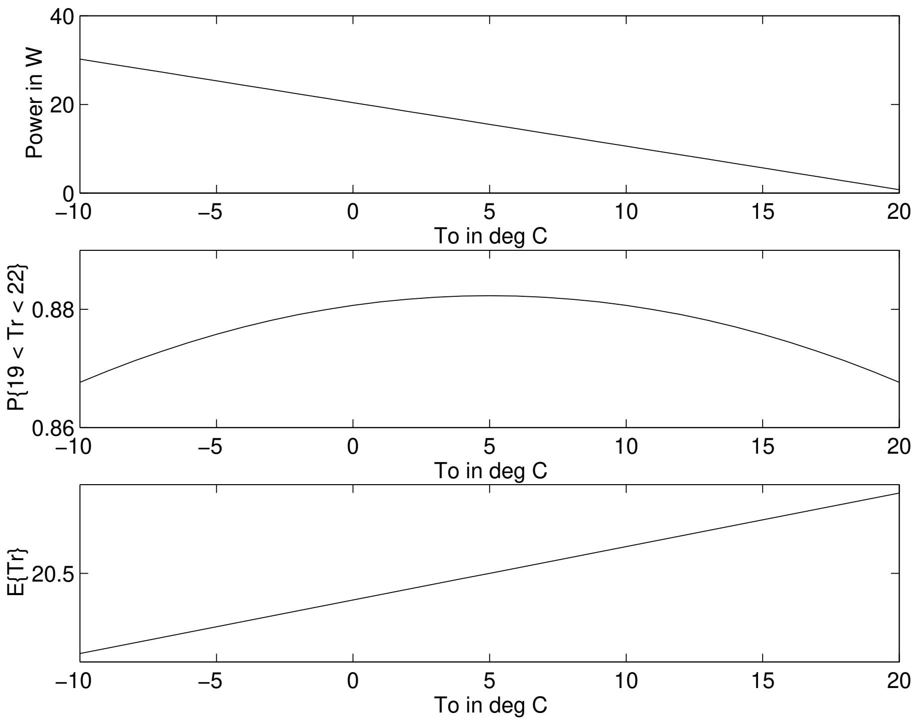

Figure 6: Objectives and expected value of in-door temperature as functions of the out-door temperature.

energy with respect to the set of out-door temperatures. The control energy is proportional to $m_1(k) = \mathrm{E}\{u(k)\}$ as long as heating and not cooling is being performed. It would be desirable to perform the optimization subject to the constraint $Y(k) = \mathrm{Prob}\left\{\underline{T}_r \leq T_r(k) \leq \bar{T}_r\right\} \geq \alpha > 0$. This is however not possible, but a sufficient condition is that $Y_2(k) = \mathrm{Prob}\left\{\underline{T}_r \leq T_r(k)\right\} \geq \frac{1}{2}(\alpha+1)$

and $Y_3(k) = \text{Prob}\left\{T_r(k) \leq \bar{T}_r\right\} \geq \frac{1}{2}(\alpha+1)$, which are the constraints that will be used. This problem can be cast in the framework of the previous sections by defining

$$
\begin{aligned}
P_{zu} &= \begin{bmatrix} 1 \\ -G \\ G \end{bmatrix}; \quad P_{zw} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -H & -1 & 0 & 0 \\ H & 1 & 0 & 0 \end{bmatrix} \\
P_{yu} &= \begin{bmatrix} G \\ 0 \end{bmatrix}; \quad P_{yw} = \begin{bmatrix} H & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\
K &= [\, K_r \quad K_o \,]; \quad w_k^T = [\, T_{ok} \quad v \quad e_r \quad e_o \,]
\end{aligned}
$$

and $\bar{z} = [\, -\underline{\mathrm{T}}_r \quad \bar{T}_r \,]$, and where the mean and covariance of w_k are given by $\bar{w}_k^T = [\, T_{ok} \quad 0 \quad 0 \quad 0 \,]$ and $\Sigma_k = \text{diag}(0, \sigma_v^2, \sigma_{e_r}^2, \sigma_{e_o}^2)$, respectively. The solution for $G = 1^{\text{o}}\text{C/W}$,$H = 1$, $\underline{\mathrm{T}}_r = 19^{\text{o}}\text{C}$, $\bar{T}_r = 22^{\text{o}}\text{C}$, $\alpha = 0.80$, $\sigma_v = 3^{\text{o}}\text{C}$, $\sigma_{e_r} = 1^{\text{o}}\text{C}$, $\sigma_{e_o} = 2^{\text{o}}\text{C}$, and the set of out-door temperatures $\{-10, 0, 10, 20\}^{\text{o}}\text{C}$ is given by

$$u = -12.1 y_r - 0.76 y_o + 266$$

The closed loop behavior as a function of the out-door temperature is shown in Figure 6.

7 Conclusions

In this paper is at been described how linear static control problems involving first and second moments and yield objectives can be solved with efficient solvers for SOCP problems. Both feed-forward, feedback and feedback/feed-forward control has been addressed. It has been seen that about 200 variables are possible to optimize with respect to about 20 constraints. The computational time is a few minutes on a standard computer. Also it has been demonstrated how the optimization can be made robust with respect to different distributions of the exogenous inputs.

Acknowledgements

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Appendix A

The control variable u can be computed in two different ways. In case

$$(I + QP_{yu})K = Q$$

can be solved for K it is given by $u = Ky$. Notice that for any $\epsilon > 0$ there is a $\bar{Q}$ such that $||H(\bar{Q}) - H(Q)|| \leq \epsilon$ and such that $\left(I + \bar{Q}P_{yu}\right)\bar{K} = \bar{Q}$ has a solution. This means that it is possible to get arbitrarily close to any $H(Q)$ using the feedback structure $u = Ky$. However, the matrix K may be large in a certain subspace. There is a way to avoid this. Compute $\tilde{y}$ as

$$\tilde{y} \triangleq P_{yw}w = y_0 - P_{yu}u_0$$

for some arbitrary u_0, where y_0 is the corresponding measured output. Then compute u as

$$u = Q\tilde{y} = Q(y_0 - P_{yu}u_0)$$

How this relates to digital control is now going to be discussed. Assume that the optimal control signal designed for the static system is used in a digital control application using zero order hold, and that the sample interval is much larger than the computational time so that the control signal can be written to the D/A converters almost immediately after the measurement signal has been read from the A/D converters. Then the following set of equations are appropriate for describing the closed loop behavior:

$$\begin{aligned} z(k) &= P_{zu}u(k) + P_{zw}w(k) \\ y(k) &= P_{yu}u(k-1) + P_{yw}w(k) \\ u(k) &= Q(y(k) - P_{yu}u(k-1)) \end{aligned}$$

where k is time index, not to be confused with the k used to index the different probability measures in the previous section. Notice that the control signal can be written as

$$u(k) + QP_{yu}u(k-1) = Qy(k)$$

It is seen that the case when $I + QP_{yu}$ is not invertible corresponds to integral control, and hence it is not surprising that it cannot be implemented using just a static feedback controller $u = Ky$. By writing $\hat{y}(k) = P_{zu}u(k-1)$ it holds that $u(k) = Q\tilde{y}(k)$, where $\tilde{y}(k) = y(k) - \hat{y}(k)$ can be interpreted as the estimation error when estimating $y(k)$ with $\hat{y}(k)$. Notice that the observer is of dead-beat type, i.e. the estimation error is minimized as fast as possible. This follows from the fact that $\tilde{y}(k) = P_{yw}w(k)$. It is now clear that this problem setup is a special case of the general dynamic case described in [BB91]. Also notice that all the results in the main body of the paper can be extended to the dynamic case including plants and controllers with poles on the stability boundary. Finally it is concluded that the closed loop system is stable. This follows from the fact that $z(k) = H(Q)w(k)$, $u(k) = QP_{yw}w(k)$ and that $y(k) = P_{yu}QP_{zw}w(k-1) + P_{zw}w(k)$.

Appendix B

In this appendix the SOCP:s will be recast in their standard form

$$\min_x f^T x$$

subject to

$$||A_i x + b_i||_2 \leq c_i^T x + d_i; \quad i \in I_L$$

To this end write

$$Q = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_p \end{pmatrix}$$

where $Q_i \in \mathbb{R}^{m \times 1}$ and define

$$q = \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_p \end{pmatrix}$$

Notice that

$$Q^T P_{zu}^T e_i = \begin{pmatrix} Q_1^T P_{zu}^T e_i \\ Q_2^T P_{zu}^T e_i \\ \vdots \\ Q_p^T P_{zu}^T e_i \end{pmatrix} = \begin{pmatrix} e_i^T P_{zu} Q_1 \\ e_i^T P_{zu} Q_2 \\ \vdots \\ e_i^T P_{zu} Q_p \end{pmatrix} = S_i q$$

where $S_i = \mathrm{diag}(e_i^T P_{zu})$ with p blocks. So with $\bar{A}_i(D) = DP_{yw}^T S_j$ and $\bar{b}_j(D) = DP_{zw}^T e_i$ it holds that

$$DH_i^T(Q) = DP_{zw}^T e_i + DP_{yw}^T Q^T P_{zu}^T e_i = \bar{A}_i(D)q + \bar{b}_i(D)$$

Now write

$$P_{yw}\bar{w} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_p \end{pmatrix}$$

Then it holds that

$$QP_{yw}\bar{w} = Tq$$

where

$$T = \begin{pmatrix} t_1 I_m & t_2 I_m & \cdots t_p I_m \end{pmatrix}$$

So with $\bar{c}_i^T = e_i^T P_{zu} T$ and $\bar{d}_i = e_i^T P_{zw}\bar{w}$ it holds that

$$H_i(Q)\bar{w} = e_i^T P_{zw}\bar{w} + e_i^T P_{zu} Q P_{yw}\bar{w} = \bar{c}_i^T q + \bar{d}_i$$

For Problem 4 define $x = q$ and for the other problems define $x^T = \begin{pmatrix} t & q^T \end{pmatrix}$. The rest of the boring book-keeping is left as an exercise to the reader. Notice that the extension to linear combinations discussed at the end of Section 3 is obtained by stacking more rows to A_i and b_i. Furthermore the extension to worst case distributions discussed in Section 6 is obtained by having one constraint for each distribution.