OBSTRUCTIONS TO UNIRATIONALITY IN POSITIVE CHARACTERISTIC

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Abstract

We develop new techniques to obstruct unirationality and uniruledness in positive characteristic. As applications, we construct a counterexample to a 1977 conjecture of Shioda that supersingular simply-connected surfaces are unirational, find examples of hyperbolic surfaces in positive characteristic, and prove that almost all reductions of Hilbert modular varieties are not uniruled.

Dedication

If they be two, they are two so
   As stiffe twin compasses are two,
Thy soule the fixt foot, makes no show
   To move, but doth, if th'other doe.

And though it in the centre sit,
   Yet, when the other far doth rome,
It leanes, and hearkens after it,
   And grows erect, as that comes home.

Such wilt thou be to mee, who must
   Like th'other foot, obliquely runne;
Thy firmnes makes my circle just,
   And makes me end, where I begunne.

John Donne

To Ming

Acknowledgements

> He took the golden Compasses, prepar'd
> In Gods Eternal store, to circumscribe
> This Universe, and all created things:
> One foot he center'd, and the other turn'd
> Round through the vast profunditie obscure,
> And said, thus farr extend, thus farr thy bounds,
> This be thy just Circumference, O World.
>
> John Milton

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I often find the only way to really learn mathematics is to take some pretense of authority and write a chapter about it. Therefore, I thank you, dear reader, for providing me an excuse to actually learn the things described in this text.

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Chapter 1

Introduction

> It was stated at the outset, that this system would not be here, and at once, perfected. You cannot but plainly see that I have kept my word. But I now leave my cetological System standing thus unfinished, even as the great Cathedral of Cologne was left, with the crane still standing upon the top of the uncompleted tower. For small erections may be finished by their first architects; grand ones, true ones, ever leave the copestone to posterity. God keep me from ever completing anything. This whole book is but a draught—nay, but the draught of a draught. Oh, Time, Strength, Cash, and Patience!
>
> Herman Melvile

1.1 Rationality Problems

One of the oldest and most fruitful circles of questions in algebraic geometry is rationality problems. These ask when a system of polynomials can be solved by rational functions. In the modern language of birational geometry, this is equivalent to asking when a variety is birationally equivalent to $\mathbb{P}^n$. The fundamental idea of birational geometry is that the incredible rigidity of algebraic maps means that the equivalence relation defined by cutting and pasting subvarieties along algebraic maps preserves much of the complexity of the original variety. Precisely, two varieties are said to be *birationally equivalent*, or *birational*, if there are dense Zariski open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$. Many natural geometric constructions – for example, the blowup along a subvariety – preserve the birational equivalence. The Italian school of algebraic geometry richly developed the birational geometry of surfaces in the late 19th and early 20th centuries culminating in the Enriques–Kodaira classification of algebraic surfaces.

In dimension ≤ 2, the situation for rationality is fairly well-understood. For example, an algebraic surface is rational if and only if it is the blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a Hirzebruch surface. Furthermore, Castelnuovo's theorem gives a precise numerical criterion for when a surface is rational. However, the situation in dimension ≥ 3 is already much more subtle and is an area of active research.

In this thesis, we will consider weaker properties similar to rationality, for instance, *unirationality* and *uniruledness*, which are introduced in the subsequent sections. The main object of our study will be positive characteristic where these notions behave much more wildly than we are used to over the complex numbers. The main result is a new technique to prove that certain varieties over fields of positive characteristic are not uniruled and, *a fortiori*, not unirational.

1.2 Unirationality: examples, obstructions and open problems

When varieties are provably irrational, there remain interesting gradations in their complexity. For example, we can ask if the variety is *unirational* meaning its points are parametrized by rational functions. This property ensures that many solutions to the defining equations can be easily written down.

Definition 1.2.1. Let X be a variety over k. We say that X is *unirational* (over k) if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$.

We can always arrange $n = \dim X$ by taking general linear sections of the source projective space. Unirationality can be viewed as a weaker notion of a polynomial system being "solvable" by rational functions where we do not require the parametrization by rational functions to give a precise 1-to-1 description of the set of solutions. In the 19th century it was understood that the Riemann–Hurwitz formula shows that – over the complex numbers – only genus 0 curves are unirational. From the function field perspective, Lüroth generalized this result to curves over any field. In modern notation he proved:

Theorem 1.2.2 (Lüroth, 1876)**.** *If X/k is a curve over a field of any characteristic then X is unirational iff X is rational. If moreover X is smooth and complete then $X \cong \mathbb{P}^1_k$.*

Lüroth asked if the higher-dimensional analog was true. In field-theoretic terms he asked whether, given a finitely generated field extension K/k contained in a purely-transcendental extension $k(t_1, \ldots, t_n)$, must K/k itself be purely transcendental. This question came to be known as the Lüroth problem. In more modern language, his problem asks whether a unirational variety of dimension n over k is always rational. The complex surface case, when $k = \mathbb{C}$ and $n = 2$, was also settled affirmatively in the 19th century by Castelnuovo.

Theorem 1.2.3 (Castelnuovo, 1896)**.** *Let $X/\mathbb{C}$ be a smooth projective surface. Then the following are equivalent,*

1. *X is unirational*

2. *X is rational*

3. *$p_2(X) := \dim_{\mathbb{C}} H^0(X, \omega_X^{\otimes 2}) = 0$ and $b_1(X) := \dim_{\mathbb{C}} H^1(X, \mathbb{C}) = 0$.*

Castelnuovo proved the equivalence by giving a purely numerical criterion – statement (3) – for rationality. He then showed that unirationality implies these numerical invariants vanish. The numerical invariants in question, the second *pluri-genus* p_2 and the first *Betti number*, fit into a more general framework of *obstructions* to unirationality (over $\mathbb{C}$) that apply in any dimension.

Definition 1.2.4. The m-th *pluri-genus* of X is the dimension of the space of sections of $\omega_X^{\otimes m}$,

$$p_m(X) := \dim_k H^0(X, \omega_X^{\otimes m}).$$

This is a birational invariant of smooth proper varieties. More generally, we can look at tensor forms i.e. sections of $(\Omega_X)^{\otimes m}$ and these can be pulled back along any rational map $f : X \dashrightarrow Y$ to obtain an obstruction to unirationality.

Proposition 1.2.5. *Let k be of characteristic zero and X a proper k-variety. Then if $\omega \in H^0(X, \Omega_X^{\otimes m})$ is a nonzero form for $m > 0$ then X is not unirational.*

Proof. Indeed, if $f : \mathbb{P}^n \dashrightarrow X$ is a unirational parametrization with $n = \dim X$ then f is generically étale. Therefore, on the locus $U \subset \mathbb{P}^n$ where f is a morphism, $f^* \omega$ is not zero identically. Furthermore, since X is proper, f is defined away from codimension ≥ 2 so $f^* \omega$ extends to a nonzero section of $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{\otimes m})$. No such section exists. $\square$

It turns out to be vital that k has characteristic zero here as we will see in the subsequent section. The second invariant $b_1(X) := \dim_{\mathbb{C}} H^1(X, \mathbb{C})$ also fits into a larger story. The more refined obstruction is the fundamental group $\pi_1^{\text{top}}(X_{\mathbb{C}})$ or, in positive characteristic, the étale fundamental group $\pi_1^{\text{ét}}(X_{\bar{k}})$.

Theorem 1.2.6 (Serre (1959) [Ser59], Ekedahl (1983) [Eke83])**.** *Let X be a unirational smooth projective variety over an algebraically closed field k.*

1. *if k has characteristic zero then $\pi_1^{\text{ét}}(X) = 1$ and $\pi_1^{\text{top}}(X_{\mathbb{C}}) = 1$ for any $k \hookrightarrow \mathbb{C}$*
2. *if k has characteristic $p > 0$ then $\pi_1^{\text{ét}}(X)$ is a finite group of order coprime to p.*

In the subsequent section, we will see examples (see Example 1.3.7) showing that the second statement is actually sharp. Therefore, both the conditions $p_2(X) = 0$ and $b_1(X) = 0$ fit into a larger story of *obstructions* to unirationality over $\mathbb{C}$. What turns out to be special to dimension 2 is that the vanishing of these obstructions $p_2 = 0$ and $b_1 = 0$ exactly characterizes rationality. Indeed, in dimension 3, a slew of unirational but non-rational examples were discovered in the year 1971 (some appearing in 1972). These examples are: any smooth cubic 3-fold by the work of Clemens and Griffiths [CG72], unirational smooth quartic 3-folds whose non-rationality follows from the work of Iskovskikh and Manin [IM71], and certain Brauer–Severi varieties over rational surfaces constructed by Artin and Mumford [AM72]. As illustrated by the complexity and variety of the cited works, proving that a variety is not rational is often quite subtle. The situation is even more stark for the problem of proving a variety is not unirational. For example, even in characteristic zero there is no known example of a rationally-connected variety that we can prove to be non-unirational.

1.3 The Landscape in Positive Characteristic and Shioda's Conjecture

One of the most striking aspects of geometry in positive characteristic is that a variety of general type may be unirational. Indeed, the pullback argument in the proof of Proposition 1.2.5 fails completely in positive characteristic because of the existence of inseparable maps for which f^* is identically zero. If the map $f : \mathbb{P}^n \dashrightarrow X$ were separable then f^* would be injective just like in characteristic zero so the argument would go through. However, Zariski constructed examples of surfaces over $\mathbb{F}_p$ with an inseparable unirational parametrization but no separable one due to the existence of pluri-forms.

Example 1.3.1 (Zariski)**.** [Zar58, §8] Consider a surface $X/\mathbb{F}_p$ defined as a desingularization of some compactification of the affine surface

$$\{z^p = f(x, y)\} \subset \mathbb{A}^3_{\mathbb{F}_p}$$

for some $f \in \mathbb{F}_p[x,y]$. If we choose f generically of sufficiently large degree, X will be of general type. However, notice that if we adjoin p^{th}-roots s,t of x and y then the equation becomes

$$z^p = f(s^p, t^p) = f(s,t)^p$$

therefore $k(X) \subset k(s,t)$ given by $z = f(s,t)$ and $x = s^p$ and $y = t^p$. This shows that X is always unirational via the inseparable map $\mathbb{A}^2 \dashrightarrow X$ given by

$$x \mapsto s^p, \quad y \mapsto t^p, \quad z \mapsto f(s,t).$$

This demonstrates that unirationality in positive characteristic is a phenomenon that persists even for quite "complex" varieties from the perspective of the usual classification of surfaces. This shows there are even fewer tools available to us in positive characteristic to prove that varieties are not unirational. The main goal of this thesis is to develop some new methods to accomplish this. The main ideas of this work are in conversation with two remarkable results from 1977: Shioda's classification of unirational Kummer surfaces and Bogomolov's boundedness theorem for curves on general type surfaces with $c_1^2 > c_2$. We will develop obstructions inspired by the techniques of the latter work and use them to answer questions asked in the former.

1.3.1 Shioda's Examples

Zariski's surfaces are cooked up precisely to be unirational in the specific characteristic p which is baked into the form of the equation. However, Shioda found even more remarkable examples of smooth projective general type surfaces over $\operatorname{Spec} \mathbb{Z}$ which nonetheless have unirational reduction at infinitely many primes p. Shioda showed the following by an inspired computation.

Theorem 1.3.2 (Shioda (1974), Shioda–Katsura (1979))**.** *[Shi74, SK79] Let F_m be the degree m Fermat surface over $\mathbb{F}_p$ (for $p > 2$, an odd prime) given by the equation*

$$\{X^m + Y^m + Z^m + W^m = 0\} \subset \mathbb{P}^3_{\mathbb{F}_p}$$

in characteristic p. The following are equivalent,

1. *F_m is unirational over $\overline{\mathbb{F}_p}$*
2. *$-1 \in \langle p \rangle$ inside $(\mathbb{Z}/m\mathbb{Z})^\times$ i.e. $p^\nu \equiv -1 \mod m$ for some ν*
3. *$H^2_{\text{ét}}(F_m, \mathbb{Q}_\ell)$ is spanned by algebraic cycles.*

This congruence condition is satisfied for all primes in certain arithmetic progressions and hence F_m is unirational at a set of primes of positive natural density by Dirichlet's theorem. In fact, as we change the degree m, the density can be arbitrarily close to 1. The second and third conditions are a manifestation of the Tate conjecture. Recall that the cycle map in étale cohomology

$$\gamma : Z^i(X) \to H^{2i}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(i))$$

lands inside the $\mathbb{Q}_\ell$-subspace $H^{2i}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\bar{k}/k)}$ of Galois invariants. For $k = \mathbb{F}_q$, this is the subspace where Frob_q acts via $q^i \cdot \mathrm{id}$ (due to the presence of the Tate twist). The Tate conjecture predicts equality between this subspace and the $\mathbb{Q}_\ell$-span of $\operatorname{im} \gamma$ i.e. the subspace spanned by algebraic cycles. For a variety defined over $k = \mathbb{F}_q$, the Tate conjecture thus predicts that $H^{2i}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(i))$ is spanned by algebraic cycles (defined over $\bar{k}$) if and only if Frob_q^n acts by $(q^n)^i \cdot \mathrm{id}$ for some n or equivalently if all the eigenvalues of Frob_q on $H^{2i}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(i))$ are of the form $\zeta \cdot q^i$ for ζ a root of unity. In analogy with the case of elliptic curves, we call this property of a Galois representation *supersingularity* at p. For Galois representations of geometric origin, this property is computationally accessible by point counting and the Weil conjectures.

Shioda proved that F_m is supersingular at p, i.e. the eigenvalues of Frob_p on $H^2_{\text{ét}}(F_m, \mathbb{Q}_\ell)$ are of the form $\zeta \cdot p$ for ζ a root of unity, if and only if $p^\nu \equiv -1 \mod m$ for some ν. The proof appeals to Weil's 1949 work – in which his eponymous conjectures were first introduced – that develops point counting formulae in terms of Jacobi and Gauss sums [Wei49]. Furthermore, whenever this supersingularity condition holds, Shioda used an ingenious change of variables to write down an explicit unirational inseparable parametrization of F_m.

To prove the reverse implication, Shioda needed an obstruction; this comes from the Galois-structure of étale cohomology discussed above.

Definition 1.3.3. We say a surface X is *Shioda-supersingular* if $H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ is spanned by algebraic cycles (possibly defined over $\bar{k}$).

Lemma 1.3.4. *If a variety X over k is unirational, it is (Shioda) supersingular.*

Proof. The rational map $f : \mathbb{P}^2 \dashrightarrow X$ can be resolved by blowing up $\mathbb{P}^2$ at finitely many points (possibly defined over k'/k) to obtain a morphism $\widetilde{f} : \mathrm{Bl}\mathbb{P}^2 \to X$. This dominant morphism induces a Galois-equivariant injection

$$H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell) \hookrightarrow H^2_{\text{ét}}((\mathrm{Bl}\mathbb{P}^2)_{\bar{k}}, \mathbb{Q}_\ell)$$

Over k', the Galois-module $H^2_{\text{ét}}((\mathrm{Bl}\mathbb{P}^2)_{\bar{k}}, \mathbb{Q}_\ell)$ is spanned by algebraic cycles corresponding to the hyperplane and exceptional divisors. Pushing forward cycles generating the image of

$\widetilde{f}^*$, we obtain that $H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ is also generated by algebraic cycles over k' and, *a fortiori* X is supersingular. □

Therefore, Shioda concluded that F_m is not unirational over $\overline{\mathbb{F}_p}$ unless $p^\nu \equiv -1 \mod m$ for some ν. In 1977, Shioda observed a similar phenomenon for Kummer surfaces. For an abelian surface A, let $\mathrm{Km}(A)$ be the associated Kummer surface obtained by resolving the 16 singular points of the quotient $A/\pm 1$. Shioda proved the following theorem.

Theorem 1.3.5 (Shioda, 1977)**.** *[Shi77] Let A be an abelian surface over a field k of characteristic > 2. Then the following are equivalent:*

1. $\mathrm{Km}(A)$ *is unirational*
2. $\mathrm{Km}(A)$ *is (Shioda) supersingular*
3. *A is Shioda supersingular*
4. *A is a supersingular abelian surface (isogenous to E^2 for E a supersingular elliptic curve).*

He also worked out some examples with Delsarte surfaces. Based on these examples, he conjectured:

Conjecture (Shioda, 1977). *Let X be a smooth projective surface over an algebraically closed field k of characteristic p with $\pi_1^{\text{ét}}(X) = 1$. Then X is unirational if and only if it is Shioda supersingular.*

Remark 1.3.6. In Theorem 1.2.6, we saw that unirationality in positive characteristic implies a *finite* fundamental group rather than a trivial one as is the case in characteristic zero. Therefore, one might ask why Shioda formulated his conjecture only in the simply-connected case. This is because Shioda was aware of counterexamples if the simply-connectedness assumption is relaxed to only require $\pi_1^{\text{ét}}(X)$ to be finite of order coprime to p. Indeed, in that setting, X is unirational if and only if its universal cover $\widetilde{X} \to X$ is unirational. However, $\widetilde{X}$ may have additional classes in H^2 not spanned by algebraic cycles. Such counterexamples are not particularly convincing; one should, in that case, consider the conjecture for $\widetilde{X}$ not for X.

For an explicit example, Shioda used the Godeaux surface constructed as a free μ_5 quotient of the Fermat X_5 of degree 5 which is supersingular in all characteristics but unirational if and only if X_5 is i.e. when $p^\nu \equiv -1 \mod 5$ for some ν.

Example 1.3.7. Let $G = \mu_5$ act on F_5 by $\lambda \cdot [X : Y : Z : W] = [X : \lambda Y : \lambda^2 Z : \lambda^3 W]$. This action is free and hence $X = F_5/G$ is a smooth projective surface with $\pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_p}) = \mathbb{Z}/5\mathbb{Z}$ called the Godeaux surface. Shioda's result shows that this surface is unirational for infinitely many primes $p \neq 5$. Hence the positive characteristic part of Theorem 1.2.6 is sharp.

Remark 1.3.8. One reason, known to Shioda, to doubt the validity of this conjecture is the failure of a particular characteristic zero analog. The definition of Shioda supersingular also makes sense in characteristic zero where it forces the Hodge structure on H^2 to be trivial in the sense that $H^2 = H^{1,1}$. This is also a reasonable analog of the weaker property of supersingularity. Indeed, divisibility properties of the Frob-eigenvalues are analogous to classes lying in a certain step of the Hodge filtration. In particular, we think of Frob-eigenvalues on H^2 not divisible by p as analogous to classes in $H^{2,0} \oplus H^{0,2}$. Therefore, supersingularity should correspond to having a trivial Hodge structure $H^2 = H^{1,1}$ so that H^2 is spanned by algebraic cycles by the Lefschetz $(1,1)$-theorem.

Severi asked if a complex surface X with $\pi_1^{\text{top}}(X) = 1$ and $H^2(X, \mathbb{C}) = H^{1,1}(X)$ must be unirational (hence rational). This is false. There are counterexamples due to Dolgachev of elliptic surfaces [Dol66] (cf. [Dol10] in English) and further examples due to Barlow [Bar85] of general-type simply-connected surfaces with $K_X^2 = 1$ and Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 9 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

However, the existence of inseparable unirational parametrization in positive characteristic makes this analogy less convincing. It is a very interesting question to ask if the reduction mod p of a Barlow surface satisfies Shioda's conjecture. I suspect not.

The first goal of this thesis is to exhibit a counterexample to Shioda's conjecture. In fact, we will do more: we will exhibit examples that are not even covered by genus 0 or genus 1 curves. A variety covered by rational curves is called *uniruled*. Precisely, X is uniruled if there is a dominant rational map $f : W \times \mathbb{P}^1 \dashrightarrow X$ such that f does not factor through $W \times \mathbb{P}^1 \to W$. In Chapter 3 we will prove the following theorem by exhibiting such a surface X contradicting the Shioda conjecture.

Theorem 1.3.9. *(cf. chapter 3 Theorem 3.1.1) There exists a smooth projective surface over $\overline{\mathbb{F}}_p$ with the following properties:*

1. *$H^2_{\text{ét}}(X, \mathbb{Q}_\ell)$ is spanned by algebraic cycles*

2. *$\pi_1^{\text{ét}}(X) = 1$*

3. *X is not unirational, in fact X is not uniruled or covered by genus 1 curves.*

In cases where standard obstructions to rationality fail, a sequence of intricate invariants has been employed to detect (non)rationality with tremendous success since the 1970s (cf. [IM71, AM72, CG72, Kol95, Voi15, Tot16, HPT19, Sch19]). The same cannot be said of unirationality. For example, proving the non-unirationality of any single rationally-connected variety remains a major open problem as does the unirationality problem for many varieties in positive characteristic. In the absence of now standard techniques developed up to the 1960s (étale π_1, étale cohomology, tensor forms in characteristic zero), to the author's knowledge, this is the first new demonstration that a variety is **not** unirational or even uniruled.

1.4 Unirationality of Moduli Spaces

Another motivation to go beyond the obstructions arising from étale cohomology is to prove the non-uniruledness of moduli spaces in characteristic p. In characteristic 0, pluri-forms have a long and storied history establishing non-uniruledness of moduli spaces such as $M_{g,n}$. These methods are insufficient to make any headway in positive characteristic because they cannot rule out inseparable maps.

The (non)-unirationality of moduli spaces is particularly interesting because the abstract birational geometry has implications for the behavior of the generic objects parametrized in the universal family. A moduli space is unirational exactly when the generic object it parametrizes can be "written down" by an explicit algebraic recipe. Explicitly, this means there is a versal family over a rational variety whose generic point gives the generic object written down in terms of "free parameters", the transcendental coordinates $t_1, \dots, t_n$ on the fraction field $k(t_1, \dots, t_n)$.

Example 1.4.1. Every genus 2 curve is hyperelliptic meaning there is a degree 2 map $C \to \mathbb{P}^1$ ramified over exactly 6 branch points in $\mathbb{P}^1$. This means it can be written in the form

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

choosing coordinates on $\mathbb{P}^1$ so that the branch points occur at $0, 1, \infty, \lambda_1, \lambda_2, \lambda_3$. Since the parameters $\lambda_1, \lambda_2, \lambda_3$ are freely varying in an open set of $\mathbb{A}^3$, this shows that M_2 is 3-dimensional and unirational.

For small values of g, the moduli space of curves M_g is known to be unirational. However, a celebrated result of Harris and Mumford proves that the moduli space of curves of genus g is of general type for $g > 23$. In characteristic zero, this implies that M_g is not unirational or even uniruled. More generally, the following result shows that only finitely many $M_{g,n}$ are uniruled.

Theorem 1.4.2 (Harris–Mumford (1982) and many others)**.** *[HM82, Har84, Log03, FJP24, FJP25, dP25] $\kappa(M_{g,n}) \geq 0$ for $g > 22$ or $n > 15$. Hence, in these cases, $(M_{g,n})_{\mathbb{C}}$ is not uniruled.*

A particularly nice case is $g = 1$ and $n \geq 11$ where the cusp form Δ proves $\kappa(M_{1,n}) \geq 0$. Sawin used more delicate arguments using the Galois representation on étale cohomology as an obstruction to prove certain cases of the non-uniruledness of $M_{1,n}$ in positive characteristic.

Theorem 1.4.3 (Sawin, 2019)**.** *[Saw19] $(M_{1,n})_{\overline{\mathbb{F}}_p}$ is not uniruled in the range $n \geq p \geq 11$.*

Even in this case, for fixed n, we do not know if $M_{1,n}$ is unirational or uniruled in all but finitely many characteristics. To the author's knowledge, next to nothing is known about the uniruledness or unirationality of $M_{g,n}$ for $g > 1$ in positive characteristic.

The difficulty of using techniques based on étale cohomology to prove uniform obstruction results over different primes partially inspired the investigation of the non-cohomological obstructions we consider in this thesis. Our techniques do not yet apply to the moduli spaces of curves $M_{g,n}$. However, we are able to apply them to Hilbert modular varieties to obtain uniform obstruction results.

Theorem 1.4.4. *(cf. chapter 6 Theorem 6.6.1) Let $Y_0^F(N)$ be the Hilbert modular variety at level $\Gamma_0(N)$ associated to a totally real field F. Then if either N or the discriminant of F is sufficiently large, the reduction $Y_0^F(N)_{\overline{\mathbb{F}}_p}$ mod p is not uniruled or covered by genus 1 curves for all primes p unramified in F and not dividing N.*

In particular, the obstruction applies to all but finitely many characteristics simultaneously.

1.5 Hyperbolicity

Many of the techniques in this thesis were motivated by methods used to study hyperbolicity of complex manifolds. Our investigations go beyond rationality questions to problems more directly connected to hyperbolicity. Here we review some of the history, definitions, and main conjectures of this area.

1.5.1 The Kobayashi Pseudodistance

In complex analysis, hyperbolic spaces such as bounded symmetric domains often carry a canonical *Bergman metric* characterized by the remarkable property that every biholomorphism is an isometry. For example, the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ which is the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ equipped with a certain metric of constant negative holomorphic bisectional curvature satisfies this property. The miracle is that the complex structure determines a metric, unique up to all holomorphic reparametrizations, whose metric geometry encodes the complex geometry. For positively curved spaces such as $\mathbb{P}^n$, there can be no such biholomorphism invariant metric. This is because maps like the homothety $z \mapsto \lambda z$ on $\mathbb{P}^1$ are fundamentally non-area preserving conformal transformations. More precisely, any group of isometries of a compact connected Riemannian manifold satisfies two properties (1) it is a compact real Lie group (2) any element fixing a point and the tangent space at that point must be trivial. The biholomorphism group PGL_{n+1} of $\mathbb{P}^n$ violates both properties: it is noncompact, witnessed by the subgroup $\mathbb{C}^\times$ consisting of homotheties such as $z \mapsto \lambda z$, and it contains unipotent elements such as $z \mapsto z + 1$ which fixes ∞ and the tangent space at ∞; hence it cannot be contained in any group of isometries.

However, motivated by the Schwarz lemma (1869) and its generalization by Ahlfors (1938), Kobayashi discovered there is always a canonical *pseudo-metric* determined only by the complex structure which is invariant under all biholomorphisms. Here let $\mathbb{D} \subset \mathbb{C}$ be the unit disk and ρ the Poincaré metric on $\mathbb{D}$.

Definition 1.5.1. Let X be a complex manifold. The *Kobayashi pseudo-distance* is the pseudometric defined as follows:

$$d_X(p, q) = \inf_{\alpha} \ell(\alpha)$$

where α is a chain of holomorphic disks $f_i : \mathbb{D} \to X$ and points $p = p_0, p_1, \dots, p_k = q$ of X and pairs $a_1, b_1, \dots, a_k, b_k \in \mathbb{D}$ such that,

$$f_i(a_i) = p_{i-1} \quad f_i(b_i) = p_i$$

and the length $\ell(\alpha)$ of the chain is defined as,

$$\ell(\alpha) := \rho(a_1, b_1) + \cdots + \rho(a_k, b_k).$$

Example 1.5.2. Let $X = \mathbb{C}$ then $d_X = 0$. Indeed, by choosing larger and larger discs containing p, q their pullback to the unit disk is then closer and closer to the origin and hence has vanishing Poincaré distance.

Example 1.5.3. The Schwarz–Pick lemma shows any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is a contraction for the Poincaré metric. Therefore, $d_{\mathbb{D}} = \rho$.

Definition 1.5.4. We say that a complex manifold X is *(Kobayashi)-hyperbolic* if d_X is an honest metric meaning if $d_X(x, y) = 0$ then $x = y$.

The Kobayashi pseudodistance is not only intrinsic to X and hence preserved by all biholomorphic reparametrizations, it is actually length non-increasing (i.e. *short*) for all holomorphic maps.

Lemma 1.5.5. *Let $f : X \to Y$ be holomorphic. Then $d_Y(f(x), f(y)) \le d_X(x, y)$*

Proof. Indeed, choosing any chain of disks $g_i : \mathbb{D} \to X$ computing $d_X(x, y)$ we see that $f \circ g_i$ is a chain of disks connecting $f(x)$ and $f(y)$ of the same length. Therefore,

$$d_Y(f(x), f(y)) = \inf_{\alpha} \ell(\alpha) \le d_X(x, y)$$

□

Corollary 1.5.6. *If $f : \mathbb{C} \to X$ is an entire curve then for $x, y \in f(\mathbb{C})$ we have $d_X(x, y) = 0$ meaning if f is nonconstant then d_X is degenerate along the image of f.*

Proof. Indeed, let $z_1, z_2 \in \mathbb{C}$ map to x, y respectively. Then,

$$d_X(x, y) \le d_{\mathbb{C}}(z_1, z_2) = 0$$

□

Brody provided a converse to this result.

Theorem 1.5.7 (Brody)**.** *Let X be a compact complex manifold. If d_X is degenerate then there exists a nonconstant entire curve $f : \mathbb{C} \to X$.*

Definition 1.5.8. A complex manifold is *Brody-hyperbolic* if there is no nonconstant entire map $f : \mathbb{C} \to X$.

In particular, if X is Brody hyperbolic then X does not contain any rational or elliptic curves. We will return to this observation in Demailly's notion of *algebraically hyperbolic*. Therefore, as a consequence of Brody's theorem we obtain the following equivalence.

Theorem 1.5.9 (Brody)**.** *Let X be a compact complex manifold. Then X is Kobayashi hyperbolic if and only if it is Brody hyperbolic.*

Therefore, we will call compact manifolds with this property just "hyperbolic" or "analytically hyperbolic" for emphasis. In the noncompact case, these two notions do differ [Bar80].

1.5.2 Algebraic Hyperbolicity

The importance for us of Brody's reinterpretation of hyperbolicity for compact manifolds is that it resembles a form more closely touching algebraic geometry. This connection is made concrete by another redefinition due to Demailly which is conjecturally equivalent to hyperbolicity for smooth projective varieties.

Definition 1.5.10. Let X be a projective variety and H an ample divisor. We say X is *algebraically hyperbolic* if there exists $\epsilon > 0$ such that for every complete integral curve $C \subset X$ we have,

$$2g(C) - 2 \geq \epsilon \deg_H C$$

where $g(C)$ is the geometric genus of C.

Demailly proved that algebraic hyperbolicity is at least a weaker notion than analytic hyperbolicity in the projective setting.

Theorem 1.5.11 (Demailly)**.** *Let X be a smooth projective variety. Then*

$$X \textit{ is analytically hyperbolic} \implies X \textit{ is algebraically hyperbolic}.$$

Moreover, he conjectured the converse to hold as well.

Conjecture 1.5.12 (Demailly)**.** *Let X be a smooth projective variety. Then*

$$X \textit{ is algebraically hyperbolic} \implies X \textit{ is analytically hyperbolic}.$$

Algebraic hyperbolicity, being amenable to attack via techniques in algebraic geometry, is often easier to study. For example, the famous conjecture of Kobayashi predicting the hyperbolicity of general hypersurfaces remains open.

Conjecture 1.5.13 (Kobayashi)**.** *Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n+1$. Then X is hyperbolic.*

After a considerable effort by many authors, the conjecture is now known for $d \geq d_n$ where $d_n \in O(n^4)$ [BK24]. However, its algebraic variant is known almost completely [Voi96, CR19, Yeo25] by much more elementary means.

1.5.3 Green–Griffiths–Lang conjecture

It is expected that general type varieties are *close* to being hyperbolic in the following precise sense. We say that a projective variety X is *algebraically quasi-hyperbolic* if the images of all genus 0 and genus 1 curves lie in a proper Zariski closed subset $Z \subsetneq X$. In characteristic zero, this is the algebraic analog of a weakening of Brody hyperbolicity: we say that X is *quasi-hyperbolic* if the images of all nonconstant entire curves $f : \mathbb{C} \to X$ lie in a proper Zariski closed subset. The Green–Griffiths–Lang conjecture predicts this is always true for varieties of general type.

Conjecture 1.5.14 (Green–Griffiths–Lang)**.** *Let $X/\mathbb{C}$ be a smooth projective variety of general type. Then X is quasi-hyperbolic i.e. there is a proper Zariski closed subset $Z \subsetneq X$ containing the images of all nonconstant entire curves $f : \mathbb{C} \to X$.*

The minimum such Z is often called the *exceptional set* of X. The remarkable feature of this conjecture is that we expect every entire curve to be *algebraically degenerate* meaning it satisfies a polynomial relation (i.e. it lives in Z). Moreover, we expect that these relations can be chosen independently of the map!

The Green–Griffiths–Lang conjecture is highly resonant with another wide open problem: the Bombieri–Lang conjecture predicting some control on the locus of rational points on general type varieties.

Conjecture 1.5.15 (Bombieri–Lang)**.** *Let X/K be a smooth projective variety over a number field K. Assume X is general type. Then there is a proper Zariski closed subset $Z \subsetneq X$ such that for each finite extension K'/K the set of K'-points outside Z is finite.*

An even more ambitious claim is that we should be able to take Z in the Bombieri–Lang conjecture to be the exceptional set of $X_{\mathbb{C}}$. This conjecture would imply a deep connection between the hyperbolic geometry of X and its distribution of rational points. To the author, no enduring mystery in mathematics is more compelling than understanding the nature of this connection. Unfortunately, we will have nothing to say about it in this thesis.

The strategy developed by Bogomolov, Green, Griffiths, Demailly and many others towards the Green–Griffiths–Lang conjecture is to produce algebraic differential equations that entire curves must satisfy. We can think of a first-order differential equation as a closed subspace of the tangent bundle of X. If we could produce enough differential equations so that the intersection of these loci inside the tangent bundle (or appropriate higher-order jet spaces) does not dominate X then we win.

Unfortunately, in many cases of interest known to be hyperbolic, this strategy fails because there are provably not enough algebraic differential equations to cut out a proper subvariety of X. For example, this occurs for products of hyperbolic curves and most Hilbert modular surfaces [DR15]. These are exactly the types of objects we need to study in this thesis, so a more sophisticated approach is needed.

1.5.4 Bogomolov's Boundedness Theorem

The first real progress towards the Green–Griffiths–Lang conjecture is Bogomolov's celebrated boundedness theorem for complex surfaces of general type satisfying $c_1^2 > c_2$. Bogomolov's remarkable proof also provides a route around the roadblock mentioned above.

Theorem 1.5.16 (Bogomolov, 1977)**.** *[Bog77] Let X be a smooth projective surface with $s_2(X) = c_1(X)^2 - c_2(X) > 0$. Then the curves on X with fixed genus form a bounded family. In particular, X has finitely many genus 0 and genus 1 curves i.e. it is algebraically quasi-hyperbolic.*

This was later improved by McQuillan to also constrain non-algebraic entire curves $f : \mathbb{C} \to X$ and thus prove the full Green–Griffiths–Lang conjecture.

Theorem 1.5.17 (McQuillan, 1998)**.** *[McQ98] If X is a smooth projective surface of general type with $s_2(X) = c_1(X)^2 - c_2(X) > 0$ then X satisfies the Green–Griffiths–Lang conjecture. More precisely, there is a proper Zariski closed subset $Z \subsetneq X$ such that every entire curve $f : \mathbb{C} \to X$ has image contained in Z. Consequently, if X contains no genus 0 or genus 1 curves, then X is hyperbolic.*

Bogomolov is able to sidestep the issues discussed above – that the algebraic differential equations on X may have a base locus dominating X – by showing that this base locus carries a natural foliation to which any rational curve must be tangent. He uses the condition $c_1^2 > c_2$ to ensure that X carries a symmetric differential form $\omega \in H^0(X, \operatorname{Sym}^m \Omega_X)$ for some $m > 0$. A key input to this proof is a vanishing theorem for the cohomology of $\operatorname{Sym}^m \Omega_X$ which follows from Bogomolov's study of the stability of the cotangent bundle of a general type surface. From there, the existence of symmetric forms follows from a Hirzebruch–Riemann–Roch calculation.

Now, since $\mathbb{P}^1$ carries no symmetric differential forms, for any map $f : \mathbb{P}^1 \to X$ we must have $f^*\omega = 0$. We can think of this as a nonlinear first-order differential equation satisfied by the map f. Bogomolov uses the following covering construction to turn this nonlinear differential equation into a linear one i.e. into a foliation that f is tangent to.

Let us recall Bogomolov's construction in detail. Let X be a smooth projective surface. For any generically immersive map $f : C \to X$ from a smooth projective curve to X, there is a unique "holonomic" lift $t_f : C \to \mathbb{P}_X(\Omega_X)$ defined by the map of bundles

$$f^* : f^*\Omega_X \twoheadrightarrow \operatorname{im} f^* \subset \omega_C$$

where $\operatorname{im} f^*$ is torsion-free of rank 1 and hence a line bundle. This defines t_f in such a way that $t_f^*\mathscr{O}_{\mathbb{P}}(1) = \operatorname{im} f^*$. At the points $t \in C$ where f is an immersion, the map t_f is defined by $s \mapsto (f(s), \mathrm{d}f_s(\partial_s))$, viewing $\mathbb{P} := \mathbb{P}(\Omega_X)$ as the projectivization of the physical tangent bundle. Since C is a smooth projective curve and $\mathbb{P}_X(\Omega_X)$ is proper, the extension over the points where f is not an immersion is unique.

Now let $\omega \in H^0(X, \operatorname{Sym}^m \Omega_X)$ be a nonzero symmetric form. We can view

$$\omega \in H^0(X, \operatorname{Sym}^m \Omega_X) = H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(m))$$

as a section of the line bundle $\mathscr{O}_{\mathbb{P}}(m)$ hence its vanishing locus on $\mathbb{P}$ is a divisor Z. This represents the locus $(x, v) \in \mathbb{P}$ where $\omega_x(v) = 0$. Recall that for any map $f : \mathbb{P}^1 \to X$ we must have $f^*\omega = 0$. By definition, this means $t_f : \mathbb{P}^1 \to \mathbb{P}$ factors through Z. Similarly, for an elliptic curve C and a map $f : C \to X$, either $f^*\omega$ is zero identically or is zero nowhere. Hence, if $V(\omega) = \{x \in X \mid \omega_x = 0\}$ contains a big divisor then at most finitely many curves may not meet $V(\omega)$. Except for a finite number of exceptions, for each elliptic curve, $f^*\omega$ is therefore zero.

Since $\dim X = 2$, the divisor Z is a union of surfaces. Let $S \to Z$ be a resolution of singularities of a component of Z dominating X. Except for finitely many rational or elliptic curves – namely those which equal the image in X of a component of Z or which do not meet $V(\omega)$ – all rational or elliptic curves on X lift to a map $\widetilde{f} : C \to S$. Let $g : S \to X$ be the induced morphism. The quotient map $\pi^*\Omega_X \twoheadrightarrow \mathscr{O}_{\mathbb{P}}(1)$ defines a foliation on each component of Z.

Definition 1.5.18. The *induced holonomic foliation* on S is defined as follows. Consider the pushout diagram,

$$\begin{tikzcd}
g^*\Omega_X \arrow[r, two heads] \arrow[d, hook] & \iota^*\mathscr{O}_{\mathbb{P}}(1) \arrow[d, hook] \\
\Omega_S \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner"] & \mathscr{Q}
\end{tikzcd}$$

where $\iota : S \to \mathbb{P}$ is the obvious map. Since $\Omega_S \twoheadrightarrow \mathcal{Q}$ is surjective, its dual $\mathcal{F} \hookrightarrow \mathcal{T}_S$, with $\mathcal{F} = \mathcal{Q}^\vee$, is a saturated reflexive subsheaf of generic rank 1. Therefore, $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$ because $\mathcal{F}$ has rank 1 so it defines a foliation.

Lemma 1.5.19. *Let $f : C \to X$ be a map from a smooth proper curve C such that $f^*\omega = 0$. Then the lift $\widetilde{f} : C \to S$ is a leaf of $\mathcal{F}$.*

Proof. The lifted map $t_f : C \to \mathbb{P}$ is induced by the factorization

$$f^*\Omega_X \to t_f^*\mathcal{O}_{\mathbb{P}}(1) \to \omega_C$$

using that t_f factors through S, the following diagram shows that $\widetilde{f}^*\Omega_S \to \omega_C$ factors through $\widetilde{f}^*\Omega_S \twoheadrightarrow \widetilde{f}^*\mathcal{Q}$,

$$\begin{tikzcd}
\widetilde{f}^* g^* \Omega_X \arrow[r] \arrow[d] & \widetilde{f}^* \iota^* \mathcal{O}_{\mathbb{P}}(1) \arrow[d] \arrow[ddr, bend left] & \\
\widetilde{f}^* \Omega_S \arrow[r] \arrow[drr, bend right] & \widetilde{f}^* \mathcal{Q} \arrow[dr, dashed] & \\
 & & \omega_C
\end{tikzcd}$$

□

From here, Bogomolov concludes by showing that each foliation can have at most finitely many rational leaves. This requires analytic input (e.g. a theorem of Jouanolou [Jou78]) that fails in positive characteristic. In the subsequent chapters, we will replace this step with a detailed study of the successive inseparable quotients by these induced foliations. This study culminates in a new criterion for algebraic quasi-hyperbolicity in positive characteristic.

Bogomolov's method requires the existence of a symmetric differential form (which is where the condition $c_1^2 > c_2$ is required). Subsequently, many authors sought to generalize Bogomolov's arguments, replacing the symmetric form with a section of a higher jet bundle, leading to, for example, the proof of Kobayashi's conjecture for hypersurfaces of very large degree. Likewise, in chapter 7, we will sketch methods employing higher jets as unirationality obstructions in positive characteristic.

1.5.5 Positive Characteristic

The situation regarding hyperbolicity is rather different in positive characteristic. While the analytic notions of hyperbolicity do not make sense in positive characteristic, Demailly's notion of algebraic hyperbolicity does. However, the analog of the Green–Griffiths–Lang conjecture fails spectacularly. In fact, we have already seen it fail maximally since a general type variety may be uniruled, even unirational! Likewise, in chapter 3 §3.5 we will construct examples of smooth projective surfaces of general type in positive characteristic that are provably not covered by genus 0 or genus 1 curves yet are nevertheless not algebraically quasi-hyperbolic. Explicitly, such a surface contains an infinite set of genus 0 or genus 1 curves that are "discrete" in the sense that none move in families and whose degrees grow without bound. In the same section, we will give a new criterion for algebraic quasi-hyperbolicity in positive characteristic. Then we use this criterion to show that analogous constructions to the above examples lead to surfaces that *are* algebraically quasi-hyperbolic showing the subtlety of this property in positive characteristic.

Remark 1.5.20. In characteristic zero, in Demailly's definition of algebraic hyperbolicity it is equivalent to consider all nonconstant maps $f : C \to X$ from a smooth curve to X or restrict to those which are birational onto their image. The situation changes in characteristic $p > 0$ because any morphism $f : C \to X$ may be composed with Frobenius to produce a morphism of higher degree. Therefore, in positive characteristic, the former definition is defective; the correct definition of algebraic hyperbolicity is the one given in Definition 1.5.10 or we may equivalently consider all *generically immersive* maps $f : C \to X$ from a smooth curve.

The techniques we develop to control the images of genus 0 and 1 curves in positive characteristic parallel, and are largely inspired by, the techniques mentioned above employing algebraic differential equations towards the Green–Griffiths–Lang conjecture or complex hyperbolicity. The Green–Griffiths strategy of building enough algebraic differential equations on X to force any genus ≤ 1 curve to be algebraically degenerate works just as well in positive characteristic using Hasse-Schmidt derivations [Cad26]. However, techniques based on foliations to push beyond the Green–Griffiths locus, such as in Bogomolov's argument, encounter major new difficulties in characteristic $p > 0$.

The main difficulty is that solutions to differential equations behave quite differently in positive characteristic, losing much of their rigidity. In particular, there is no uniqueness theorem for solutions to ODEs with prescribed boundary conditions due to the basic fact that both 1 and x^p satisfy $f'(x) = 0$ in characteristic p. However, the solutions to differential equations are parametrized by subvarieties of a certain inseparable quotient (see chapter 2 Lemma 2.3.3). This weak description of the solutions is enough to jump-start our technique. Specifically, we try to force the curves on the inseparable quotient to themselves satisfy additional algebraic differential equations. If we can ensure this happens *ad infinitum* then we reach a contradiction by infinite descent (or equivalently by the fact that an algebraic map has finite degree).

Another way to view our technique is that it builds a mechanism to force higher and higher derivatives of a map $f : C \to X$ to be degenerate. A toy model for what happens is as follows: if f is a polynomial whose divided derivatives $\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n} f = 0$ all vanish, then f is a constant. Of course, this triviality holds equally well in positive characteristic. In the main applications of this text to the Shioda conjecture and Hilbert modular varieties, we achieve this control over the higher order derivatives of a map $f : C \to X$ in a fairly roundabout way using the inseparable quotients. We have direct control only over the first derivative of f using the first-order differential equation determined by a symmetric form ω. However, after descending f through an inseparable quotient, the new map will have its first-order derivatives controlling p-th order divided derivatives of the original map. In this way, we effectively control divided derivatives of arbitrarily large orders. A more direct method to force arbitrarily high derivatives of a map $f : C \to X$ to vanish is to find algebraic differential equations of arbitrarily high order that f must satisfy in such a way as to impose nondegenerate conditions on the higher (divided) derivatives $f^{(n)}$. A version using higher-order differential equations arising from jet bundles is explored in chapter 7.

It is worth mentioning that Bogomolov's theorem gives control not only over genus 0 and 1 curves but also curves of any fixed genus. We have focused on the perspective of obstructing uniruledness (or obstructions to covering by genus 1 curves) because the analogous boundedness results usually fail in positive characteristic due to the existence of partial Frobenii. For example, if $X = C \times C$ is a self-product of a curve C over $\mathbb{F}_p$, then the sequence of graphs $\Gamma_n \subset C \times C$ of the n-th iterated Frobenius $F^n : C \to C$ will not be bounded. Indeed, they are all curves isomorphic to C but with degree growing without bound. These curves are exactly the pullback by the partial Frobenius $\mathrm{id}\times F : C\times C \to C\times C$ iterated n-times to the diagonal $\Delta \subset C \times C$. However, in the cases of interest in this thesis, we will be able to show that such constructions are the *only* source of failure of boundedness. Precisely, we will show that every family of curves of fixed genus g is pulled back via a finite sequence of partial Frobenii from a bounded family of curves (cf. chapter 6 §6.6.9).

1.6 Outline of Chapters

We now give an overview of the organization of the chapters to come.

1. Chapter 2 develops the basic positive-characteristic theory of foliations used throughout the thesis. In particular, it reviews relative Frobenius, p-curvature, p-closed

foliations, and Ekedahl's inseparable quotient construction in the form needed for later descent arguments.

2. Chapter 3 introduces the first infinite-descent argument for product-quotient surfaces. Using diagonal symmetric differentials and the foliations they induce, it proves boundedness results for rational and elliptic curves in positive characteristic and constructs counterexamples to Shioda's conjecture from suitable product-quotient surfaces.

3. Chapter 4 supplies the arithmetic input needed to prove supersingular reduction for the preceding constructions in infinitely many characteristics. It studies supersingular reduction for Hurwitz curves of genera 3, 7, 14, and 17, as well as related genus 17 curves, with special attention to the first Hurwitz triplet and auxiliary covers built from the Klein and Fermat quartics.

4. Chapter 5 abstracts the mechanism underlying the previous arguments into a more flexible descent algorithm. The main results are a general degree bound verifying an infinite descent contradiction when the requisite symmetric differentials can be constructed indefinitely.

5. Chapter 6 applies this framework to Hilbert modular varieties. After reviewing their moduli interpretations, integral models, special fibers, Hasse invariants, and tautological foliations, it proves non-uniruledness results for reductions mod p of Hilbert modular varieties of large level or discriminant. When the obstruction applies, it will yield non-unirationality results for all but finitely many characteristics (depending on the level and totally real field).

6. Chapter 7 sketches a higher-order version of the obstruction theory using jet bundles. The goal is to replace first-order differential equations by higher-order conditions that survive larger inseparable degree, and thereby indicate a possible route beyond the foliation-based arguments developed earlier.

Conventions. A *variety* is a geometrically integral separated scheme of finite type over a field k. Unfortunately, this terminological choice means that Hilbert modular varieties – being often disconnected – are *not* varieties. When we say that a variety X is *covered* by rational curves, elliptic curves, or curves of genus g, we mean there is a family of smooth proper curves $\mathscr{C} \to T$ of genus g along with a dominant map $f : \mathscr{C} \to X$ not contracting the fibers. A resolution of surface singularities $\pi : \tilde{S} \to S$ is *minimal* if there is no (-1)-curve (possibly defined over $\bar{k}$) contracted by π. When we say a surface X is *minimal*, we mean it is smooth projective and contains no (-1)-curve (possibly defined over $\bar{k}$). When

X is smooth projective of nonnegative Kodaira dimension, minimality is equivalent to the condition that K_X is nef. In characteristic $p > 0$, we write $X^{(p^e)}$ for the e-fold Frobenius twist of X over the ground field and $F_X^e : X \to X^{(p^e)}$ for the relative Frobenius map. We write $\mathrm{Frob}_X : X \to X$ for the absolute Frobenius map of X. When we say that a variety is *simply-connected*, we mean the geometric étale fundamental group $\pi_1^{\text{ét}}(X_{\bar{k}})$ is trivial. The group D_n is the dihedral group with $2n$ elements. For example, D_6 is the symmetries of the regular hexagon.

Chapter 2

Foliations in Positive Characteristic and p-Curvature

Fate is partial to repetitions.

Luis Borges

In this chapter we review the definition and basic properties of foliations in positive characteristic. Since there is no reference known to the author containing all these statements in the precise form given here (e.g. for general foliations as opposed to flat connections), we spell out some of the non-commutative algebra involved in working with p-curvatures.

First we make a few remarks about conventions. Let p be a prime number. For a scheme X of pure characteristic p, let $\mathrm{Frob}_X : X \to X$ denote the absolute Frobenius, the morphism given by the identity on the underlying topological space and with $\mathrm{Frob}_X^\# : \mathcal{O}_X \to \mathcal{O}_X$ given by $f \mapsto f^p$. Let $f : X \to S$ be a morphism of schemes in characteristic p, then the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Frob}_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ S \arrow[r, "\mathrm{Frob}_S"] & S \end{tikzcd}$$

commutes. The fiber product $X^{(p/S)} := X \times_{S,\mathrm{Frob}_S} S$ is called the *Frobenius twist* of X relative to S. This diagram implies the existence of a morphism $F_{X/S} : X \to X^{(p/S)}$ via the universality of the pullback square

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"] \arrow[rr, bend left, "\mathrm{Frob}_X"] \arrow[dr] & X^{(p/S)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ & S \arrow[r, "\mathrm{Frob}_S"] & S \end{tikzcd}$$

which we call the *relative Frobenius* of X over S. When there is no ambiguity about the base scheme, such as when X lives over a ground field k, we will write F_X for $F_{X/k}$.

On a (possibly singular) S-scheme, we define the *relative tangent sheaf*

$$\mathscr{T}_{X/S} := \Omega_{X/S}^{\vee} = \mathscr{D}er_{\mathscr{O}_S}(\mathscr{O}_X, \mathscr{O}_X)$$

as the dual of the sheaf of Kähler differentials which equals the sheaf of S-derivations on X. This is a sheaf of $\mathscr{O}_S$-linear Lie algebras via the Lie bracket of derivations. When X is a variety over a field k we will write Ω_X for $\Omega_{X/k}$ and likewise $\mathscr{T}_X$ for $\mathscr{T}_{X/k}$. Note that the tangent sheaf enjoys much worse functorial properties than the cotangent sheaf. If $f : X \to Y$ is a morphism of S-schemes with Y smooth over S then dualizing the pullback map

$$f^*\Omega_{Y/S} \to \Omega_{X/S}$$

we obtain a map called the *differential*

$$\mathrm{d}f : \mathscr{T}_{X/S} \to f^*\mathscr{T}_{Y/S}.$$

Since the map goes the wrong way, we do not obtain a map of sections of the tangent sheaf. Moreover, if $\Omega_{Y/S}$ is not locally free, the differential map is not well-defined since duals do not commute with pullbacks. In general we only have the following diagram of $\mathscr{O}_X$-modules,

$$\begin{tikzcd} \mathscr{T}_{X/S} \arrow[r, "\mathrm{d}f"] & (f^*\Omega_{Y/S})^{\vee} \\ & f^*\mathscr{T}_{Y/S} \arrow[u] \end{tikzcd}$$

Therefore, care must be taken when making functoriality arguments in the singular setting. In the following, we usually assume the target is smooth for simplicity.

2.1 Foliations

Definition 2.1.1. Let X be a normal variety. A *foliation* $\mathscr{F}$ is a coherent subsheaf $\mathscr{F} \subset \mathscr{T}_X$ of the tangent sheaf $\mathscr{T}_X$ of X which is

1. *involutive*: $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$ meaning $\mathscr{F}$ is closed under the Lie bracket

2. *saturated*: $\mathscr{T}_X / \mathscr{F}$ is torsion free.

We will almost always only consider foliations on smooth varieties. In this text, we will only consider *algebraic* foliations where the objects in the above definition lie in the algebraic category and the sheaves are coherent sheaves of $\mathscr{O}_X$-modules on finite type schemes. One can also consider *analytic* or *smooth* foliations taking subsheaves of the tangent bundle in those corresponding categories. Over $\mathbb{C}$, the involutive condition ensures that $\mathscr{F}$ is locally *integrable*. Integrability means that through any smooth point of X where $\mathscr{F}$ is a sub-bundle, there is a unique analytic submanifold "integrating" $\mathscr{F}$ in the sense that its tangent space everywhere agrees with the subspaces $\mathscr{F} \subset \mathscr{T}_X$. More precisely, in a small analytic neighborhood of such a point, the leaves look like fibers of a smooth (submersive) morphism. This motivates the following definition.

Definition 2.1.2. The *singular locus* of a foliation $\mathscr{F} \subset \mathscr{T}_X$ denoted as $\mathrm{Sing}(\mathscr{F})$ is the union of X^{sing} and the locus where $\mathscr{F} \subset \mathscr{T}_X$ is not a sub-bundle.

The saturation condition implies that a foliation on a normal variety is determined by its restriction to any dense open set. Hence algebraic foliations are of a birational nature and often inform the birational study of X. This, along with the following functoriality result, implies that foliations can be pulled back along rational maps, confirming their birational nature.

Definition 2.1.3. Let $f : X \to Y$ be a morphism of varieties and $\mathscr{F} \subset \mathscr{T}_Y$ a foliation on Y. Then we define the *pullback foliation* $f^+\mathscr{F} \subset \mathscr{T}_X$ as the saturation of the image of $\mathscr{G} \to \mathscr{T}_X$ arising in the pullback square

$$\begin{tikzcd} \mathscr{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{T}_X \arrow[d, "\mathrm{d}f"] \\ (f^*\mathscr{F}^\vee)^\vee \arrow[r] & (f^*\Omega_Y)^\vee \end{tikzcd}$$

If f is generically smooth, this operation preserves the corank, $\operatorname{rank} \mathscr{T}_X - \operatorname{rank} \mathscr{F}$, of the foliation.

Often these analytic leaves are very far from being algebraic as they have wild global behavior. For example, they can be dense in the analytic topology and hence only form *immersed* submanifolds (when restricted away from the singular locus of the foliation). It will also be important to study *algebraic leaves* which are subvarieties tangent to the foliations. This is exactly the special case when some analytic leaf is Zariski closed.

Definition 2.1.4. Let $\mathscr{F} \subset \mathscr{T}_X$ be a foliation on X. We say a subvariety $Z \hookrightarrow X$ is *tangent* to $\mathscr{F}$ if the map $\mathscr{T}_Z \to \mathscr{T}_X|_Z$ factors through $\mathscr{F}_Z \subset \mathscr{T}_X|_Z$, the saturation of $(\mathscr{F}|_Z)^{\vee\vee} \hookrightarrow \mathscr{T}_X|_Z$. If additionally $\mathscr{T}_Z \to \mathscr{T}_X|_Z$ is identified with $\mathscr{F}_Z$ we say that Z *is a leaf* of $\mathscr{F}$.

Note that the property of being tangent depends only on behavior along a dense open set (e.g. along the smooth locus of Z).

2.2 p-Curvature

In positive characteristic, there is a new invariant of a foliation called the p-curvature. To motivate its definition, we take a moment to consider the ordinary curvature of a subsheaf $\mathscr{F} \subset \mathscr{T}_X$. The *curvature* is the obstruction to $\mathscr{F}$ being a sheaf of sub Lie algebras of $\mathscr{T}_X$ i.e. it measures the failure of the involutivity condition $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$. In analogy with connections, we sometimes call this condition *flatness*.

Definition 2.2.1. If $\mathscr{F}$ is not involutive, then there is a $\mathscr{O}_X$-linear map called the *curvature* or *residual Lie bracket* witnessing the failure of involutivity. It is defined as the map of $\mathscr{O}_X$-modules,

$$\ell_{\mathscr{F}_X} : \bigwedge^2 \mathscr{F} \to \mathscr{T}_X / \mathscr{F} \quad \text{via} \quad \ell_{\mathscr{F}_X} : X \wedge Y \mapsto [X, Y] \mod \mathscr{F}.$$

The map $\ell_{\mathscr{F}_X}$ is well-defined and $\mathscr{O}_X$-linear because for any $f \in \mathscr{O}_X$

$$f[X, Y] = [fX, Y] + Y(f) \cdot X = [X, fY] - X(f) \cdot Y$$

and therefore

$$f[X, Y] \equiv [fX, Y] \equiv [X, fY] \mod \mathscr{F}$$

because if $X, Y \in \mathscr{F}$ then so are $Y(f) \cdot X$ and $X(f) \cdot Y$ since $\mathscr{F}$ is an $\mathscr{O}_X$-module.

In characteristic $p > 0$, the tangent bundle $\mathscr{T}_X$ is not just a Lie algebra, it is a p-Lie algebra meaning there is a Frobenius semi-linear operation $(-)^p$ commuting with the Lie-bracket defined by $\partial \mapsto \partial^p$ where ∂^p is the differential operator given by iteratively applying ∂.

Lemma 2.2.2. *Let A be an algebra over a field k of characteristic p and $\partial : A \to A$ a k-derivation. Then the differential operator ∂^p given by*

$$x \mapsto \underbrace{\partial \cdots \partial}_{p \text{ times}} x$$

is a k-derivation.

Proof. It is clear that ∂^p is k-linear. The Leibniz rule implies that

$$\partial^n(fg) = \sum_{j=0}^{n} \binom{n}{j} \partial^j(f)\partial^{n-j}(g).$$

Taking $n = p$ and using the divisibility of binomial coefficients gives

$$\partial^p(fg) = \partial^p(f)g + f\partial^p(g).$$

□

The p-curvature is the obstruction to $\mathscr{F} \subset \mathscr{T}_X$ being a subsheaf of p-Lie algebras given that it is a Lie algebra subsheaf.

Definition 2.2.3. Given an involutive subsheaf $\mathscr{F} \subset \mathscr{T}_X$, the *$p$-curvature* map is the map

$$\psi_{\mathscr{F},p} : \mathrm{Frob}^* \mathscr{F} \to \mathscr{T}_X/\mathscr{F} \quad \text{via} \quad \psi_{\mathscr{F},p} : \partial \mapsto \partial^p \mod \mathscr{F}.$$

measuring the failure of $\mathscr{F}$ to be *p-closed* i.e. $\mathscr{F}^p \subset \mathscr{F}$ meaning $\mathscr{F} \subset \mathscr{T}_X$ is a sub p-Lie algebra.

The p-curvature turns out to be an $\mathscr{O}_X$-linear map of $\mathscr{O}_X$-modules. The proof of this fact is surprisingly tricky. The easiest route is via the following Jacobson relations and Deligne's formula for p-powers in noncommutative rings. By a *Lie polynomial* we mean a noncommutative polynomial formed from addition and compositions of commutators.

Lemma 2.2.4 (Jacobson, Deligne)**.** *(cf. [Kat70, Theorem 5.3]) Let R be a (possibly noncommutative) associative ring of characteristic p. Then,*

1. *there are universal Lie polynomials $s_i(x, y) \in \mathbb{F}_p \langle x, y \rangle$ such that for all $a, b \in R$*

$$(a + b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b)$$

where s_i is defined by the universal relation in $\mathbb{F}_p \langle x, y\rangle [t]$,

$$(\mathrm{ad}_{at+b})^{p-1}(a) = \sum_{i=1}^{p-1} i\, s_i(a,b)\, t^{i-1}$$

2. *for all $g, \theta \in R$ such that $\{\mathrm{ad}_\theta^n(g^m)\}_{n,m\geq 0}$ commute with each other,*

$$(g\theta)^p = g^p\theta^p + g \cdot \mathrm{ad}_\theta^{p-1}(g^{p-1})\theta.$$

Remark 2.2.5. Since they arise from applying ad_{xt+y}^{p-1} to x, the polynomials $s_i(x,y)$ lie in $\mathrm{Lie}(x,y)_{p-1}$, the $(p-1)^{\mathrm{th}}$-stage of the lower central series. Recall that for a Lie algebra $\mathfrak{g}$ the lower central series is the sequence of Lie subalgebras: $\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}]$ setting $\mathfrak{g}_0 = \mathfrak{g}$.

Example 2.2.6. For example, if $p = 2$ then $(a+b)^2 = a^2 + ab + ba + b^2$ so $s_1(a,b) = ab + ba = [a,b]$ because the characteristic is 2.

Lemma 2.2.7. *If $\mathscr{F}$ is involutive, the p-curvature map $\psi_{\mathscr{F},p}$ is an $\mathscr{O}_X$-linear map of $\mathscr{O}_X$-modules.*

Proof. To show $\psi_{\mathscr{F},p}(\partial_1 + \partial_2) = \psi_{\mathscr{F},p}(\partial_1) + \psi_{\mathscr{F},p}(\partial_2)$ we use the relation

$$(\partial_1 + \partial_2)^p = \partial_1^p + \partial_2^p + \sum_{i=1}^{p-1} s_i(\partial_1, \partial_2)$$

where the s_i are Lie polynomials so $s_i(\partial_1, \partial_2) \in \mathscr{F}$ because $\mathscr{F}$ is closed under Lie bracket. We further need to show that $\psi_{\mathscr{F},p}(f\partial) = f^p \psi_{\mathscr{F},p}(\partial)$ so that it becomes $\mathscr{O}_X$-linear viewed as a map on $\mathrm{Frob}_X^* \mathscr{F}$. Indeed, since $\mathrm{ad}_\partial^n(f^m) = \partial^n(f^m)$ is a scalar, these commute in the ring of differential operators. Hence by Jacobson's lemma,

$$(f\partial)^p = f^p\partial^p + f\, \mathrm{ad}_\partial^{p-1}(f^{p-1})\partial$$

and because $\mathrm{ad}_\partial^{p-1}(f^{p-1}) = \partial^{p-1}(f^{p-1}) \in \mathscr{O}_X$, the second term lies in $\mathscr{F}$. □

2.2.1 Compatibility

It will be important to know that p-curvature is compatible with pullbacks. One of the main applications of the p-curvature map is that it imposes a restriction on the algebraic leaves of a foliation. When $\mathscr{F}$ is not involutive, the residual Lie bracket map ℓ gives a restriction on the locus of integrable subvarieties.

Proposition 2.2.8. *Let X be a smooth variety and $f : Z \to X$ a morphism invariant under a subsheaf $\mathscr{F} \subset \mathscr{T}_X$ in the sense that the tangent map factors as*

$$\mathrm{d}f : \mathscr{T}_Z \to f^*\mathscr{F} \to f^*\mathscr{T}_X$$

then the composition of $\mathscr{T}_Z \to f^*\mathscr{F}$ with the pullback of the curvature,

$$\bigwedge^2 \mathscr{T}_Z \to \bigwedge^2 f^*\mathscr{F} \xrightarrow{f^*\ell_{\mathscr{F}}} f^*(\mathscr{T}_X/\mathscr{F})$$

is identically zero.

Proof. This follows immediately from the following Proposition 2.2.9 expressing the compatibility of curvature with pullback applied to the map $f : (Z, \mathscr{T}_Z) \to (X, \mathscr{F})$ and using that the subsheaf $\mathscr{T}_Z \subset \mathscr{T}_Z$ trivially has zero curvature. □

Of course, if Z is a curve, the curvature of $\mathscr{F}$ imposes no restriction since $\bigwedge^2 \mathscr{T}_Z = 0$. However, this is a nontrivial obstruction to finding invariant subvarieties (or immersed submanifolds in the analytic setting) of dimension > 1.

Proposition 2.2.9. *Suppose $f : X \to Y$ is a map of S-schemes where $Y \to S$ is smooth. Suppose there is a commutative diagram*

$$\begin{tikzcd} \mathscr{F}_X \arrow[r, "\alpha"] \arrow[d, "\kappa"] & \mathscr{T}_{X/S} \arrow[d, "\mathrm{d}f"] \\ \mathscr{F}_Y \arrow[r, "f^*\beta"] & f^*\mathscr{T}_{Y/S} \end{tikzcd}$$

where $\mathscr{F}_X$ and $\mathscr{F}_Y$ are coherent sheaves on X and Y respectively. As before consider the curvature maps

$$\ell_{\mathscr{F}_X} : \bigwedge^2 \mathscr{F}_X \to \mathscr{T}_{X/S}/\mathscr{F}_X \quad \textit{via} \quad \ell_{\mathscr{F}_X} : X \wedge Y \mapsto [X, Y] \mod \mathscr{F}$$

and likewise for $\ell_{\mathscr{F}_Y}$. These maps are compatible in the sense that the following diagram relating the curvatures

$$\begin{tikzcd} \bigwedge^2 \mathscr{F}_X \arrow[r, "\ell_{\mathscr{F}_X}"] \arrow[d, "\wedge^2\kappa"'] & \mathscr{T}_{X/S}/\mathscr{F}_X \arrow[d, "\mathrm{d}f"] \\ f^* \bigwedge^2 \mathscr{F}_Y \arrow[r, "f^*\ell_{\mathscr{F}_Y}"] & f^*\mathscr{T}_{Y/S}/\mathscr{F}_Y \end{tikzcd}$$

commutes.

Proof. We turn this into the corresponding algebra statement: let $\phi : A \to B$ be a map of R-algebras, N an A-module and M a B-module equipped with a commutative diagram of B-modules,

$$\begin{tikzcd} \mathrm{Der}_R(B,B) \arrow[r, "\partial \mapsto \partial\circ\phi"] & \mathrm{Der}_R(A,B) \arrow[r, equal] & B \otimes_A \mathrm{Der}_R(A,A) \\ M \arrow[u, "\alpha"] \arrow[rr, "\kappa"] & & B \otimes_A N \arrow[u, "\mathrm{id}_B \otimes \beta"'] \end{tikzcd}$$

then we need to show that $[\alpha(m), \alpha(m')] = [\beta(\kappa(m)), \beta(\kappa(m'))]$ modulo M where the second Lie bracket is defined by taking the image of $\kappa(m)$ in $B \otimes_A \mathrm{Der}_R(A, A)$ and extending the bracket B-linearly. This is well-defined up to elements of $N \otimes_A B$. Explicitly, write,

$$\beta(\kappa(m)) = \sum f_i \otimes \partial_i$$

for some $\partial_i \in \mathrm{Der}_R(A, A)$ in the image of $\beta : N \to \mathrm{Der}_R(A, A)$ and $f_i \in B$ and likewise

$$\beta(\kappa(m')) = \sum f_i' \otimes \partial_i.$$

Then,

$$[\beta(\kappa(m)), \beta(\kappa(m'))] = \sum f_i f_j' [\partial_i, \partial_j] \in \mathrm{Der}_R(A, B) / (N \otimes_A B).$$

However, $\beta(\kappa(m))$ has image $\alpha(m) \circ \phi$ in $\mathrm{Der}_R(A, B)$. Therefore, for $a \in A$,

$$\begin{aligned} [\alpha(m), \alpha(m')](a) &= \alpha(m)\alpha(m')\phi(a) - \alpha(m')\alpha(m)\phi(a) \\ &= \alpha(m)\left[\sum_j f_j'\partial_j a\right] - \alpha(m')\left[\sum_i f_i\partial_i a\right] \\ &= \sum_j \left[f_j'\alpha(m)(\partial_j a) + \alpha(m)(f_j') \cdot (\partial_j a)\right] - \sum_i \left[f_i\alpha(m')(\partial_i a) + \alpha(m')(f_i) \cdot (\partial_i a)\right] \\ &= \sum_{i,j} f_i f_j' [\partial_i, \partial_j](a) + \left[\sum_j \alpha(m)(f_j') \cdot \partial_j - \sum_i \alpha(m')(f_i) \cdot \partial_i\right] \end{aligned}$$

but ∂_i is in the image of $N \to \mathrm{Der}_R(A, A)$ so the last term is zero in $\mathrm{Der}_R(A, B) / B \otimes_A N$ proving the claim. ◻

Likewise, when $\mathscr{F}$ is not p-closed, its p-curvature also provides an obstruction to the existence of algebraic leaves. For example, the following elementary observation is quite powerful.

Proposition 2.2.10. *Let X be a smooth variety and $f : Z \to X$ a morphism invariant under* *a foliation $\mathscr{F} \subset \mathscr{T}_X$ in the sense that the tangent map factors as*

$$\mathrm{d}f : \mathscr{T}_Z \to f^*\mathscr{F} \to f^*\mathscr{T}_X$$

then the composition of $\mathscr{T}_Z \to f^*\mathscr{F}$ with the pullback of the p-curvature

$$\mathscr{T}_Z \to f^*\mathscr{F} \xrightarrow{f^*\psi_{\mathscr{F},p}} f^*(\mathscr{T}_X/\mathscr{F})$$

is identically zero.

Proof. This follows immediately from the following Proposition 2.2.12 expressing the compatibility of p-curvature with pullbacks applied to the map $f : (Z, \mathscr{T}_Z) \to (X, \mathscr{F})$ of foliated varieties noting that the p-curvature of the trivial foliation $\mathscr{T}_Z \subset \mathscr{T}_Z$ is zero. □

Our primary application of this result is in the case of (families of) curves. We have assumed previously that $f : Z \to X$ is invariant in the strong sense that $\mathrm{d}f$ factors globally through the foliation. When we defined a leaf, it was important to relax this condition along the singular locus of the foliation. Similarly, here we only require that the tangent space lies in $\mathscr{F}$ generically. This is only a generalization in the presence of singularities. Indeed, since $\mathscr{T}_Z$ is torsion-free, along the smooth locus of $\mathscr{F} \subset \mathscr{T}_X$ where $\mathscr{F}$ is a sub-bundle, the tangent map $\mathrm{d}f$ factors globally if and only if it factors at the generic point.

Corollary 2.2.11. *Let X be a smooth variety and $f : C \to X$ a generically immersed map from a smooth curve generically invariant under a foliation $\mathscr{F}$. Then*

$$f(C) \subset \operatorname{Sing} \mathscr{F} \cup \overline{K^1(\psi_{\mathscr{F},p})}$$

where K^1 is the locus where a map of $\mathscr{O}_X$-modules has nontrivial kernel.

Proof. Assume $f(C) \not\subset \operatorname{Sing}(\mathscr{F})$ then by definition there is a dense open subset $U \subset C$ such that $f(U)$ is contained in the smooth locus of X and $\mathscr{F}$ and there is a factorization

$$\begin{tikzcd} \mathscr{T}_U \arrow[r] \arrow[dr, "\mathrm{d}f"'] & f|_U^*\mathscr{F} \arrow[d, hook] \\ & f|_U^*\mathscr{T}_X \end{tikzcd}$$

where $\mathrm{d}f$ is nonvanishing on U. By the previous proposition, the composition,

$$\mathscr{T}_U \to f|_U^*\mathscr{F} \xrightarrow{f|_U^*\psi_{\mathscr{F},p}} f|_U^*(\mathscr{T}_X/\mathscr{F})$$

is identically zero. Since $\mathscr{T}_U \to f|_U^* \mathscr{F}$ is nonvanishing, $f(U) \subset K^1(\psi_{\mathscr{F},p})$ and therefore $f(C) \subset \overline{K^1(\psi_{\mathscr{F},p})}$. □

For example, if $\mathscr{F}$ is a rank 1 foliation then,

$$\operatorname{im} f \subset \operatorname{Sing} \mathscr{F} \cup V(\psi_{\mathscr{F},p})$$

If X is a surface and $\mathscr{F}$ is not p-closed then $\operatorname{Sing} \mathscr{F}$ is a finite collection of points so any invariant curve is contained in $\Delta_{\mathscr{F},p} := V(\psi_{\mathscr{F},p})$. In particular, on a surface, there are only finitely many curves tangent to a foliation with nonvanishing p-curvature. Now we prove the compatibility of p-curvature with pullback.

Proposition 2.2.12. *Let $f : (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y)$ be a map of foliated S-schemes with $Y \to S$ smooth. Explicitly, this means there is a commutative diagram*

$$\begin{tikzcd} \mathscr{F}_X \arrow[r, "\alpha"] \arrow[d, "\kappa"] & \mathscr{T}_{X/S} \arrow[d, "\mathrm{d}f"] \\ \mathscr{F}_Y \arrow[r, "f^*\beta"] & f^*\mathscr{T}_{Y/S} \end{tikzcd}$$

and the subsheaves $\mathscr{F}_X$ and $\mathscr{F}_Y$ are assumed to be involutive. Then the diagram

$$\begin{tikzcd} \operatorname{Frob}_X^* \mathscr{F}_X \arrow[r, "\psi_{\mathscr{F},p}"] \arrow[d] & \mathscr{T}_{X/S}/\mathscr{F}_X \arrow[d] \\ f^* \operatorname{Frob}_Y^* \mathscr{F}_Y \arrow[r, "f^*\psi_{\mathscr{F},p}"] & f^*(\mathscr{T}_{Y/S}/\mathscr{F}_Y) \end{tikzcd}$$

of $\mathscr{O}_X$-linear maps commutes.

First we need a lemma.

Lemma 2.2.13. *Let A, B be (possibly noncommutative) associative rings of characteristic p and let M be an (A, B)-bimodule. Suppose that for some $a \in A$ and $m \in M$,*

$$a \cdot m = \sum_i f_i \cdot m \cdot b_i$$

such that the $f_i \in A$ commute with each other and $b_i \in B$. Then the difference,

$$a^p \cdot m - \sum_i f_i^p \cdot m \cdot b_i^p$$

lies in the subspace

$$\left\langle t \cdot m \cdot s \mid t \in \mathbb{Z}[f_1, \ldots, f_r, \operatorname{Lie}(a, f_1, \ldots, f_r)_{p-1}] \text{ and } s \in \operatorname{Lie}(b_1, \ldots, b_r)_{p-1} \right\rangle$$

where $\mathrm{Lie}(x_1, \dots, x_r)_{p-1}$ *denotes the* $(p-1)$*-th stage of the lower central series of the Lie algebra generated by* $x_1, \dots, x_r$.

Proof. M is an $A \otimes B^{\mathrm{op}}$-module and $(a \otimes 1 - \sum_i f_i \otimes b_i)$ acts as zero. Furthermore,

$$(a \otimes 1 - \sum_i f_i \otimes b_i)^p = a^p \otimes 1 + \left(-\sum_i f_i \otimes b_i\right)^p + \sum_{j=1}^{p-1} s_j(a \otimes 1, -\sum_i f_i \otimes b_i)$$

so the last term is in the span of $\mathrm{Lie}(a, f_1, \dots, f_r)_{p-1} \otimes b_i$. By Jacobson again,

$$\left(-\sum_i f_i \otimes b_i\right)^p = -\sum_i f_i^p \otimes b_i^p + g$$

where g is in the span of $f_i \otimes \mathrm{Lie}(b_1, \dots, b_r)_{p-1}$. Therefore, applying this element to $m \in M$ we obtain,

$$a^p \cdot m - \sum_i f_i^p \cdot m \cdot b_i^p = -\sum_{j=1}^{p-1} s_j(a \otimes 1, \sum_i f_i \otimes b_i) \cdot m - g \cdot m.$$

□

Proof of proposition. This is a local statement so we may reduce to an R-algebra map $\phi : A \to B$ with $R \to A$ smooth along with a diagram of B-modules

$$\begin{array}{ccc} \mathrm{Der}_R(B, B) & \xrightarrow{\partial \mapsto \partial \circ \phi} & \mathrm{Der}_R(A, B) \\ \uparrow & & \uparrow \\ M & \xrightarrow{\kappa} & B \otimes_A N \end{array}$$

Suppose that $\partial \in M$ we need to show that

$$\psi_p(\mathrm{Frob}^* \partial) \circ \phi - \psi_p(\mathrm{Frob}^* \kappa(\partial)) \in \mathrm{im}\,(B \otimes_A N \to \mathrm{Der}_R(A, B)).$$

We can expand the $\kappa(\partial) \in B \otimes_A N$

$$\kappa(\partial) = \sum_i b_i \otimes \partial_i$$

in terms of elements $\partial_i \in N$ and scalars $b_i \in B$. Recall that ψ_p is Frob-semilinear so $f^* \psi_p$ is B-semilinear meaning,

$$\psi_p(\kappa(\partial)) = \psi_p\left(\sum_i b_i \otimes \partial_i\right) = \sum_i b_i^p \otimes \partial_i^p.$$

Let $\mathscr{D}_A$ be the algebra consisting of differential operators on A generated by derivations. We apply the lemma to $B \otimes_A \mathscr{D}_A$ viewed as a $(\mathscr{D}_B, \mathscr{D}_A)$-bimodule. The left $\mathscr{D}_B$ action is defined via

$$g \cdot (b \otimes D) = gb \otimes D \quad \text{and} \quad \partial \cdot (b \otimes D) = (\partial b) \otimes D + b \cdot (\partial \circ \phi) D$$

where, using smoothness of A over R, $\partial \circ \phi \in \mathrm{Der}_R(A, B) = B \otimes_A \mathrm{Der}_R(A, A)$ so the element $(\partial \circ \phi)D$ is a well-defined element of $B \otimes_A \mathscr{D}_A$. This $(\mathscr{D}_B, \mathscr{D}_A)$-bimodule is sometimes called the "transfer module" in D-module theory. Commutativity of the diagram involving κ shows that, viewing ∂ and ∂_i as differential operators in $\mathscr{D}_B$ and $\mathscr{D}_A$ respectively, we have the relation,

$$\partial \cdot 1 = \sum_i b_i \cdot 1 \cdot \partial_i$$

in $B \otimes_A \mathscr{D}_A$. Therefore, applying the lemma we see that

$$\psi_p(\partial) \circ \phi - \psi_p(\kappa(\partial)) = \partial^p \circ \phi - \sum_i b_i^p \otimes \partial_i^p \in B \otimes_A \mathscr{D}_A$$

is in the submodule spanned by elements of the form $r \cdot 1 \cdot s = rs$ where r is in the subalgebra generated by iterated application of ∂ to the scalars $b_i \in B$ and products of the b_i and s is generated by commutators of ∂_i which lie in N. Hence the difference lies in the image of $B \otimes_A N$ completing the proof. $\square$

2.3 Ekedahl's theory of positive characteristic foliations

We now review the theory of p-closed foliations, what Ekedahl calls 1-foliations, as developed by Ekedahl [Eke87]. In particular, there is an equivalence between p-closed foliations and certain purely inseparable morphisms. This correspondence was first understood by Jacobson in the context of developing a Galois theory for purely inseparable extensions [Jac44] and realized in its modern form by Ekedahl in [Eke87]. From now on in this section, let k be a perfect field of characteristic p and all varieties will be varieties over k. As before, we let $\mathrm{Frob}_X : X \to X$ denote the (k-semilinear) absolute Frobenius and $F_{X/k} : X \to X^{(p)}$ denote the (k-linear) relative Frobenius.

Definition 2.3.1. Let $f : X \dashrightarrow Y$ be a dominant rational map of varieties. We say that f is *purely inseparable* if the extension of function fields $k(Y) \hookrightarrow k(X)$ is purely inseparable. If, moreover, f is generically finite, then $k(X)^{p^h} \subset k(Y)$ for some h. The smallest such h is called the *height* of f. Note that if f has height h then by definition there is a rational map $g : Y \dashrightarrow X^{(p^h)}$ so that $g \circ f : X \dashrightarrow Y \dashrightarrow X^{(p^h)}$ is the h-iterated relative Frobenius $F^h_{X/k} : X \to X^{(p^h)}$ giving a factorization $g \circ f = F^h_{X/k}$.

Theorem 2.3.2 (Ekedahl)**.** *Let X be a smooth variety. There is an equivalence between the following two posets*

1. *p-closed foliations $\mathcal{F} \subset \mathcal{T}_X$*

2. *finite height-1 purely inseparable morphisms $f : X \to Y$ with Y normal*

constructed as follows: given a p-closed foliation $\mathcal{F} \subset \mathcal{T}_X$ we define $f : X \to X/\mathcal{F}$ where f is the identity of the underlying topological spaces and $\mathcal{O}_{X/\mathcal{F}} = \mathrm{Ann}_{\mathcal{F}}(\mathcal{O}_X)$ where explicitly

$$\mathcal{O}_{X/\mathcal{F}}(U) := \{s \in \mathcal{O}_X(U) \mid \forall \partial \in \mathcal{F}(U) : \partial s = 0\}.$$

Therefore, f is the map $X \to \mathbf{Spec}_X(\mathrm{Ann}_{\mathcal{F}}(\mathcal{O}_X))$. In reverse, given $f : X \to Y$ the p-closed foliation is the subsheaf $\mathcal{T}_{X/Y} \subset \mathcal{T}_X$ of relative tangent fields or equivalently $\mathrm{Ann}_{\mathcal{T}_X}(f^{-1}\mathcal{O}_Y)$. Furthermore, the correspondence has the following properties:

1. *Y is smooth iff $\mathcal{F}$ is nonsingular ($\mathcal{F} \subset \mathcal{T}_X$ is a subbundle) iff f is flat*

2. *over the open set $U := X \setminus \mathrm{Sing}(\mathcal{F})$ where $\mathcal{F} \subset \mathcal{T}_X$ is a sub-bundle, there is an exact sequence of $\mathcal{O}_X$-modules*

$$0 \to \mathcal{F} \to \mathcal{T}_X \to f^*\mathcal{T}_Y \to \mathrm{Frob}_X^* \mathcal{F} \to 0$$

   which implies that, over U,

$$K_X = f^*K_Y + (p-1)c_1(\mathcal{F})$$

3. *over U, the map $f^*\mathcal{T}_Y \to \mathrm{Frob}_X^* \mathcal{F}$ is the pullback of a map $\mathcal{T}_Y \to g^*\sigma^*\mathcal{F}$ where $g : Y \to X^{(p)}$ is such that $g \circ f = F_X$ and $\sigma : X^{(p)} \to X$ is the canonical k-semilinear isomorphism. Furthermore, the kernel of $\mathcal{T}_Y \to g^*\sigma^*\mathcal{F}$ is a p-closed foliation defining g.*

In general, Y can have quite nasty singularities so K_Y may only be represented by a Weil divisor. In this case, f^*K_Y may not be well-defined except over U. We will use Ekedahl's constructions to understand the leaves of a foliation $\mathcal{F}$ in characteristic p. When $\mathcal{F}$ is p-closed, we get the following useful characterization of the leaves of $\mathcal{F}$.

Lemma 2.3.3. *Let $\mathscr{F}$ be a p-closed foliation on a smooth variety X. Let $Z \subset X$ be a subvariety of dimension* $\operatorname{rank} \mathscr{F}$ *not contained in the singular locus of $\mathscr{F}$. Then Z arises as the (generically reduced) pullback of a subvariety $Z' \subset X/\mathscr{F}$ if and only if Z is a leaf of $\mathscr{F}$.*

Proof. There is always a commutative diagram of relative Frobenii,

$$\begin{tikzcd} Z \arrow[r, "F_Z"] \arrow[d] & Z^{(p)} \arrow[d] \\ X \arrow[r, "F_X"] & X^{(p)} \end{tikzcd}$$

This diagram is not Cartesian: the fiber product is rather the extension of Z by nilpotents along its conormal sheaf. If Z is a leaf, the map $\mathscr{T}_Z \to \mathscr{T}_X|_Z$ is identified with (the reflexive hull) of $\mathscr{F}_X|_Z \to \mathscr{T}_X$ therefore there is a compatible diagram of quotients by foliations

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z^{(p)} \arrow[r, equal] \arrow[d] & Z^{(p)} \arrow[d] \\ X \arrow[r] & X/\mathscr{F} \arrow[r] & X^{(p)} \end{tikzcd}$$

so the unique map $Z \to Z^{(p)} \times_{X/\mathscr{F}} X$ must be an isomorphism at the generic point by degree considerations.

Since Z is not contained in the singular locus of $\mathscr{F}$, we may shrink to assume that $Y = X/\mathscr{F}$ is smooth and $f : X \to X/\mathscr{F} = Y$ is flat. Now, let $Z' \subset X/\mathscr{F}$ be a subvariety and consider the fiber product $F := Z' \times_{X/\mathscr{F}} X$. By base change, the relative cotangent bundle $\Omega_{F/Z'}$ is the pullback of $\Omega_{X/Y} \cong \mathscr{F}^\vee$ where $f : X \to X/\mathscr{F}$ is the quotient map. Hence, if F is generically reduced, i.e., the case when Z is the preimage of Z', then $Z \to Z'$ must have zero differential. Therefore, from the diagram

$$\begin{tikzcd} & & \mathscr{T}_Z \arrow[r, "0"] \arrow[d] \arrow[dl, dashed] & \mathscr{T}_{Z'}|_Z \arrow[d] \\ 0 \arrow[r] & (\mathscr{F}|_Z)^{\vee\vee} \arrow[r] & \mathscr{T}_X|_Z \arrow[r] & \mathscr{T}_{X/\mathscr{F}}|_Z \end{tikzcd}$$

we see that $\mathscr{T}_Z \to \mathscr{T}_X|_Z$ factors through $(\mathscr{F}|_Z)^{\vee\vee} \to \mathscr{T}_X|_Z$. $\square$

Chapter 3

Descent for Product-Quotient Surfaces

This hallway seems to be on a decline.
In fact all I do now is brake.

Will Navison

3.1 Introduction

Let X be a smooth projective variety over $\overline{\mathbb{F}}_p$. In analogy with the case of elliptic curves, we say that X is *supersingular* if the q-Frobenius action on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$, for some $\ell \neq p$, has all its eigenvalues equal to $q^{i/2}$ times a root of unity for each i. The Tate conjecture predicts that, when odd cohomology vanishes, supersingularity is equivalent to *Shioda supersingularity*: that $H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell(i))$ is spanned by algebraic cycles.

In 1977, Shioda made the following conjecture predicting when a surface over an algebraically closed field k of characteristic $p > 0$ is unirational [Shi77].

Conjecture (Shioda). *Let X be a smooth projective surface over k with $\pi_1^{\text{ét}}(X) = 1$. Then X is unirational if and only if it is Shioda supersingular.*

In this chapter, we construct a counterexample to Shioda's conjecture. The examples, whose invariants are given in the D_7, D_6, and A_4 cases below, will be the minimal resolution X of certain diagonal quotients $(C \times C)/G$ where C is a Hurwitz curve of genus 14 with automorphism group $\mathrm{PSL}_2(\mathbb{F}_{13})$ and G is any copy of the groups D_6, D_7 or A_4 inside $\mathrm{PSL}_2(\mathbb{F}_{13})$. These surfaces are defined over the number field $K_+ = \mathbb{Q}(\cos \frac{2\pi}{7})$ and their reduction at infinitely many places will violate Shioda's conjecture. The first few such characteristics are $1091, 2339, 6551, 7643, \ldots$ (see §4.2). We will actually conclude the following stronger result about curves covering the reduction of X.

Theorem 3.1.1. *Let X be one of the surfaces described above. For each prime $\mathfrak{p} \subset K_+$ of good reduction, its reduction $X_{\mathfrak{p}}$ is not covered by rational or elliptic curves. However, X is simply-connected and $X_{\mathfrak{p}}$ is Shioda supersingular for infinitely many $\mathfrak{p}$.*

First, we demonstrate new methods to prove that a surface in positive characteristic is not unirational or even covered by elliptic curves. This uses the strategy of Bogomolov's proof [Bog77] of finiteness for rational and elliptic curves on a general type surface satisfying $c_1^2 > c_2$ along with an argument by infinite descent. This strategy leads to the following criterion for product-quotient surfaces, those surfaces birational to a diagonal quotient $(C_1 \times C_2)/G$ where C_i are smooth projective curves equipped with faithful G-actions.

Theorem 3.1.2 (Theorem 3.2.3 + Theorem 3.2.7)**.** *Let $\pi : X \to (C_1 \times C_2)/G$ be a product-quotient surface defined over $\mathbb{F}_p$. If X carries a symmetric $2m$-form $\omega \in H^0(X, \mathrm{Sym}^{2m} \Omega_X)$ that is "diagonal", i.e. locally $\omega = f(z_1, z_2)\mathrm{d}{z_1}^m \mathrm{d}{z_2}^m$ (see Definition 3.2.1), then X has only finitely many rational curves of fixed degree. If the vanishing locus of ω contains a big divisor then there are only finitely many elliptic curves on X of fixed degree.*

The reason for the assumption that C_1, C_2 and the G-actions are defined over $\mathbb{F}_p$ is so that we do not need to worry about "asymmetric Frobenius twists" of X (see the discussion before Theorem 3.2.3). In the main text, we will remove this unnecessary assumption at the cost of additional notation and strengthening the hypothesis by requiring symmetric forms on these related "asymmetric twists". This more general statement is necessary in the proof of Theorem 3.1.1.

Once we know that certain symmetric forms provide an obstruction to unirationality in positive characteristic, we can view Shioda's conjecture as a claim about the geography of complex surfaces using spreading out techniques. In this way, the problem of constructing examples becomes primarily one of complex geometry. Indeed, Shioda would predict that the following three properties are in tension:

1. X is simply-connected
2. X carries a diagonal (see Definition 3.2.1) symmetric form $\omega \in H^0(X, \mathrm{Sym}^{2m} \Omega_X)$
3. X has primes of supersingular reduction which, via the philosophy of the Mumford–Tate conjecture, we consider as a constraint on its Hodge structure.

Product-quotient surfaces are the only case where the author knows enough tools to check all three conditions. Still, in searching for examples, one runs into an apparent dilemma: to show (3) one wants G to be very large to control the Galois representations of C_1, C_2 but the larger G is, the more difficult (2) is to achieve. The trick needed to have our cake and eat it too is to consider curves C_i with extremely large automorphism groups but to choose G to be only a modest subgroup of $\mathrm{Aut}(C_i)$ just big enough that $C_i/G \cong \mathbb{P}^1$. This can be viewed as choosing a highly symmetric special point on the Hurwitz space of G-covers $C \to \mathbb{P}^1$ so that it has infinitely many primes of supersingular reduction although most other G-covers likely do not. A reasonable choice is to consider *Hurwitz curves*, those achieving the Hurwitz bound $\#\mathrm{Aut}(C) = 84(g(C) - 1)$. The first time the numerics are in our favor occurs at genus 14 with the so-called "first Hurwitz triplet" consisting of the three isomorphism classes of complex genus 14 curves with $\mathrm{Aut}(C) \cong \mathrm{PSL}_2(\mathbb{F}_{13})$ with $84 \cdot (14 - 1) = 1092$ automorphisms. Exploiting these symmetries, we can actually find an example breaking this tension predicted by Shioda's conjecture.

Shioda's conjecture is often restricted to the case of K3 surfaces, this case sometimes being attributed to Artin. Shioda verified the case of Kummer surfaces [Shi77] in 1977. To the author's knowledge, this conjecture remains open in general. However, the techniques developed in this thesis cannot say anything about the K3 surface case because, as recently shown by Gounelas and Liedtke, the only K3 surfaces carying a nontrivial symmetric form are, in fact, unirational [GL26].

3.2 Rational curves on product-quotient surfaces

In this section, we will use the existence of a particular type of symmetric differential form – which we refer to as *diagonal* – to obstruct unirationality. In particular, we will actually show there are only finitely many rational curves of bounded degree. The idea is inspired by techniques used to establish the hyperbolicity of complex varieties, particularly Bogomolov's proof of boundedness for fixed genus curves on surfaces of general type satisfying $c_1^2 > c_2$. In the particular case of rational curves, Bogomolov uses the fact that any symmetric differential must vanish when restricted to the curve and hence the curve lifts "holonomically" to the vanishing locus Z inside $\mathbb{P}_X(\Omega_X)$ of the symmetric form. The fact that the curve on Z arose as a holonomic lift turns out to imply it lies tangent to a particular "holonomic" foliation on Z. Bogomolov concludes his argument by showing (or invoking a theorem of Jouanolou) that algebraic leaves of a foliation form a bounded family. Up until the last step, the argument works perfectly well in any characteristic. However, the final step fails miserably for foliations in characteristic p because there are far "too many" curves tangent to a p-closed foliation. Indeed, curves tangent to the foliation are exactly those pulled back from Ekedahl's inseparable quotient by the foliation (see Chapter 2 §2.3). Often, e.g. if the Ekedahl quotient is not of general type, these curves form unbounded families.

Our replacement for the final step uses the special "diagonal" property of the chosen symmetric differential. The diagonal property will imply a certain self-similarity between the problem of finding rational curves on X and on quotients by the induced foliations. This observation sets up an argument by infinite descent.

Definition 3.2.1. Let C_1 and C_2 be smooth projective curves. The decomposition

$$\Omega_{C_1\times C_2} = \pi_1^*\Omega_{C_1} \oplus \pi_2^*\Omega_{C_2}$$

induces a map

$$\omega_{C_1\times C_2} = \pi_1^*\Omega_{C_1} \otimes \pi_2^*\Omega_{C_2} \hookrightarrow \operatorname{Sym}^2 \Omega_{C_1\times C_2}$$

realizing $\omega_{C_1\times C_2}$ as a direct summand. We call a symmetric $2m$-form *diagonal* if it arises from a section of $\omega_{C_1\times C_2}^{\otimes m}$ along the m-fold power of this map.

Definition 3.2.2. Let $f : C_1 \times C_2 \dashrightarrow X$ be a generically étale dominant rational map from a product of smooth curves to a smooth surface X. Call a section $\omega \in H^0(X, \operatorname{Sym}^{2m} \Omega_X)$ *diagonal* if $f^*\omega$ is diagonal.

Let $\pi : X \to (C_1 \times C_2)/G$ be the minimal resolution of a product-quotient surface over a perfect field k. Denote by E the exceptional divisor of π. We also need to consider associated "asymmetrically Frobenius twisted" surfaces, the minimal resolution

$$\pi^{a,b} : X^{a,b} \to (C_1^{(p^a)} \times C_2^{(p^b)})/G$$

of the diagonal quotient of the twisted curves with their induced G-actions. Here $C^{(p^a)}$ denotes the a-th Frobenius twist of C meaning the pullback of $C \to \operatorname{Spec} k$ along $\operatorname{Frob}^a : \operatorname{Spec} k \to \operatorname{Spec} k$ with its functorially induced G-action. Note that $X^{a,a} = X^{(p^a)}$. When C_1, C_2, and the G-actions are defined over $\mathbb{F}_p$, then actually X and $X^{a,b}$ are isomorphic because C and $C^{(p)}$ are G-equivariantly isomorphic. Usually $X^{a,b}$ has very similar geometry to X but they are not always isomorphic or even have the same numerical invariants (see Example 3.6.2). Because G acts diagonally, the maps $F_1 : C_1 \times C_2 \to C_1^{(p)} \times C_2$ and $F_2 : C_1 \times C_2 \to C_1 \times C_2^{(p)}$ applying relative Frobenius in the first or second coordinate respectively are G-equivariant. Therefore, they induce maps $F_1 : X^{a,b} \to X^{a+1,b}$ and $F_2 : X^{a,b} \to X^{a,b+1}$.

Theorem 3.2.3. *If, for all $a, b \geq 0$, the surface $X^{a,b}$ carries a diagonal symmetric form, then X has only finitely many rational curves of fixed degree.*

In particular, this sequence of diagonal symmetric forms gives an obstruction to unirationality of X. Any section of a tensor power of Ω_X (or of any Schur functor) obstructs separable unirationality via pulling back to $\mathbb{P}^2$. The difficulty is in ruling out inseparable maps $\mathbb{P}^2 \dashrightarrow X$. Instead of worrying about this problem, we study the rational curves individually. This has the advantage that, up to reparameterization, we may assume the curves are generically immersed and hence restriction of forms works as in characteristic zero.

Example 3.2.4. It is important to note that the existence of some (not necessarily diagonal) symmetric form is not sufficient to obstruct unirationality. For example, Mumford [Mum61] constructed Enriques surfaces X in characteristic 2 carrying a 1-form ω. These X are all unirational, in fact an Enriques surface in characteristic 2 has a 1-form if and only if it is unirational by [CDL25, Proposition 1.3.8] and [Bla82]. In this case, the foliation $\mathscr{F}$ associated to ω defines a quotient map $X \to X/\mathscr{F}$ with $X/\mathscr{F}$ either rational or a unirational K3. We learn that all but finitely many rational curves on X descend to $X/\mathscr{F}$, but of course there are many such curves.

3.2.1 Proof of Theorem 3.2.3

The idea of the proof is as follows. For a rational curve $h : C \to X$ consider its lift $t_h : C \to \mathbb{P}_X(\Omega_X)$. Because $h^*\omega = 0$, the lift lands inside the vanishing locus Z of ω viewed as a section of $\mathscr{O}_{\mathbb{P}}(2m)$. Either h is contained in a component that does not dominate X, which comprises a finite number of curves that can be set aside, or it lies tangent to the holonomic foliation. We will identify these holonomic foliations with those induced on X by the projections $\pi_i : C_1 \times C_2 \to C_i$ through the quotient structure. So h must be tangent to, for example, the foliation induced by π_2. Applying Lemma 2.3.3, it arises as the pullback of $h' : C' \to X'$ along the quotient map $X \to X'$. This step is captured in Figure 3.1. It turns out that X' is also a product-quotient surface, birationally equivalent to $X^{1,0}$. We assume that $X^{1,0}$ also carries a diagonal symmetric form so we can run the argument again. Since $\deg \pi_1 \circ h$ or $\deg \pi_2 \circ h$ drops at each stage, this sets up a contradiction by infinite descent.

Proof of Theorem 3.2.3. Choose diagonal symmetric forms $\omega^{a,b} \in H^0(X^{a,b}, \mathrm{Sym}^{2m} \Omega_{X^{a,b}})$ on each surface $X^{a,b}$. Denote by $\omega := \omega^{0,0}$ the form on X. Let $Z = Z(\omega) \subset \mathbb{P}_X(\Omega_X)$ be the vanishing locus of ω viewed as a section of $\mathscr{O}_{\mathbb{P}}(2m)$. Denote by $V(\omega) \subset X$ the vanishing locus of ω viewed as a section of $\mathrm{Sym}^{2m} \Omega_X$. We will first set aside a finite collection of rational curves on each $X^{a,b}$ grouped into the following types:

(1) exceptional curves of the resolution $\pi^{a,b}$

(2) images under q of the fibers of the projection maps π_1 and π_2 on $C_1 \times C_2$

(3) images of components of Z not dominating X

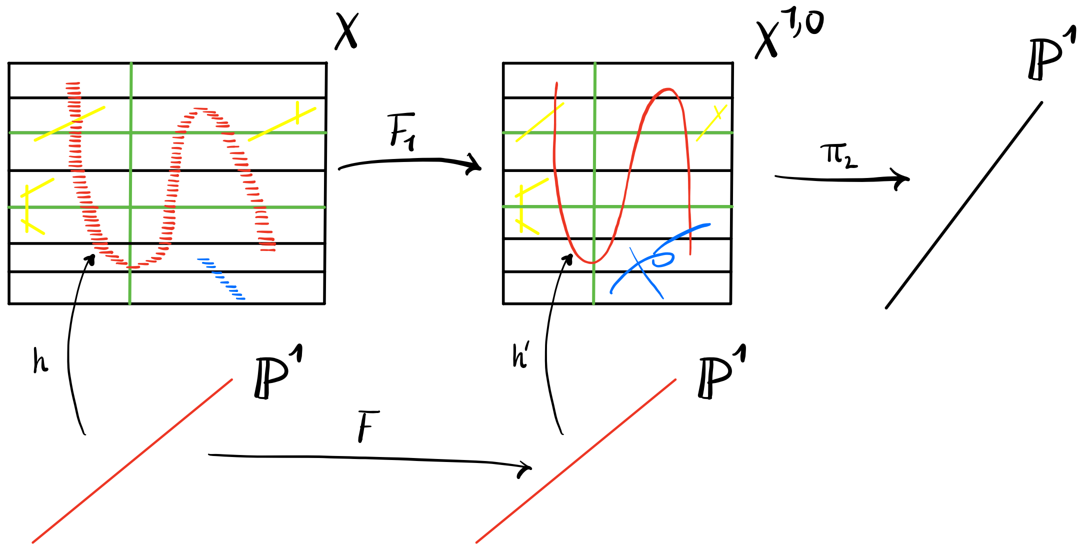


Figure 3.1: Descent of a rational curve (red) along $X \longrightarrow X' = X^{1,0}$. Beginning with a rational curve $h : \mathbb{P}^1 \longrightarrow X$ lifting to a curve on S tangent to the foliation $\mathscr{F}$ – represented here by the curve being formed from segments tangent to the fibers of π_2 – it is the pullback of a rational curve h' on X' with $\deg \pi_2 \circ h = p \deg \pi_2 \circ h'$. Exceptional divisors of the resolution (curves of type (1) in yellow) and fibers of either projection that are rational (curves of type (2) in green) are pulled back to themselves. However, rational components of $V(\omega^{1,0})$ (curves of type (3) in blue) have their pullback under partial Frobenius again rational of p times the degree. Such a pullback is denoted on X (blue, marked as p-th order tangent to the fibers of π_2).

A curve is labeled uniquely by the first applicable type in the list (i.e., curves of type (3) are by definition not type (1) or (2)). Note that there are only finitely many curves of type (2) since otherwise $q^*\omega$ would have to be zero, but q is assumed to be étale. We run Bogomolov's construction of a foliated cover induced by ω (see Chapter 1 §1.5.4) on $\mathbb{P}_X(\Omega_X)$ compatibly with the covering product surface. This gives a diagram,

$$\begin{tikzcd} Z(q^*\omega) \arrow[r, dashed] \arrow[d, hook] & Z(\omega) \arrow[d, hook] \\ \mathbb{P}(\Omega_{C_1\times C_2}) \arrow[r, dashed] \arrow[d] & \mathbb{P}_X(\Omega_X) \arrow[d] \\ C_1 \times C_2 \arrow[r, dashed, "f"] & X \end{tikzcd}$$

We are only interested in components of Z which dominate X since there are at most finitely many curves that lift to the other components (since their image is a union of finitely many curves in X). Each of those components of Z is dominated by a corresponding component of $Z(q^*\omega)$ dominating $C_1 \times C_2$. However, $q^*\omega$ is a diagonal form, meaning in local coordinates it has the form $f(z_1, z_2)\mathrm{d}{z_1}^m\mathrm{d}{z_2}^m$. If $[u : v]$ are the corresponding homogeneous fiber coordinates on $\mathbb{P}(\Omega_{C_1\times C_2})$, then this section is represented by $f(z_1, z_2)u^m v^m$. Its vanishing locus is therefore the union of the inverse image of $V(f) \subset C_1 \times C_2$ and the two divisors $u = 0$ and $v = 0$. The latter are precisely the two sections $C_1 \times C_2 \to \mathbb{P}(\Omega_{C_1\times C_2})$ corresponding to the quotients $\Omega_{C_1\times C_2} \twoheadrightarrow \pi_i^*\Omega_{C_i}$. Therefore, the resolution $g : S \to X$ of each component of Z that dominates X fits into a diagram

$$C_1 \times C_2 \dashrightarrow S \to X$$

factoring the rational map q. Moreover, the induced holonomic foliations are compatible so $\mathscr{F} \subset \mathscr{T}_S$ pulls back to one of the standard foliations $\pi_i^*\mathscr{T}_{C_i} \subset \mathscr{T}_{C_1\times C_2}$. In particular, $\mathscr{F}$ is p-closed since this can be checked étale-locally. Indeed, the diagonal action makes the foliations $\pi_i^*\mathscr{T}_{C_i} \subset \mathscr{T}_{C_1\times C_2}$ equivariant and they descend to the foliations $\mathscr{F}_i$ on X induced by the projection maps $\sigma_i : X \to C_i/G$. Furthermore, $S \to X$ is birational to the quasi-section of $\mathbb{P}(\Omega_X) \to X$ induced by one of the foliations $\mathscr{F}_i \subset \mathscr{T}_X$ by commutativity of the diagram,

$$\begin{tikzcd} \mathbb{P}(\Omega_{C_1\times C_2}) \arrow[r, dashed, "\mathrm{d}q"] \arrow[d, shift left] & \mathbb{P}(\Omega_X) \arrow[d, shift left] \\ C_1 \times C_2 \arrow[r, dashed, "q"] \arrow[u, bend left, "\pi_i^*\mathscr{T}_{C_i}"] & X \arrow[u, dashed, bend left, "\mathscr{F}_i"] \end{tikzcd}$$

and the sections form a cartesian diagram over an open set. Over that open set $S \to X$ agrees with the section defined by $\mathscr{F}_i$ since they are both closed inside $\mathbb{P}(\Omega_X)$ with the same preimage in $\mathbb{P}(\Omega_{C_1 \times C_2})$.

In particular, $S \to X$ is birational and we may assume that the foliation $\mathscr{F}$ on S pulls back to the foliation $\pi_1^* \mathscr{T}_{C_1} \subset \mathscr{T}_{C_1 \times C_2}$ on $C_1 \times C_2$ defined by the fibration $\pi_2 : C_1 \times C_2 \to C_2$. The other case is completely symmetric. The quotient of $C_1 \times C_2$ by this foliation is the geometric Frobenius on C_1 i.e., the map $F_1 : C_1 \times C_2 \to C_1^{(p)} \times C_2$ applying geometric Frobenius in the first coordinate. Because G acts diagonally, the map $F_1 : C_1 \times C_2 \to C_1^{(p)} \times C_2$ is G-equivariant so it produces a map $X \to X'$ of the minimal resolutions of the quotients. In our previous notation, $X' = X^{1,0}$. Because the partial Frobenii are G-equivariant, there is a diagram,

$$\begin{tikzcd}
C_1 \times C_2 \arrow[r, "F_1"] \arrow[d, dashed, "q"] & C_1^{(p)} \times C_2 \arrow[r, "F_2"] \arrow[d, dashed, "f'"] & C_1^{(p)} \times C_2^{(p)} \arrow[d, dashed, "q^{(p)}"] \\
X \arrow[r] & X' \arrow[r, dashed] & X^{(p)} \\
S \arrow[r] \arrow[u] & S/\mathscr{F} \arrow[r] \arrow[u, dashed] & S^{(p)} \arrow[u]
\end{tikzcd}$$

Each of the top squares is generically Cartesian because q and f' are generically étale. The map $S \to X \dashrightarrow X'$ must factor through $S/\mathscr{F}$ because its tangent map kills $\mathscr{F} \subset \mathscr{T}_S$. This is because $X \dashrightarrow X'$ is étale-locally identified with F_1 and $\mathscr{F}$ pulls back to the foliation $\pi_1^* \mathscr{T}_{C_1} \subset \mathscr{T}_{C_1 \times C_2}$ on $C_1 \times C_2$ which induces F_1. Now, since both $S \to S/\mathscr{F}$ and $S \to X \dashrightarrow X'$ have degree p, we conclude that $S/\mathscr{F} \dashrightarrow X'$ is birational.

This is relevant for the following reason. Let $h : C \to X$ be a rational curve not of type (1). Since $h^* \omega = 0$, the lift $t_h : C \to \mathbb{P}_X(\Omega_X)$ factors through one of the components of Z. As before, we restrict our attention to those curves h factoring through $S \to X$. By Lemma 1.5.19, $\tilde{h} : C \to S$ is a leaf of $\mathscr{F}$ and therefore arises as the pullback of a curve $h' : C' \to X'$ via an application of Lemma 2.3.3. Because $C \to C'$ is a dominant map of curves, C' is also rational.

Starting with $h : C \to X$, we have produced a "simpler" curve $h' : C' \to X'$ on a similar surface X'. Precisely, $\deg \pi_2 \circ h' = \frac{1}{p} \cdot \deg \pi_2 \circ h$ because the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[r, bend left, "\pi_2"] & X' \arrow[r, "\pi_2"] & C_2 \\
C \arrow[r, "F"] \arrow[u, "h"] & C' \arrow[u, "h'"] &
\end{tikzcd}$$

commutes (see also Figure 3.1). Intuitively, this is because, for each point of intersection between h' and a fiber, the corresponding intersection point of h on X has multiplicity p because the curve is made p-th order tangent to the fibers. Furthermore, $X' = X^{1,0}$ also carries a diagonal symmetric form by assumption, so we may repeat this process with h'. Restoring the symmetry of the situation, at each step either the sum $\deg \pi_1 \circ h + \deg \pi_2 \circ h$ drops or one of $\deg \pi_i \circ h = 0$, meaning h lives inside a fiber of one of the projections. Those rational curves are already accounted for in type (1) or (2). Otherwise, as we iterate this process, the degree against one of the π_i must decrease so eventually $h^{(i)} : C^{(i)} \to X^{a,b}$ must terminate at one of the finitely many rational curves already set aside where the process cannot continue.

Let us take stock of the possibilities for rational curves on X not currently ruled out. There are the curves of type (1) - (3) set aside earlier and all rational curves which can be produced by pulling back these curves a finite number of times along a sequence of partial Frobenii. Notice that only curves of type (3) may proliferate after applying Frobenii since fibers of π_i pull back to fibers and exceptional curves pull back to exceptional curves. Since the remaining curves have positive degree against both projections π_1 and π_2, the partial Frobenius pullback increases their degree against the ample divisor $F_1 + F_2$. Since there are only finitely many curves of type (1), and at each stage we apply either F_1^{-1} or F_2^{-1}, there are only finitely many rational curves of each fixed degree on X. □

Remark 3.2.5. Explicitly, every rational curve on X is either of type (1) - (3) or is the pull-back of a curve of type (3) on $X^{a,b}$. Note that, $F_1 \circ F_2 = F_X$ is simply the relative Frobenius of X so any curve pulled back along both partial Frobenii becomes non-reduced. Alternatively, one can say that no reduced curve can be tangent to both foliations simultaneously. Therefore, in total the rational curves on X are

$$\{\text{type (1) - (3) on } X\} \cup \bigcup_{a \geq 1} (F_1^a)^{-1} \{\text{type (3) on } X^{a,0}\} \cup \bigcup_{b \geq 1} (F_2^b)^{-1} \{\text{type (3) on } X^{0,b}\}$$

so this analysis produces an upper bound of

$$\sum_{a \leq \log_p d} \#\{\text{type (3) on } X^{a,0}\} + \sum_{b \leq \log_p d} \#\{\text{type (3) on } X^{0,b}\}$$

for the number of rational curves on X not of type (1) or (2) of degree $\leq d$. Notice moreover, that any curve of type (3) on X' not tangent to F_2 does pull back along F_1 to a reduced rational curve on X with p times the degree (it remains rational because $F_1^{-1}(C) \to C$ factors Frobenius $F_C : C^{(1/p)} \to C$ since $F_X^{-1}(C)$ is a non-reduced copy of $C^{(1/p)}$). If X is defined over $\overline{\mathbb{F}}_p$, then only finitely many $X^{a,b}$ are non-isomorphic so the collection of curves $\{\text{type (3) on } X^{a,0}\}$ has uniformly bounded size. In this case, if any such rational curve of type (3) exists, then X genuinely does contain infinitely many rational curves although those curves will have exponentially growing degree.

The explicit description of curves on X immediately gives the following corollary.

Theorem 3.2.6. *Suppose on each $X^{a,b}$ the diagonal symmetric forms span a line bundle $\mathscr{L} \hookrightarrow \mathrm{Sym}^{2m} \Omega_{X^{a,b}}$ whose stable base locus $\mathbb{B}(\mathscr{L}) := \bigcap_{m \geq 0} B(\mathscr{L}^m)$ contains no rational curves outside the exceptional locus of $\pi^{a,b}$, then X contains only finitely many rational curves.*

Proof. Since the argument can be run for any choices of diagonal symmetric forms $\omega^{a,b}$ we can restrict the curves of type (3) further to only include curves in the stable base locus formed by intersecting $V(\omega^{a,b})$ over all diagonal symmetric forms. The assumption means there are no rational curves of type (3) on any $X^{a,b}$. Hence the descent process must terminate in curves of type (1) or (2) but these are mapped to themselves under partial Frobenius preimage so X has only the finite number of rational curves of types (1) and (2). □

In Remark 3.2.5, we saw this result is almost sharp. If the vanishing locus of the symmetric form $\omega^{a,b}$ contains a rational curve (not of type (1) or (2)) then its partial Frobenius iterates (under whichever of F_1 or F_2 it is not tangent to) form a sequence of rational curves of unbounded degree. We will return to this observation in more detail in §3.5.

In characteristic zero, the same methods apply much more directly since curves tangent to the standard foliations must actually be fibers. In that case, we obtain finiteness without disallowing curves of type (3), recovering the main results of [GRVR18] by essentially the same argument.

3.2.2 Elliptic curves

A similar argument applies to elliptic curves as well.

Theorem 3.2.7. *Suppose, for all $a, b \geq 0$, the surface $X^{a,b}$ carries a diagonal symmetric form. Then X has only finitely many rational curves of fixed degree. If the vanishing loci of these diagonal forms contain a big divisor then there are only finitely many elliptic curves on X of fixed degree.*

Proof. All the descent steps are identical. However, for an elliptic curve $h : C \to X$, it is not automatic that $h^*\omega = 0$. What is true is that if $h^*\omega$ has a zero then $h^*\omega = 0$ identically. To force the lift $t_h : C \to \mathbb{P}_X(\Omega_X)$ to factor through $Z(\omega)$, we will want $h(C)$ to meet $V(\omega) \subset X$ the locus where ω is zero as a section of $\mathrm{Sym}^{2m} \Omega_X$. This is how the extra assumption is used. If $V(\omega)$ contains a big divisor $D \subset X$ then all but finitely many curves must meet $V(\omega)$. Therefore, we must amend the types of curves on $X^{a,b}$ set aside at the beginning. Indeed, consider the finitely many elliptic curves on $X^{a,b}$ grouped into the following types:

(1) exceptional curves of the resolution $\pi^{a,b}$

(2) images under q of the fibers of the projection maps $\pi_i : C_1 \times C_2 \to C_i$

(3) images of components of Z not dominating X

(4) curves not meeting $V(\omega)$.

Now the argument proceeds exactly as before by forcing any curve not of type (1) - (4) above to arise as the pullback along a sequence of partial Frobenii. Now both curves of type (3) and (4) may proliferate under these operations. However, since the sum of their degrees increases at each stage by at least a factor of p, there can only be finitely many rational and elliptic curves in each fixed degree. □

3.3 Invariants of product-quotient surfaces

Throughout this section, fix a finite group G and two smooth projective curves C_1, C_2 defined over the complex numbers, each equipped with a faithful action of G. Let $Y = (C_1 \times C_2)/G$ be the quotient by the diagonal action. Since G acts faithfully on each curve, the fixed locus of any nontrivial element of G on $C_1 \times C_2$ is a finite set of points. Moreover, the stabilizer of any point on C_i must be cyclic, since it acts faithfully on the one-dimensional tangent space of the curve at that point. Therefore, the singularities of Y are isolated cyclic quotient singularities (see Definition 3.3.1). Let $\varphi : X \to Y$ be the minimal resolution of these isolated cyclic quotient singularities. Write Y° for the smooth locus of Y, which is exactly the locus over which $C_1 \times C_2 \to Y$ is étale. We write $\overline{C}_i := C_i/G$. These varieties fit into a diagram

$$\begin{tikzcd}
 & & & & X \arrow[d, "\varphi"] \arrow[ddl, bend right, "\sigma_1"'] \arrow[ddr, bend left, "\sigma_2"] & \\
 & C_1 \times C_2 \arrow[rrr, "q"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & Y := (C_1 \times C_2)/G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
C_1 \arrow[rrr, bend right] & & C_2 \arrow[rrr, bend right] & \overline{C}_1 & & \overline{C}_2
\end{tikzcd}$$

We usually assume that $C_i/G \cong \mathbb{P}^1$; otherwise, the fundamental group of X will always be infinite. However, the only place this substantively changes the discussion is when working with a monodromy presentation in §§3.3.3–3.3.4. In this section, we compile computations of the various invariants of the surface X in terms of the data G, C_1, C_2 and the actions. We state these results over the complex numbers for simplicity and because the proofs, especially those related to the fundamental group, rely on topological methods. Later, we will use base change and spreading out techniques to apply these results in arithmetic settings of interest.

3.3.1 Finite quotient singularities

Definition 3.3.1. A surface singularity is called *finite quotient* if it is analytically isomorphic to the singularity produced as the quotient $\mathbb{A}^2/G$ where $G \subset \mathrm{GL}_2(\mathbb{C})$ is a finite group. The singularity is called *cyclic* if G can be taken to be a cyclic group. In that case, the action is always equivalent to one generated by the diagonal matrix

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^a \end{pmatrix}$$

with $\zeta_n := e^{2\pi i/n}$ and $\gcd(a, n) = 1$. We call this a $\frac{1}{n}(1, a)$ singularity.

To see why we need only consider diagonal matrices of the above form, note that any finite-order element of $\mathrm{GL}_2(\mathbb{C})$ is diagonalizable. Furthermore, a finite abelian subgroup is simultaneously diagonalizable; see, for example, [Ful93, §2.6]. Therefore, a cyclic singularity is equivalent to the quotient of $\mathbb{A}^2$ by the group generated by a matrix

$$\begin{pmatrix} \zeta & 0 \\ 0 & \xi \end{pmatrix}$$

where ζ and ξ are roots of unity. Let n be the order of the subgroup generated by this matrix. If ζ, ξ are not primitive n-th roots of unity, then this singularity is isomorphic to a singularity of the form $\frac{1}{n'}(1, a')$ for some $n' < n$. Indeed, suppose $\mathrm{ord}(\zeta) = k$ for $k < n$. Then the subgroup generated by the k-th power

$$\begin{pmatrix} 1 & 0 \\ 0 & \xi^k \end{pmatrix}$$

fixes the x-axis and then its quotient is the map $\mathbb{A}^2 \xrightarrow{(x,y)\mapsto(x,y^{\frac{n}{k}})} \mathbb{A}^2$. Therefore, our singularity is isomorphic to the quotient of $\mathbb{A}^2$ by the group generated by

$$\begin{pmatrix} \zeta & 0 \\ 0 & \xi^{\frac{n}{k}} \end{pmatrix}.$$

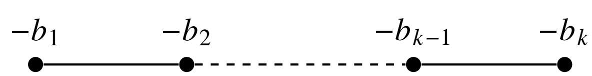
Repeating this process, we reduce to the case that ζ and ξ are both primitive n-th roots of unity. Then $\xi = \zeta^a$ for some a coprime to n.

Remark 3.3.2. Note that among these quotient singularities, the *canonical* or *ADE* surface singularities are those for which we can choose $G \subset \mathrm{SL}_2(\mathbb{C})$. Among these, the cyclic canonical singularities are exactly those of type $\frac{1}{n+1}(1, n)$ called type A_n.

Cyclic quotient singularities are always klt, hence rational $\mathbb{Q}$-Gorenstein. For the reader's convenience, we recall some properties of the well-known Hirzebruch–Jung resolution, but the interested reader might also consult [Ful93, §2.6]. The minimal resolution of a quotient singularity has, as its exceptional divisor, a tree of rational curves. For the cyclic quotient singularities, this tree is a chain of rational curves with self-intersections determined by the arithmetic of the fraction $\frac{n}{a}$. Let $[\![b_1, b_2, \cdots, b_\ell]\!]$ denote the continued fraction representation of $\frac{n}{a}$ using the convention

$$[\![b_1, b_2, \cdots, b_\ell]\!] := b_1 - \cfrac{1}{b_2 - \frac{1}{b_3 - \cdots}} = \frac{n}{a}.$$

Proposition 3.3.3. *The exceptional divisor of the minimal resolution of a $\frac{1}{n}(1, a)$ singularity is a chain $Z_1 + \cdots + Z_\ell$ of smooth rational curves Z_i ordered such that $Z_i \cdot Z_{i+1} = 1$ and $Z_i \cdot Z_j = 0$ for $|i - j| > 1$ and with self-intersections $Z_i^2 = -b_i$ where $b_1, \ldots, b_\ell$ are the numbers appearing in the continued fraction representation $\frac{n}{a} = [\![b_1, b_2, \cdots, b_\ell]\!]$. Therefore, the dual graph of E is*



3.3.2 Chern numbers

Denote the genera $g(C_i)$ by g_i. Following [BP12, §1], we refer to the multi-set of *types* $\frac{1}{n}(1, a)$ (where the types are symbols subject to the relation $\frac{1}{n}(1, a) = \frac{1}{n}(1, a')$ iff $a' \equiv a^{\pm 1}$ mod n) of singularities on Y as the *basket* $\mathscr{B}(Y)$ of singularities. Furthermore, we define the following numerical invariants of a cyclic quotient singularity.

Definition 3.3.4. (cf. [BP12, Definition 1.4]) Let $\frac{n}{a} = [\![b_1, \ldots, b_\ell]\!]$ be the continued fraction representation and $0 < a' < n$ the inverse of a in $(\mathbb{Z}/n\mathbb{Z})^\times$. Then define,

1. $k(\frac{n}{a}) := -2 + \frac{2+a+a'}{n} + \sum_i (b_i - 2)$
2. $e(\frac{n}{a}) := \ell + 1 - \frac{1}{n}$
3. $B(\frac{n}{a}) := 2e(\frac{n}{a}) + k(\frac{n}{a})$

Let $\mathscr{B}$ be a basket of singularities. Then we define the invariants of the basket by summation,

$$k(\mathscr{B}) := \sum_{x\in\mathscr{B}} k(x) \quad e(\mathscr{B}) := \sum_{x\in\mathscr{B}} e(x) \quad B(\mathscr{B}) := \sum_{x\in\mathscr{B}} B(x) = 2e(\mathscr{B}) + k(\mathscr{B})$$

Proposition 3.3.5. *[BP12, Proposition 1.5] In the notation of this section*

$$K_X^2 = \frac{8(g_1-1)(g_2-1)}{\#G} - k(\mathscr{B}) \quad \textit{and} \quad c_2(X) = \frac{4(g_1-1)(g_2-1)}{\#G} + e(\mathscr{B})$$

3.3.3 The fundamental group

In this section, we suppose that the curve C with a G-action satisfying $C/G \cong \mathbb{P}^1$ is presented, via the Riemann existence theorem, as the topological covering space associated to a map $\varphi : \pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}) \twoheadrightarrow G$ sending the simple loop around p_i to the monodromy $g_i = \varphi(\gamma_i)$ around p_i. Call the sequence of elements $g_1, \dots, g_r$ a *spherical* list of generators, meaning a list that generates and satisfies $g_1 \dots g_r = 1$. Write m_i for the order of the monodromy $g_i = \varphi(\gamma_i)$ around p_i. It is more convenient to consider the finitely presented group:

Definition 3.3.6. The *polygonal group* of *signature* $(m_1, \dots, m_r)$ is the following finitely presented group:

$$\mathbb{T}(m_1, \dots, m_r) = \left\langle c_1, \dots, c_r \mid c_1^{m_1} = 1, \dots, c_r^{m_r} = 1, c_1 \cdots c_r = 1 \right\rangle.$$

When the signature $(m_1, m_2, m_3) = (a, b, c)$ has length 3 we call this group the *triangle group* and write $\Delta(a, b, c) := \mathbb{T}(a, b, c)$ for emphasis. The name is given because $\Delta(a, b, c)$ is the group of oriented isometries preserving the triangular hyperbolic tiling with angles $(\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c})$. Sometimes this group is called the *von Dyck group* and the term "triangle group" is reserved for its index 2 cover

$$\left\langle x, y, z \mid x^2, y^2, z^2, (xy)^a, (yz)^b, (zx)^c \right\rangle$$

consisting of all (not necessarily orientation-preserving) isometries of the (a, b, c) triangular hyperbolic tiling generated by reflections x, y, z about the three sides of one of the triangles.

Remark 3.3.7. $\mathbb{T}(m_1, \ldots, m_r)$ is the fundamental group of the orbifold curve $[C/G]$ which is $\mathbb{P}^1$ with a cyclic orbifold stabilizer at p_i of order m_i.

The cover $C \to \mathbb{P}^1$ can be interpreted as the étale cover of orbifolds $C \to [C/G]$ and is determined by $\varphi : \mathbb{T}(m_1, \ldots, m_r) \twoheadrightarrow G$ sending c_i to the monodromy around p_i. In what follows, we have two curves C_1, C_2 each presented as a G-cover of $\mathbb{P}^1$. Let $m_1, \ldots, m_r$ be the orders of monodromy for $C_1 \to \mathbb{P}^1$ and let $n_1, \ldots, n_s$ be the orders of monodromy for $C_2 \to \mathbb{P}^1$ and let $\varphi_1 : \mathbb{T}(m_1, \ldots, m_r) \twoheadrightarrow G$ and $\varphi_2 : \mathbb{T}(n_1, \ldots, n_s) \twoheadrightarrow G$ be the classifying maps defined by spherical generators $g_1, \ldots, g_r \in G$ and $h_1, \ldots, h_s \in G$ respectively.

Lemma 3.3.8. *The fundamental group of Y° fits into a diagram of exact sequences*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(C_1) \times \pi_1(C_2) \arrow[r] \arrow[d, equal] & \pi_1(Y^\circ) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[r] \arrow[d, hook, "\Delta"] & 1 \\
1 \arrow[r] & \pi_1(C_1) \times \pi_1(C_2) \arrow[r] & \mathbb{T}(m_1, \ldots, m_r) \times \mathbb{T}(n_1, \ldots, n_s) \arrow[r] & G \times G \arrow[r] & 1
\end{tikzcd}$$

Proof. This arises from the diagram of étale covers of orbifolds,

$$\begin{tikzcd}
(C_1 \times C_2)^\circ \arrow[r, hook] \arrow[d, "\text{ét}"] & C_1 \times C_2 \arrow[r, equal] \arrow[d, "\text{ét}"] & C_1 \times C_2 \arrow[d, "\text{ét}"] \\
Y^\circ \arrow[r] & [(C_1 \times C_2)/G] \arrow[r] & [C_1/G] \times [C_2/G]
\end{tikzcd}$$

where the left downward map is a G-cover and the right is a $(G \times G)$-cover compatible with the diagonal inclusion $\Delta : G \hookrightarrow (G \times G)$. Furthermore, since the inclusion of a codimension ≥ 2 subset of a smooth variety is an isomorphism on π_1, we conclude that the natural map

$$\pi_1(Y^\circ) \to \pi_1([(C_1 \times C_2)/G])$$

is an isomorphism because $\pi_1((C_1 \times C_2)^\circ) \xrightarrow{\sim} \pi_1(C_1 \times C_2)$ is an isomorphism and the fibration exact sequences form a commutative diagram. □

Write $\mathbb{H}(G; \varphi_1, \varphi_2) := (\varphi_1 \times \varphi_2)^{-1}(\Delta)$ which is identified with $\pi_1(Y) = \pi_1(Y^\circ)$.

Lemma 3.3.9. $\pi_1(X) = \pi_1(Y) = \pi_1(Y^\circ)/\mathrm{Tors}(\pi_1(Y^\circ))$ *where* $\mathrm{Tors}(G)$ *denotes the normal subgroup of G generated by all elements of finite order.*

Proof. That the map $\pi_1(X) \to \pi_1(Y)$ is an isomorphism is well-known for rational surface singularities. Since Y has klt singularities, it also follows from [Kol93, Theorem 7.8]. The second statement follows from the main result of [Arm65, Arm68]; see also [BCGP12, Proposition 3.4]. □

We will need some notions from [BCGP12]. Denote by $\varphi_i : \mathbb{T}_i \twoheadrightarrow G$ the presentations by polygonal groups and $R_i = \ker \varphi_i \triangleleft \mathbb{T}_i$ their kernels which are identified with $\pi_1(C_i)$. Let $C_i \subset \mathbb{T}_i$ denote the set of generators $\{c_1, \ldots, c_r\}$. By [BCGP12, Lemma 4.9] there are finite sets $\mathcal{N}_i \subset \mathbb{T}_1 \times \mathbb{T}_2$ so that

1. $\mathcal{N}_1 \subset \{(c^\ell, zd^nz^{-1}) \in \mathbb{T}_1 \times \mathbb{T}_2 \mid c \in C_1, d \in C_2, \ell, n \in \mathbb{N}, z \in \mathbb{T}_2\}$
2. $\mathcal{N}_2 \subset \{(zc^\ell z^{-1}, d^n) \in \mathbb{T}_1 \times \mathbb{T}_2 \mid c \in C_1, d \in C_2, \ell, n \in \mathbb{N}, z \in \mathbb{T}_1\}$
3. $\mathcal{N}_i \subset \mathbb{H}(G; \varphi_1, \varphi_2)$
4. the normal closure of $\mathcal{N}_i$ in $\mathbb{H}$ is $\mathrm{Tors}(\mathbb{H})$.

Let $L_i := \pi_i(\mathcal{N}_i)$ be the set of first/second coordinates of the finite set $\mathcal{N}_i$. Thus we can take $L_i \subset \mathbb{T}_i$ to be the set of powers of generators of $\mathbb{T}_i$ whose image in G is conjugate to the image of a power of a generator of the other presentation. The following lemma is proved in [BCGP12, p.25].

Lemma 3.3.10. *There is an exact sequence*

$$1 \to E_2 \to \mathbb{H}/\mathrm{Tors}(\mathbb{H}) \to \mathbf{H} \to 1$$

where E_2 is a finite group and $\mathbf{H} \subset \mathbb{T}_1/\langle\langle L_1 \rangle\rangle_{\mathbb{T}_1} \times \mathbb{T}_2/\langle\langle L_2 \rangle\rangle_{\mathbb{T}_2}$ is a finite index subgroup. Here $\langle\langle L_i \rangle\rangle_{\mathbb{T}_i}$ denotes the subgroup normally generated by $L_i \subset \mathbb{T}_i$.

Corollary 3.3.11. $\mathbb{H}/\mathrm{Tors}(\mathbb{H})$ *is finite if and only if the orbifold groups $\mathbb{T}_i/\langle\langle L_i \rangle\rangle_{\mathbb{T}_i}$ are finite for $i = 1, 2$.*

In the remainder of the section, we prove a sufficient condition to check that a product-quotient surface with $C_1 = C_2$ (really, we just need that they have the same monodromy representation over $\mathbb{P}^1$ although the branch points may be different) is simply-connected. This will be useful in examples.

From now on, we are working with $(g_1, \ldots, g_r) = (h_1, \ldots, h_s)$. Recall that $E := \pi_1(Y^\circ)$ is the subgroup of $\mathbb{T}(m_1, \cdots, m_r)^2$ on which the two projections to G agree. We want conditions on $(g_1, \ldots, g_r)$ for which E is normally generated by torsion elements.

Definition 3.3.12. We say that a list of spherical generators $(g_1, \ldots, g_r)$ of a finite group G *extends a good presentation* if there exists a finite presentation

$$G \cong \langle a_1, \ldots, a_s \mid R \rangle$$

where R is a list of words in $a_1, \ldots, a_s$ so that the following hold:

1. $\{a_1, \ldots, a_s\} \subset \{g_1, \ldots, g_r\}$ under the isomorphism
2. each $g_i = h_i a_{j(i)}^{\ell_i} h_i^{-1}$ for some integers $j(i), \ell_i$ and $h_i \in G$
3. each word $w \in R$ is conjugate to one of the following forms
   (a) the words $w = a_i^{e_1} h(\underline{a}) a_j^{e_2} h(\underline{a})^{-1}$ where $h(\underline{a})$ is any word in the variables $a_1, \ldots, a_s$ and e_1, e_2 are integers
   (b) for any choice $\tilde{h}_i(\underline{a})$ of a word evaluating to h_i (for some choice of h_i as in (2)), the words
   $$\prod_{i=1}^{r} \tilde{h}_i(\underline{a}) a_{j(i)}^{\ell_i} \tilde{h}_i(\underline{a})^{-1}$$

Proposition 3.3.13. *Suppose that $(g_1, \ldots, g_r)$ are spherical generators of G extending a good presentation. Then E is normally generated by torsion, hence $\pi_1(X) = 1$ for the corresponding surface.*

Proof. We need to show that every element of E is conjugate to one that can be written as a product of elements of finite order. First, we reduce to the subgroup $\{(1, k)\}_{k \in \ker \varphi}$. For any $(w_1, w_2) \in E$, towards induction on the word length of w_1, write $w_1 = c_1 w_1'$ then $(w_1, w_2) = (c_1, c_1) \cdot (w_1', c_1^{-1} w_2)$ with both terms inside E. Since each $c_i \in \mathbb{T}$ has finite order, the first term has finite order. Repeating this process we reduce to terms of the form $(1, k)$ with $k \in \ker \varphi$.

Now we use the good presentation extending $g_1, \ldots, g_r$. Since $\varphi : c_i \mapsto g_i$, the kernel is normally generated[1] by the relations between the g_i in G. A generating set is R plus the relations $g_i = \tilde{h}_i(\underline{a}) a_{j(i)}^{\ell_i} \tilde{h}_i(\underline{a})^{-1}$. It suffices to show that each of these relations is in the normal subgroup generated by elements of finite order in E. The relations in (2) give elements $(c_i, \tilde{h}_i(\underline{c}) c_{j(i)}^{\ell_i} \tilde{h}_i(\underline{c})^{-1}) \in E$. But this element has finite order $\operatorname{lcm}(m_i, m_{j(i)})$ since c_i has order m_i and the second term is conjugate to $c_{j(i)}$ which has order $m_{j(i)}$. Similarly, the words of type (a) give elements of finite order $(c_i^{-e_1}, h(\underline{c}) c_j^{e_2} h(\underline{c})^{-1})$. Finally, the words of type (b) use the relation $c_1 \cdots c_r = 1$. Using the elements of finite order

$$(c_i, \tilde{h}_i(\underline{c}) c_{j(i)}^{\ell_i} \tilde{h}_i(\underline{c})^{-1}) \in E$$

and taking the product over i we get $(1, w) \in E$ is a product of finite-order elements. □

[1] One might worry that $\ker \varphi$ being normally generated in $\mathbb{T}$ by some set is not the same as saying that $\{(1, k)\}_{k \in \ker \varphi}$ is normally generated in E by the same set. There is no issue because if $k = wk'w^{-1}$ for $w \in \mathbb{T}$ then $(1, k) = (w, w) \cdot (1, k') \cdot (w^{-1}, w^{-1})$ and each term is in E.

Example 3.3.14. Let $G = C_n$ be a cyclic group and $(g_1, \ldots, g_r)$ a list of spherical generators and suppose that g_1 generates. Then taking $a_1 = g_1$ and $R = \{a_1^n\}$ gives a good presentation since each $g_i = a_1^{e_i}$ and the relation a_1^n is of the form (a).

The following example will be used in the "D_7 case" of Theorem 3.1.1.

Example 3.3.15. Let $G = D_n = \langle r, s \mid r^n, s^2, (rs)^2 \rangle$ be the dihedral group of order $2n$ with n odd. Suppose we take a list of spherical generators $g_1, \ldots, g_r$ so that r and s appear among the g_i. Up to Hurwitz moves and relabeling r and s by an automorphism, we may assume $g_1 = r$ and $g_2 = s$. Then $a_1 = r$ and $a_2 = s$ makes the above presentation good. Indeed, because n is odd, every element is conjugate to s or a power of r, so (2) is satisfied. Finally, the relations are all of the form $a_i^{e_i} = 1$, which are of type (a), except the relation $rsrs = 1$ which is actually a conjugation relation $rsrs^{-1}$ since $s = s^{-1}$ and hence also of type (a).

Finally, we give an example showing the use of the "product relation" in a good presentation.

Example 3.3.16. Let $G = D_n$ using the Coxeter presentation

$$D_n = \langle f_1, f_2 \mid f_1^2, f_2^2, (f_1 f_2)^n \rangle .$$

Consider the set of spherical generators $g_1, \ldots, g_{2n}$ where

$$g_i = \begin{cases} f_1 & i \text{ odd} \\ f_2 & i \text{ even} \end{cases}$$

Then we have $g_i = f_{j(i)}$ for some $j(i)$, so (2) is satisfied. Furthermore, the relations in the presentation are either $f_i^2 = 1$, which is type (a), or $(f_1 f_2)^n = 1$, which is type (b) using $\widetilde{h}_1 = 1$ and $\ell_i = 1$.

3.3.4 Computing the basket of singularities

Here we describe an algorithm for computing the basket of singularities $\mathscr{B}$ of a surface $Y = (C_1 \times C_2)/G$ where the curves C_i are presented as G-covers $f_i : C_i \to \mathbb{P}^1$ prescribed by spherical generators $g_1, \ldots, g_r$ and $h_1, \ldots, h_s$. This is described in §1.2 of [BP12]; in particular, see [BP12, Proposition 1.17] for additional details. In §3.6, we used the Magma implementation of this algorithm included in [BP12] to actually perform calculations of the basket and its invariants.

The fixed points of the actions $G \circlearrowright C_i$ lie in the fibers of $f_i : C_i \to \mathbb{P}^1$ over the ramification points $p_1, \dots, p_r$ and $p'_1, \dots, p'_s$ respectively. The fiber $(f_1 \times f_2)^{-1}(p_i \times p'_j)$ is identified with $G/\langle g_i \rangle \times G/\langle h_j \rangle$. Therefore, the fiber of the stack $[C_1 \times C_2/G] \to C_1/G \times C_2/G = \mathbb{P}^1 \times \mathbb{P}^1$ over (p_i, p'_j) is identified with the groupoid

$$[(G/\langle g_i \rangle \times G/\langle h_j \rangle)/G] \cong \bigsqcup_{r \in \langle g_i \rangle \backslash G / \langle h_j \rangle} B(\langle g_i \rangle \cap r \langle h_j \rangle r^{-1})$$

so these groups $\langle g_i \rangle \cap r\langle h_j \rangle r^{-1}$ form the stabilizers of the $\#(\langle g_i \rangle \backslash G / \langle h_j \rangle)$ (potentially) singular points of Y in the fiber over $p_i \times p'_j$. What remains is to compute the type of the singularities corresponding to the above stabilizers. To do this, we need a local model. Let $(x, y) \in C_1 \times C_2$ be the point in the fiber over $p_i \times p'_j$ corresponding to $(1, 1) \in G/\langle g_i \rangle \times G/\langle h_j \rangle$. This means g_i fixes x and h_j fixes y. Since the monodromy element g_i acts by lifting the simple clockwise closed loop downstairs and f_i, locally near x, takes the form $z \mapsto z^{\operatorname{ord}(g_i)}$ we see that g_i acts by $e^{\frac{2\pi i}{\operatorname{ord}(g_i)}}$ on $T_x C_1$. Similarly, h_j acts by $e^{\frac{2\pi i}{\operatorname{ord}(h_j)}}$ on $T_y C_2$. Now the point $(1, r) \in G/\langle g_i \rangle \times G/\langle h_j \rangle$ corresponding to $(x, r \cdot y) \in C_1 \times C_2$ has stabilizer generated by g_i^γ where γ is the smallest positive integer such that $g_i^\gamma \in r\langle h_j \rangle r^{-1}$. Write $g_i^\gamma = r h_j^\delta r^{-1}$ for some $\delta \le \operatorname{ord}(h_j)$. Therefore g_i^γ acts on $T_x C_1 \oplus T_{r \cdot y} C_2$ via $(e^{\frac{2\pi i \gamma}{\operatorname{ord}(g_i)}}, e^{\frac{2\pi i \delta}{\operatorname{ord}(h_j)}})$. The quotient is a type $\frac{1}{n}(1, a)$ singularity where

$$n = \frac{\operatorname{ord}(g_i)}{\gamma} \quad \text{and} \quad a = \frac{\operatorname{ord}(g_i)\delta}{\operatorname{ord}(h_j)\gamma} = \frac{\delta}{\gcd(\delta, \operatorname{ord}(h_j))}.$$

Remark 3.3.17. Notice that the basket of singularities depends only on the two multisets of conjugacy classes $\operatorname{cl}(g_1), \dots, \operatorname{cl}(g_r)$ and $\operatorname{cl}(h_1), \cdots \operatorname{cl}(h_s)$, not on the individual elements of either list of spherical generators or the order in which they appear in the list. This is because the singularities over $p_i \times p'_j$ are determined by the set of pairs (γ, δ) appearing in relations of the form $g_i^\gamma = r h_j^\delta r^{-1}$ as we vary over r. Conjugating g_i and h_j by elements ℓ and ℓ' respectively simply permutes the value of r for each pair (γ, δ) since

$$(\ell g_i \ell^{-1})^\gamma = \ell g_i^\gamma \ell^{-1} = \ell r h_j^\delta r^{-1} \ell^{-1} = (\ell r \ell'^{-1})(\ell' h_j^\delta \ell'^{-1})(\ell r \ell'^{-1})^{-1}$$

so conjugation has the effect of permuting the set of r via $r \mapsto \ell r \ell'^{-1}$ while preserving the multiset of pairs (γ, δ).

3.3.5 Supersingularity and étale cohomology

One advantage of product-quotient surfaces is that their cohomology is nicely controlled by the cohomology of the curves. In particular, working over an algebraically closed field of characteristic $p > 0$, the following lemma gives a very simple criterion for (Shioda) supersingularity.

Lemma 3.3.18. *Let $f : X \dashrightarrow Y$ be a dominant rational map of smooth proper surfaces. If X is (Shioda) supersingular then so is Y.*

Proof. Resolving the map $\widetilde{f} : \widetilde{X} \to Y$, we get an inclusion $H^i_{\text{ét}}(Y, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(\widetilde{X}, \mathbb{Q}_\ell)$. Moreover, the pullback map $H^i_{\text{ét}}(X, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(\widetilde{X}, \mathbb{Q}_\ell)$ is an isomorphism for $i \neq 2$ and for $i = 2$ adds a copy of $\mathbb{Q}_\ell(-1)$ for each blowup. Hence, assuming X is supersingular, we conclude the same for $\widetilde{X}$ and also Y. If X is Shioda supersingular, then $H^2_{\text{ét}}(\widetilde{X}, \mathbb{Q}_\ell(1))$ is spanned by algebraic cycles so, using the push-pull formula, $H^2_{\text{ét}}(Y, \mathbb{Q}_\ell(1))$ is spanned by the pushforward of the cycle classes on $\widetilde{X}$ spanning $\operatorname{im}(H^2_{\text{ét}}(Y, \mathbb{Q}_\ell) \to H^2_{\text{ét}}(\widetilde{X}, \mathbb{Q}_\ell))$. □

Returning to the case that X is a product-quotient surface defined over a number field K, the dominant rational map $C_1 \times C_2 \dashrightarrow X$ immediately provides the following corollary.

Corollary 3.3.19. *If C_1 and C_2 have good supersingular reduction at some prime $\mathfrak{p} \subset K$ and X has good reduction at $\mathfrak{p}$, then $X_\mathfrak{p}$ is Shioda supersingular.*

Proof. The Tate conjecture is known for products of curves by [Tat66, Theorem 4]. Hence, by the Künneth formula, $C_1 \times C_2$ is Shioda supersingular if C_1 and C_2 are each supersingular. Then we conclude by Lemma 3.3.18. □

In fact, we know the Tate conjecture for X at all primes of good reduction by [Tat94, Theorem 5.2(b)].

3.4 Existence of symmetric differential forms

In this section, we largely follow the strategy used in [GRVR18] to construct diagonal symmetric forms which the authors use to prove quasi-hyperbolicity results in characteristic zero.

In a different direction, [RR14] developed a general method based on orbifold techniques for producing symmetric forms. In particular, they proved the following.

Theorem 3.4.1. *[RR14, Theorem 1] Let $\pi : X \to Y$ be the minimal resolution of a surface Y with ADE singularities and suppose that*

$$\tfrac{1}{2}(K_X^2 - c_2(X)) + \sum_{n\geq 1}\left[(n+1)(a_n+d_n+e_n) - \frac{a_n}{n+1} - \frac{d_n}{4(n-2)}\right] - \frac{e_6}{24} - \frac{e_7}{48} - \frac{e_8}{120} > 0$$

where a_n, d_n, and e_n are the numbers of A_n, D_n, and E_n singularities respectively. Then Ω_X is big. In particular, X carries a nonzero symmetric form.

These results are not adequate for our purposes for two reasons. First, their method relies on Y having only canonical singularities (so that π is crepant), but the surfaces we will analyze have, for example, $\frac{1}{3}(1,1)$ singularities which are not canonical. Second, the symmetric forms produced this way need not be diagonal.

3.4.1 Constructing a bundle of diagonal forms

Let E be the exceptional divisor for $\pi : X \to Y$. In this section, we construct a map

$$\gamma : \omega_X(-E) \to \operatorname{Sym}^2 \Omega_X$$

so that $q^* \operatorname{im}\gamma$ is diagonal. To construct this map, we need a few lemmas.

Lemma 3.4.2. *Let Y be a variety with klt singularities and $\pi : X \to Y$ a resolution of singularities. Then $\operatorname{im}((\pi^*\pi_*\omega_X)^{\vee\vee} \to \omega_X) = \omega_X(\lfloor -\sum a_i E_i \rfloor)$ where the E_i are the exceptional divisors of π and the a_i are the associated discrepancies.*

Proof. By [Kov00, Theorem 3 + Theorem 4], Y has rational singularities and $\pi_*\omega_X = \omega_Y$ which is a reflexive sheaf. Then we use the defining relation

$$K_X = \pi^* K_Y + \sum a_i E_i$$

and the fact that $\pi^{[*]}\mathcal{O}_Y(D) = \mathcal{O}_X(\lfloor \pi^* D \rfloor)$ where $\pi^{[*]}$ is the reflexive pullback, D is a Weil $\mathbb{Q}$-Cartier divisor, and f^*D is the pullback as a $\mathbb{Q}$-Cartier divisor. To prove this, let ℓD be Cartier. Then any local section s of $\mathcal{O}_Y(D)$ has $s^{[\ell]}$ a local section of $\mathcal{O}_Y(\ell D)$ and hence $\pi^* s^{[\ell]} = (\pi^{[*]}s)^\ell$ is a section of $\pi^*\mathcal{O}_Y(\ell D)$ so $\ell \operatorname{div}(\pi^{[*]}s) + f^*\ell D \geq 0$. Likewise, if $\operatorname{ord}_E(\pi^{[*]}s) \geq \operatorname{ord}_E(\pi^* D) + 1$ then dividing s by a local parameter for E, we can ensure that $\mathcal{O}_Y(D)$ has local sections generating $\mathcal{O}_X(\lfloor \pi^* D \rfloor)$. □

In particular, if $\pi : X \to Y$ is the minimal resolution of a surface with klt singularities, then the discrepancies satisfy $-1 < a_i \leq 0$ so we conclude that $(\pi^*\pi_*\omega_X)^{\vee\vee} \to \omega_X$ is an isomorphism. We also need Miyaoka's extension theorem [Miy84], which, in the modern version due to Greb, Kebekus, and Kovács, says that reflexive tensor-forms on Y extend to logarithmic tensor-forms on X. We use the notation $(-)^{[n]}$ and $\mathrm{Sym}^{[n]}(-)$ for the reflexive tensor power and reflexive symmetric power, meaning the reflexive hull of the usual tensor power or symmetric power respectively.

Theorem 3.4.3. *[GKK10, Corollary 3.2] Let $\pi : X \to Y$ be a resolution of a variety with isolated finite quotient singularities and let E be the reduced exceptional divisor. Then*

$$\mathrm{Sym}^{[m]}\,\Omega_Y = \pi_*\,\mathrm{Sym}^m\,\Omega_X(\log E).$$

In particular, $\pi_*\,\mathrm{Sym}^m\,\Omega_X(\log E)$ is reflexive and there is a pullback map

$$H^0(Y, \mathrm{Sym}^{[m]}\,\Omega_Y) = H^0(Y^\circ, \mathrm{Sym}^m\,\Omega_{Y^\circ}) \to H^0(X, \mathrm{Sym}^m\,\Omega_X(\log E))$$

To construct a map γ we first build a map $\omega_X \to \mathrm{Sym}^2\,\Omega_X(\log E)$ and prove that, when restricted to $\omega_X(-E)$ it factors through $\mathrm{Sym}^2\,\Omega_X$. This map is built as follows:

$$\pi_*\omega_X \xrightarrow{\sim} \omega_Y \to (q_*\omega_{C_1\times C_2})^G \to (q_*\,\mathrm{Sym}^2\,\Omega_{C_1\times C_2})^G = \mathrm{Sym}^{[2]}\,\Omega_Y = \pi_*\,\mathrm{Sym}^2\,\Omega_X(\log E)$$

where the last equality uses the logarithmic extension theorem 3.4.3. Note that the diagonal inclusion map $\omega_{C_1\times C_2} \hookrightarrow \mathrm{Sym}^2\,\Omega_{C_1\times C_2}$ is G-equivariant because G acts diagonally on $C_1 \times C_2$. Adjunction produces a map

$$\alpha : \pi^*\pi_*\omega_X \to \mathrm{Sym}^2\,\Omega_X(\log E)$$

but $(\pi^*\pi_*\omega_X)^{\vee\vee} \xrightarrow{\sim} \omega_X$ by the remark after Lemma 3.4.2 and $\mathrm{Sym}^2\,\Omega_X(\log E)$ is reflexive, in fact locally free, so α factors through the reflexive hull of the source, giving the desired map

$$\alpha' : \omega_X \to \mathrm{Sym}^2\,\Omega_X(\log E).$$

We want to show that α' lands in honest symmetric forms when restricted to $\omega_X(-E)$. This holds immediately when restricting to $\omega_X(-2E)$ but the refined result is necessary for the numerics of our examples. This is achieved via the subsequent lemma that follows the philosophy of a result due to Greb, Kebekus, Kovács, and Peternell [GKKP11] establishing the existence of pullback maps $\pi^* : H^0(Y, \Omega_Y^{[p]}) \to H^0(X, \Omega_X^p)$ for a resolution of klt singularities $\pi : X \to Y$. In view of the logarithmic extension theorem 3.4.3, this result says exactly that the extended logarithmic form always has "one fewer pole than expected" along E. Likewise, we conclude a similar "one fewer pole" statement for symmetric forms.

Lemma 3.4.4. *Let Y be a surface with rational singularities and $\pi : X \to Y$ the minimal resolution with exceptional divisor E. Then for any integer $m > 0$,*

$$\pi_* \left(\left[\operatorname{Sym}^m \Omega_X(\log E)\right](-E)\right) = \operatorname{Sym}^{[m]} \Omega_Y.$$

Proof. Since these sheaves agree on the smooth locus Y°, it suffices to prove that any reflexive symmetric form on Y extends to a section of $\left[\operatorname{Sym}^m \Omega_X(\log E)\right](-E)$. Using the logarithmic extension theorem 3.4.3, there is a diagram

$$\begin{tikzcd}
\pi_* \operatorname{Sym}^m \Omega_X(\log E) \arrow[r, equal] & \operatorname{Sym}^{[m]} \Omega_Y \\
\pi_* \left(\left[\operatorname{Sym}^m \Omega_X(\log E)\right](-E)\right) \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

and we wish to prove the upward inclusion is surjective. This is local on the normal surface Y, so we may shrink so that Y is affine and contains a single isolated singularity. Then we claim

$$H^0(X, \left[\operatorname{Sym}^m \Omega_X(\log E)\right](-E)) \to H^0(X, \operatorname{Sym}^m \Omega_X(\log E))$$

is surjective. It would suffice to establish the following vanishing result:

$$H^0(E, \operatorname{Sym}^m \Omega_X(\log E)|_E) = 0.$$

In degree $m = 1$, this is essentially a consequence of the Hodge index theorem (see [MOP20, §5]). It can also be seen from results in mixed Hodge theory, specifically Steenbrink vanishing: $R^1\pi_*\Omega_X(\log E)(-E) = 0$ [GKKP11, Theorem 14.1], and local vanishing: $R^1\pi_*\Omega_X(\log E) = 0$ [MOP20, Theorem B]. Indeed, duality gives

$$\mathbf{R}\pi_*\Omega_X(\log E)(-E) = \mathbf{R}\mathrm{Hom}_{\mathscr{O}_X}(\mathbf{R}\pi_*\Omega_X(\log E), \omega_Y).$$

Hence, $\pi_*\Omega_X(\log E)(-E)$ is reflexive because $\mathbf{R}\pi_*\Omega_X(\log E) = \pi_*\Omega_X(\log E)$ by local vanishing and ω_Y is reflexive. Then the long exact sequence

$$H^0(X, \Omega_X(\log E)(-E)) \xrightarrow{\sim} H^0(X, \Omega_X(\log E)) \to H^0(E, \Omega_X(\log E)|_E) \to H^1(X, \Omega_X(\log E)(-E))$$

proves $H^0(E, \Omega_X(\log E)|_E) = 0$ from Steenbrink vanishing.

Now we use the $m = 1$ case to prove the result for general m. The exceptional divisor E is a tree of smooth rational curves E_i. On each component there is a sequence

$$0 \to \Omega_{E_i}(E' \cdot E_i) \to \Omega_X(\log E)|_{E_i} \to \mathscr{O}_{E_i} \to 0$$

where $E' = E - E_i$. Let $\mathscr{E}_i$ denote the vector bundle $\Omega_X(\log E)|_{E_i}$ on E_i. Note that $\mathscr{E}_i$ is a sum of line bundles $\mathscr{O}_{\mathbb{P}^1}(a)$ over twists a that are all ≥ 0 or all ≤ 0. Therefore, $\mathscr{E}_i$ has the remarkable property that

$$\operatorname{Sym}^m H^0(E_i, \mathscr{E}_i) \to H^0(E_i, \operatorname{Sym}^m \mathscr{E}_i)$$

is surjective for all m. Let W_{ij} be the fiber of $\Omega_X(\log E)$ at $E_i \cap E_j$. Then $H^0(E, \Omega_X(\log E)|_E)$ is the equalizer of the two maps

$$\bigoplus_i H^0(E_i, \mathscr{E}_i) \rightrightarrows \bigoplus_{i<j} W_{ij}$$

and likewise $H^0(E, \operatorname{Sym}^m \Omega_X(\log E)|_E)$ is the equalizer of

$$\bigoplus_i H^0(E_i, \operatorname{Sym}^m \mathscr{E}_i) \rightrightarrows \bigoplus_{i<j} \operatorname{Sym}^m W_{ij}.$$

The noted surjectivity shows it suffices to prove that

$$\bigoplus_i \operatorname{Sym}^m H^0(E_i, \mathscr{E}_i) \rightrightarrows \bigoplus_{i<j} \operatorname{Sym}^m W_{ij}$$

has a trivial equalizer where the maps are sums of $\operatorname{Sym}^m(-)$ of the maps at level $m = 1$. Now it becomes entirely a question of linear algebra. Given the data $(V_i, W_k, \varphi_{i,k})$ consisting of a list of vector spaces V_i and W_k with maps $\varphi_{i,k} : V_i \to W_k$ if there are no nonzero lists $v_i \in V_i$ so that $\varphi_{i,k}(v_i)$ are equal over all i, k then the same holds for the data $(\operatorname{Sym}^m V_i, \operatorname{Sym}^m W_k, \operatorname{Sym}^m \varphi_{i,k})$ for essentially the same reason that symmetric powers preserve injectivity. □

Our proof is highly particular to the surface case so it would be interesting to study extensions of this result in higher dimension.

3.4.2 Numerical criteria

In this section, we prove a purely numerical criterion for the existence on X of (many) diagonal symmetric forms. Using the map γ of the previous section, it suffices to show that $\omega_X(-E) = \mathscr{O}_X(K_X - E)$ has (many) sections.

Proposition 3.4.5. *Let $\pi : X \to Y$ be the minimal resolution of a klt surface with exceptional* *divisor E. Then*

$$(K_X - E)^2 = K_X^2 - 3\sum_i(-E_i^2 - 2) - 2\#Y^{\mathrm{sing}}$$

Moreover, if X is of general type and $(K_X - E)^2 > 0$, then $K_X - E$ is big.

Proof. Expand

$$(K_X - E)^2 = K_X^2 - 2K_X \cdot E + E^2$$

and write $E = E_1 + \cdots + E_r$. Since the support of E is a forest (a union of trees) consisting of smooth rational curves, we get

$$E^2 = \sum_i E_i^2 + 2\sum_{i>j} E_i \cdot E_j$$

but the intersections $E_i \cdot E_j$ correspond to edges of the dual graph Γ of E. Since Γ is a forest, there is exactly one more node (component of E) than edge for each connected component. Hence,

$$E^2 = \sum_i (E_i^2 + 2) - 2\#\pi_0(\Gamma).$$

Note also that $\pi_0(\Gamma) = \#Y^{\mathrm{sing}}$ since, by Zariski's main theorem, the exceptional divisor above an isolated singularity is connected. Furthermore, the adjunction formula gives that $(K_X + E_i) \cdot E_i = -2$ so

$$K_X \cdot E = -\sum_i (E_i^2 + 2)$$

Putting everything together,

$$(K_X - E)^2 = K_X^2 + 3\sum_i (E_i^2 + 2) - 2\#Y^{\mathrm{sing}}.$$

Now, using Hirzebruch–Riemann-Roch,

$$h^0(\mathscr{O}_X(m(K_X - E))) + h^2(\mathscr{O}_X(m(K_X - E))) \geq \frac{1}{2}m^2(K_X - E)^2 + O(m)$$

but $h^2(\mathscr{O}_X(m(K_X - E))) = h^0(\mathscr{O}_X(mE - (m-1)K_X)) = 0$ for $m > 1$ because if H is an ample divisor on Y, then $\pi^*H \cdot (mE - (m-1)K_X) = -(m-1)H \cdot K_Y < 0$ so $mE - (m-1)K_X$ cannot be effective. Thus, $K_X - E$ is big as long as $(K_X - E)^2 > 0$. □

Corollary 3.4.6. *If Y has C canonical singularities (type A, D, E singularities) and additionally $N_{n,a}$ non-canonical cyclic quotient singularities of type $\frac{1}{n}(1, a)$, then*

$$(K_X - E)^2 = K_X^2 - 2C - \sum_{n\geq 1}\sum_{a=1}^{n-2} N_{n,a}\, D(\tfrac{n}{a})$$

where, writing $\frac{n}{a} = [\![b_1, \ldots, b_\ell]\!]$, we define:

$$D(\tfrac{n}{a}) := 3\sum_{i=1}^{\ell}(b_i - 2) + 2$$

In the notation of [BP12], $D(\frac{n}{a}) = 3k(\frac{n}{a}) - 3\frac{2+a+a'}{n} + 8$ where $0 \leq a' < n$ is the multiplicative inverse of a in $(\mathbb{Z}/n\mathbb{Z})^\times$.

Corollary 3.4.7. *For a product-quotient surface,*

$$(K_X - E)^2 = \frac{8(g_1 - 1)(g_2 - 1)}{\#G} - k(\mathscr{B}) - D(\mathscr{B})$$

If X is of general type and this number is positive, then $\omega_X(-E)$ is big and X carries a diagonal symmetric form.

Proof. Combine Corollary 3.4.6 with Lemma 3.4.4 and Proposition 3.3.5. □

When Y has only ADE singularities (meaning it has only A_n singularities since all the singularities of Y are cyclic quotient), it is interesting to compare this estimate to the estimate of [RR14]. Recall that an A_n singularity is type $\frac{1}{n+1}(1, n)$, so in this case,

$$k(\mathscr{B}) = 0, \qquad e(\mathscr{B}) = \sum_{n\geq 1}\frac{n(n+2)}{n+1}a_n, \qquad B(\mathscr{B}) = 2e(\mathscr{B})$$

where a_n is the number of A_n singularities on Y. Therefore [RR14] produces symmetric forms when

$$2(K_X^2 - c_2) + \sum_{n\geq 1}\frac{n(n+2)}{n+1}a_n = \frac{8(g_1 - 1)(g_2 - 1)}{\#G} - \sum_{n\geq 1}\frac{n(n+2)}{n+1}a_n$$

is positive. On the other hand, our method produces diagonal symmetric forms when

$$\frac{4(g_1 - 1)(g_2 - 1)}{\#G} - \sum_{n\geq 1} a_n$$

is positive. A more recent computation in [BIX25] improves the result of [RR14] for A_n singularities. They show that Ω_X is big if

$$K_X^2 - c_2(X) + \sum_{n\geq 1} \frac{n(n+2)}{n+1} a_n - (\tfrac{4}{3}\pi^2 - 12) \sum_{n\geq 1} a_n > 0$$

which, in our case, is the inequality

$$\frac{4(g_1 - 1)(g_2 - 1)}{\#G} - (\tfrac{4}{3}\pi^2 - 12) \sum_{n\geq 1} a_n > 0.$$

Note that $\frac{4}{3}\pi^2 - 12 \approx 1.2 > 1$ so our inequality is more easily satisfied. However, for the particular case of only A_1 singularities, [BIX25] gives a better inequality:

$$\frac{4(g_1 - 1)(g_2 - 1)}{\#G} - \frac{66}{108} a_1 > 0.$$

3.4.3 Results for Large Groups

An interesting source of curves from the arithmetic perspective are Belyĭ curves since these are defined over a number field. We can consider the case of curves with a Galois Belyĭ map, of which the Hurwitz curves studied in the following sections are an example. We find that for large groups, symmetric forms are ubiquitous and hence our obstruction applies.

Theorem 3.4.8. *Let $C_i \to \mathbb{P}^1$ be G-Belyĭ maps meaning a Galois cover ramified only over $0, 1, \infty$ with monodromy orders a, b, c. Assume the covers are hyperbolic i.e., $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Such curves are defined over a number field K and G is equipped with a presentation*

$$\Delta(a, b, c) = \langle x, y, z \mid x^a, y^b, z^c, xyz \rangle \twoheadrightarrow G.$$

Let $X \to (C_1 \times C_2)/G$ be the minimal resolution. For any prime $\mathfrak{p} \subset \mathcal{O}_K$ away from $\#G$ for which both C_i have good reduction, X has good reduction. Assume a, b, c are distinct primes then if $\#G \geq N(a, b, c)$, then for any $\mathfrak{p}$ of good reduction, $X_{\mathfrak{p}}$ is not covered by rational or elliptic curves. We can take

$$N(a, b, c) = e^{O\left((\log abc)^4 \cdot (\log\log abc)^2\right)}$$

Remark 3.4.9. Recall that Corollary 3.3.11 shows that if the two presentations $\varphi_i : \Delta(a, b, c) \twoheadrightarrow G$ have the property that $\varphi_1(x_i)$ is conjugate to a power of $\varphi_2(x_i)$ (e.g. if φ_1 and φ_2 are conjugate) for all but one generator $x_i \in \Delta(a, b, c)$ then $\pi_1(X)$ is finite so one would need to understand the Galois representation of X to obstruct unirationality of $X_{\mathfrak{p}}$ in another way.

Remark 3.4.10. The condition that a, b, c are pairwise coprime is important. Otherwise, the Fermat curve $C_n := \{X^n + Y^n + Z^n\} \subset \mathbb{P}^2$ is a $G = \mu_n^2$-Belyĭ cover of $\mathbb{P}^1$ with orders (n, n, n). But we can see that the Fermat surface F_n dominates $(C_n \times C_n)/G$ and hence $(C_n \times C_n)/G$ is unirational for a positive density set of primes.

Proof. We will show that X and all its asymmetric Galois twists have a diagonal symmetric form. Since C_i^σ is also a G-Belyĭ curve with possibly different elements $x, y, z \in G$ generating monodromy, we do not have to worry about different twists if $x, y, z \in G$ and $x', y', z' \in G$ are allowed to be arbitrary. Let $Y = (C_1 \times C_2)/G$, we need to understand the singularities of Y. Since a, b, c are pairwise coprime, the singularities appear only over $0 \times 0, 1 \times 1, \infty \times \infty$. The singularities are of type $\frac{1}{n}(1, \ell)$ for n one of a, b, c. By §3.3.4, the number of type $(1, \ell)_n$ is the number of $r \in \langle x_i \rangle \backslash G / \langle x_i' \rangle$ such that $r x_i r^{-1} = x_i'^{\ell}$. This is either empty or of size at most $\#C_G(x_i^{m_i/n})/m_i$ the centralizer of x_i^δ. Therefore,

$$
\begin{aligned}
(K_X - E)^2 &= \frac{8(g-1)^2}{\#G} - k(\mathscr{B}) - D(\mathscr{B}) \\
&\geq 2\#G \left(1 - \left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right]\right)^2 - \sum_i \sum_{\ell \in (\mathbb{Z}/n\mathbb{Z})^\times} \frac{\#C_G(x_i)}{m_i} \left(k(\tfrac{1}{m_i}(1,\ell)) + D(\tfrac{1}{m_i}(1,\ell))\right)
\end{aligned}
$$

since Riemann–Hurwitz for a Galois cover ramified over $\mathbb{P}^1$ with orders $m_1, \ldots, m_r$ says,

$$2g - 2 = \#G \left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right)\right).$$

The theorem is proved if we can show that $\#C_G(x_i)/\#G < \epsilon$ for any $\epsilon > 0$ and large $\#G$. This is a pure group theory lemma as the following shows. To see the explicit bounds, we use the following facts. We have the uniform bound

$$\left(1 - \left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right]\right) \geq \frac{1}{42}.$$

Furthermore,

$$\sum_i \sum_{\ell \in (\mathbb{Z}/n\mathbb{Z})^\times} \frac{1}{m_i} \left(k(\tfrac{1}{m_i}(1,\ell)) + D(\tfrac{1}{m_i}(1,\ell))\right) \leq \sum_i \frac{4}{m_i} \sum_{(\ell, m_i)=1} \tilde{S}(\ell/m_i).$$

defining

$$\tilde{S}(a/n) := \sum_{i=1}^{r} (b_i - 2) \quad \text{where} \quad \tfrac{n}{a} = [\![b_1, \dots, b_r]\!]$$

If we consider the usual continued fraction expansion

$$\tfrac{a}{n} = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

then there is a bound

$$\tilde{S}(a/n) \le S(a/n) := \sum_i a_i.$$

Furthermore, Panov [Pan82] and Liehl [Lie83] independently found asymptotics for the expectation of $S(a/n)$,

$$\frac{1}{\varphi(n)} \sum_{(a,n)=1} S(a/n) \sim \frac{6}{\pi^2} (\log n)^2 \quad \text{as} \quad n \to \infty$$

combining with the following lemma produces the desired asymptotics. □

Lemma 3.4.11. *Let $\Delta(a, b, c) \twoheadrightarrow G$ with a, b, c distinct primes. Then the centralizer of any generator $g \in \{x, y, z\}$ satisfies,*

$$\frac{\#C_G(g)}{\#G} = O\left(\frac{\log \log \#G}{\sqrt{\log \#G}} \right)$$

Proof. Without loss of generality, $g = x$. First I claim that G is perfect. Suppose G has an abelian quotient G^{ab} then G^{ab} is generated by $\bar{x}, \bar{y}, \bar{z}$ with $\bar{x}\bar{y}\bar{z} = 1$. But a, b, c are coprime so $\bar{x} = (\bar{y}\bar{z})^{-1}$ has order dividing bc (since $\bar{y}$ and $\bar{z}$ have coprime orders) and also divides a, so it is trivial. The same argument shows G^{ab} is trivial. Hence $G = [G, G]$. Let $H = C_G(g)$ and consider the action $G \circlearrowright (G/H)$ giving a map $\rho : G \to S_k$ where $k := [G : H]$. Notice that $N = \ker \rho$ is a normal subgroup contained in H of index $\le k!$. Furthermore, consider the conjugation action $G \circlearrowright N$. Note that $z = y^{-1}x^{-1}$ and $N \subset H$ and x^{-1} acts trivially on H by definition. Thus z and y^{-1} act the same way on N. However, y^{-1} has order b and z has order c which are coprime so x, y, z act trivially on N meaning $N \subset Z(G)$. Now, Wiegold's effective form of Schur's theorem on the derived subgroup [Wie57, (3.1)] proves

$$\log \#G = \log \#[G, G] \in O((\log [G : Z(G)])^2)$$

Hence,

$$k! \geq \#[G:N] \geq \#[G:Z(G)] = e^{\Omega(\sqrt{\log \#G})}$$

If $k! \geq s$ then $k = \Omega(\frac{\log s}{\log \log s})$ by Stirling's approximation. Therefore,

$$k = \Omega\left(\frac{\sqrt{\log \#G}}{\log \log \#G}\right)$$

□

3.5 Quasi-hyperbolic and non-quasi-hyperbolic surfaces

Recall that we say a projective surface X is *algebraically quasi-hyperbolic* if it contains finitely many genus 0 and genus 1 curves. Here we will use a slightly modified version of the descent argument to prove that certain product-quotient surfaces are algebraically quasi-hyperbolic in characteristic p. We will also see examples of non-uniruled surfaces of general type that are not algebraically quasi-hyperbolic.

First we need to recall some terminology about the base loci of linear series. Let X here be a projective variety.

Definition 3.5.1. For a line bundle L and a subspace $V \subset H^0(X, L)$ we define the *base ideal* $\mathfrak{b}(V)$ as the image of the evaluation map

$$V \otimes L^\vee \to \mathcal{O}_X.$$

The *base scheme* $\mathrm{Bs}(V) := Z(\mathfrak{b}(V))$ is the subscheme cut out by this ideal. We call the support of $\mathrm{Bs}(V)$ the *base locus* of V.

We will often write $\mathrm{Bs}(|D|)$ for D a Cartier divisor to mean the base locus of the linear series $|D|$ viewed as a subspace of sections of $\mathcal{O}_X(D)$.

Definition 3.5.2. For D a $\mathbb{Q}$-Cartier divisor, the *stable base locus* $\mathbb{B}(D)$ is the closed subset[2] defined by the intersection

$$\mathbb{B}(D) := \bigcap_{\substack{m \geq 1 \\ mD \text{ Cartier}}} \mathrm{Bs}(|mD|)_{\mathrm{red}}.$$

[2]The scheme structure on the stable base locus does not behave well. See [Laz04, Example 2.1.25].

Choose an ample divisor A. Then we define the *augmented base locus* as follows

$$\mathbb{B}_+(D) := \bigcup_{m\geq 1} \mathbb{B}(mD - A)$$

and likewise the *diminished base locus*

$$\mathbb{B}_-(D) := \bigcap_{m\geq 1} \mathbb{B}(mD + A).$$

The stable base locus is well-named since it equals $\mathrm{Bs}(|mD|)$ for sufficiently large and divisible m [Laz04, Proposition 2.1.21]. In the definition of augmented (resp. diminished) base loci, it is equivalent to take the union (resp. intersection) of $\mathbb{B}(D \mp A)$ over all ample $\mathbb{Q}$-divisors A. The point of these various loci is two-fold. First the augmented/diminished variants behave more stably than $\mathbb{B}(D)$ with respect to equivalence. While $\mathbb{B}(|D|)$ is not an invariant of the numerical class of D, both $\mathbb{B}_\pm(D)$ do only depend on D up to numerical equivalence. Second, we can rephrase many standard properties of divisors in terms of the various stable base loci.

Proposition 3.5.3. *[ELMta+06, Example 1.7 and Example 1.18] (cf. [Laz04, §2.3]) Let D be a $\mathbb{Q}$-Cartier divisor. Then,*

1. *D is ample* $\iff \mathbb{B}_+(D) = \varnothing$
2. *D is semi-ample* $\iff \mathbb{B}(D) = \varnothing$
3. *D is nef* $\iff \mathbb{B}_-(D) = \varnothing$
4. *D is big* $\iff \mathbb{B}_+(D) \neq X$
5. *D is ($\mathbb{Q}$-)effective* $\iff \mathbb{B}(D) \neq X$
6. *D is pseudo-effective* $\iff \mathbb{B}_-(D) \neq X$

For an effective divisor E, we say that D is *ample modulo E* if $\mathbb{B}_+(D) \subset E$ and similarly for the properties *semi-ample* and *nef*. Using this terminology we can give a uniform statement of the main obstruction and its strengthening towards proving algebraic quasi-hyperbolicity.

If X is a product-quotient surface, write S_X for the union of the components of the singular fibers of the two projection maps σ_1 and σ_2. Note that S_X contains all exceptional divisors as well as the main components of the singular fibers. For a closed subset Z, we write $Z^{\text{genus}=g}$ for the union of the divisorial components of Z that have geometric genus g.

Theorem 3.5.4. *Let X range over the asymmetric Frobenius twists of a product-quotient surface over a field k (possibly of characteristic $p > 0$). Then the following hold.*

(1) If each X satisfies $\mathbb{B}_-(K_X - E) \neq X$ (i.e. $K_X - E$ is pseudo-effective) then each X is not covered by genus 0 curves.

(2) If each X satisfies $\mathbb{B}_+(K_X - E) \neq X$ (i.e. $K_X - E$ is big) then each X is not covered by genus 1 curves.

(3) If each X satisfies $\mathbb{B}_-(K_X - E)^{\text{genus}=0} \subset S_X$ then every genus 0 curve on each X is contained in S_X hence there are only finitely many rational curves on each X.

(4) If each X satisfies $\mathbb{B}_+(K_X - E)^{\text{genus}=1} \subset S_X$ then every genus 1 curve on each X is contained in S_X hence there are only finitely many genus 1 curves on each X.

In particular, (3) and (4) immediately imply

1. *If each $K_X - E$ is nef modulo S_X then each X contains only finitely many rational curves all lying inside S_X.*

2. *If each $K_X - E$ is ample modulo S_X then each X is algebraically quasi-hyperbolic with exceptional set contained in S_X.*

Proof. The first part proceeds by exactly the same argument as the proof of Theorem 3.2.3. To run the descent argument and verify that X is not covered by genus 0 (resp. genus 1) curves, it suffices to show that on each twist X, there are only finitely many genus 0 (resp. genus 1) curves of each degree not tangent to one of the canonical foliations. Fix an ample divisor A on X. First assume that $K_X - E$ is pseudo-effective, meaning $(K_X - E) + \epsilon A$ is $\mathbb{Q}$-effective for any $\epsilon \in \mathbb{Q}_{>0}$. Let $f : C \to X$ be a generically immersed map from a smooth rational curve. For each integer $a > 0$, setting $\epsilon = \frac{1}{a}$ produces a section $s \in H^0(X, \mathcal{O}_X(ma(K_X - E) + mA))$ for $m \gg 0$. Using the map $\gamma : \omega_X(-E) \to \operatorname{Sym}^2 \Omega_X$ we get a pullback map

$$f^* \circ \gamma : \omega_X(-E) \to \omega_C^{\otimes 2}$$

which is zero if only if $f : C \to X$ is tangent to one of the canonical foliations. Therefore, s produces a section

$$f^*\gamma(s) \in H^0(C, \mathcal{O}_C(2amK_C + mf^*A)).$$

Since C is rational, $\deg(2amK_C + mf^*A) = -4am + m \deg f^*A$ and therefore if $f^*\gamma(s)$ is nonzero $\deg f^*A \geq 4a$. Hence if f is a curve satisfying $\deg f^*A < 4a$ then either f is tangent to one of the foliations or $f(C) \subset V(s) \cup E$. Since there are only finitely many curves contained in $V(s) \cup E$, for each degree d we can choose $a > 0$ so that $d < 4a$ and thus there are only finitely many rational curves of degree d on X not tangent to one of the foliations.

The case when $K_X - E$ is big and we need to show that X is not covered by genus 1 curves is exactly Theorem 3.2.7.

Items (3) and (4) follow from the more detailed analysis of genus 0 and 1 curves outlined in Remark 3.2.5. For (3), assume that $\mathbb{B}_-(K_X - E)^{\text{genus}=0} \subset S_X$ on each twist. This means for any rational curve $f : C \to X$ whose image is not contained in S_X then $f(C) \not\subset \mathbb{B}_-(K_X - E)$. Explicitly, this means for any integer $a > 0$, there is an integer $m > 0$ and a section $s \in H^0(X, \mathscr{O}_X(am(K_X - E) + mA))$ so that $f(C) \not\subset V(s)$. Therefore, either f is tangent to one of the canonical foliations or $f^*\gamma(s)$ is nonzero which, as before, implies that $\deg f^*A \geq 4a$. Sending $a \to \infty$ we see that f is tangent to one of the canonical foliations. By the descent argument, f must be an iterated partial Frobenius pullback of a rational curve $f' : C' \to X'$ on another asymmetric Frobenius twist which is not tangent to either foliation on X'. By the same argument applied to X', we must have $f'(C') \subset S_{X'}$. However, the singular set $S_{X'}$ is stable under partial Frobenius pullback so we conclude that $f(C) \subset S_X$.

The argument for (4) is nearly identical. Assume that $\mathbb{B}_+(K_X - E)^{\text{genus}=1} \subset S_X$ on each twist. This means for any elliptic curve $f : C \to X$ whose image is not contained in S_X then $f(C) \not\subset \mathbb{B}_+(K_X - E)$. Explicitly, this means there exists integers $a > 0$ and $m > 0$ and a section $s \in H^0(X, \mathscr{O}_X(am(K_X - E) - mA))$ so that $f(C) \not\subset V(s)$. Using the pullback map we obtain a section

$$f^*\gamma(s) \in H^0(C, \mathscr{O}_C(2amK_C - mf^*A))$$

that is zero if and only if f is tangent to the canonical foliations. However, $K_C = 0$ and $m > 0$ so $\deg(2amK_C - mf^*A) < 0$. Thus f is tangent to one of the canonical foliations. As before, this means that f is pulled back along a sequence of partial Frobenii from an elliptic curve $f' : C' \to X'$ not tangent to the canonical foliations. As before, repeating the argument to $f' : C' \to X'$ implies that $f'(C') \subset S_{X'}$ but $S_{X'}$ is stable under partial Frobenius pullback so we conclude that $f(C) \subset S_X$. □

3.5.1 Quasi-hyperbolic examples

We will be able to find product-quotient surfaces so that $K_X - E$ is ample modulo S_X for all asymmetric Frobenius twists. Thus by applying Theorem 3.5.4 we conclude that X is algebraically quasi-hyperbolic. In fact, we will find examples where $K_X - E$ is ample on all twists. To show this, we use the canonical bundle formula of Serrano for isotrivial curve fibrations.

Theorem 3.5.5. *[Ser96, Theorem 4.1] Let X be a product-quotient surface (i.e. an isotrivial curve fibration over a curve) with $\overline{C}_i := C_i/G$. Then*

$$K_X = \sigma_1^* K_{\overline{C}_1} + \sigma_2^* K_{\overline{C}_2} + \sum_{\sigma_1(L_1)=*} (m(L_1) - 1)L_1 + \sum_{\sigma_2(L_2)=*} (m(L_2) - 1)L_2 + E$$

where in the two sums L_1 and L_2 run over the irreducible components of the fibers (equivalently the multiple components of the singular fibers) of σ_1 and σ_2 respectively, with their multiplicities $m(G_i)$ inside the respective fiber attached.

Serrano also describes the structure of each singular fiber in great detail.

Theorem 3.5.6. *[Ser96, Theorem 2.1] Let $y \in C_2$ and let G_y be its stabilizer. Let F be a fiber of $\sigma_2 : X \to \overline{C}_2$ over the image $\bar{y} \in \overline{C}_2$. Then:*

1. *The reduced structure of F is the union of an irreducible smooth curve A, called the central or main component of F, and either none or at least two disjoint Hirzebruch–Jung strings, each meeting A transversely at one point.*

2. *The central component A is isomorphic to C_1/G_y and it has multiplicity $\#G_y$ in F. Each string intersects A at exactly one point transversally and the strings are attached exactly at the branch points of $C_1 \to C_1/G_y$.*

3. *If $L = \sum_{i=1}^n E_i$ is a H–J string in F and A' is the central component of the fiber of $\sigma_1 : X \to \overline{C}_1$ over $\sigma_1(L)$, then L meets A' and A at opposite ends, i.e. either $A \cdot E_1 = A' \cdot E_n = 1$ or $A \cdot E_n = A' \cdot E_1 = 1$ and the other components do not intersect A or A'.*

The main components enter the formula for K_X with their multiplicities $(\#G_y - 1)$. However, exceptional components can also contribute to the multiple fiber terms; see the following remark.

Remark 3.5.7. One should be cautious and note that the components G_i appearing in the formula run not just over main components but also over components of H–J strings which, inside a fiber F_i of σ_i, may come with multiplicity. Therefore, the exceptionals E_i may appear in the sums over the singular components, not only in the total term E. Concretely, near an $A_2 = \frac{1}{3}(1, 2)$ singularity there are main components A_1 and A_2 of the fibers of σ_1 and σ_2 over the image of the exceptional and there is a chain $A_1 - E_1 - E_2 - A_2$ where E_i

are (-2)-curves arising as the exceptional locus of the resolution. The fiber F_1 of σ_1 takes the form

$$F_1 = 3A_1 + aE_1 + bE_2.$$

Since F_1 is a fiber class, it intersects E_i trivially:

$$F_1 \cdot E_1 = 0 \implies 3 - 2a + b = 0$$

and

$$F_1 \cdot E_2 = 0 \implies a - 2b = 0$$

and thus $a = 2, b = 1$. In particular, E_1 contributes to F_1 with multiplicity 2. Similarly,

$$F_2 = 3A_2 + 2E_2 + E_1.$$

In total the contribution to K_X is

$$(2A_1 + E_1) + (2A_2 + E_2) + (E_1 + E_2) = 2(A_1 + E_1 + E_2 + A_2).$$

In [GRVR18] the authors remark that Serrano's formula can be rewritten in terms of the foliations $\mathscr{F}_1, \mathscr{F}_2$ defined by the morphisms σ_1, σ_2 to read

$$\omega_X \cong \mathscr{N}_{\mathscr{F}_1}^{\vee} \otimes \mathscr{N}_{\mathscr{F}_2}^{\vee} \otimes \mathscr{O}_X(E)$$

where $\mathscr{N}_{\mathscr{F}} := (\mathscr{T}_X/\mathscr{F})^{\vee\vee}$ is called the *normal bundle* of the foliation $\mathscr{F} \subset \mathscr{T}_X$. This formula will reappear in chapter 6 §6.5 (cf. Theorem 6.5.4 of chapter 6) in the context of Hilbert modular surfaces. Using the inclusion $\mathscr{N}_{\mathscr{F}_i}^{\vee} \hookrightarrow \Omega_X$, this observation gives another proof that the map

$$\gamma : \omega_X(-E) \to \operatorname{Sym}^2 \Omega_X$$

of §3.4.1 exists.

Using Serrano's formula

$$D := K_X - E = \sigma_1^* K_{\overline{C}_1} + \sigma_2^* K_{\overline{C}_2} + \sum_{\sigma_1(L_1)=*} (m(L_1) - 1)L_1 + \sum_{\sigma_2(L_2)=*} (m(L_2) - 1)L_2$$

and we will try to show this divisor is nef. We will index the components of singular fibers as follows. Let I be the index set of the ramification points $\bar{x}_i \in \overline{C}_1$ and J the index set of the ramification points $\bar{y}_j \in \overline{C}_2$. Let $A_{1,i}$ and $A_{2,j}$ be the main components of the fibers over $\bar{x}_i$ and $\bar{y}_j$ respectively. Let $E_{1,i,k}$ and $E_{2,j,k}$ run over the exceptional strings in those respective fibers indexed by sets $\mathrm{Exc}_{1,i}$ and $\mathrm{Exc}_{2,j}$ respectively. Note that the same divisors will appear among the $E_{1,i,k}$ and $E_{2,j,k}$ because each H–J string connects a main component $A_{1,i}$ to some other main component $A_{2,j}$. However, with this indexing they will enter the canonical bundle formula in a simple way:

$$\begin{aligned} K_X - E = \sigma_1^* K_{\overline{C}_1} + \sigma_2^* K_{\overline{C}_2} &+ \sum_{i \in I} \left((m(A_{1,i}) - 1) A_{1,i} + \sum_{k \in \mathrm{Exc}_{1,i}} (m(E_{1,i,k}) - 1) E_{1,i,k} \right) \\ &+ \sum_{j \in J} \left((m(A_{2,j}) - 1) A_{2,j} + \sum_{k \in \mathrm{Exc}_{2,j}} (m(E_{2,j,k}) - 1) E_{2,j,k} \right) \end{aligned}$$

See Figure 3.2 for the indexing and for the clean multiple-fiber intersection count used below. Two collections of curves clearly intersect $D = K_X - E$ nonnegatively. For a component $E_i \subset E$, we may compute

$$(K_X - E) \cdot E_i = (K_X + E_i - 2E_i - E') \cdot E_i = (-2) + 2b_i - \#\text{neighbors} \geq 0$$

where $E' = E - E_i$ and $E' \cdot E_i$ is the number of neighboring components in its H–J string. The inequality holds because $b_i \geq 2$ and #neighbors ≤ 2. Furthermore, if F_1 is a smooth fiber of σ_1 (or a curve numerically equivalent to the fiber class) then compute,

$$D \cdot F_1 = \#G \deg K_{\overline{C}_2} + \sum_{\sigma_2(L_2) = *} (m(L_2) - 1) L_2 \cdot F_1$$

but F_1 misses all components L_2 except the central components $A_{2,j}$ and $m(A_{2,j}) = \#G_{y_j}$ where $y_j \in C_2$ is any preimage of $\sigma_2(A_{2,j})$ and

$$A_{2,j} \cdot F_1 = \deg (A_{2,j} \to \overline{C}_1) = \#G / \#G_{y_j}$$

by commutativity of the diagram

$$\begin{tikzcd} C_1 \times \{y_j\} \arrow[r, hook] \arrow[d] & C_1 \times C_2 \arrow[r, "\pi_1"] \arrow[d, "q"] & C_1 \arrow[d] \\ A_{2,j} \arrow[r, hook] & Y \arrow[r, "\sigma_1"] & \overline{C}_1 \end{tikzcd}$$

and the fact that $\deg (C_1 \to \overline{C}_1) = \#G$ and $\deg (C_1 \to A_{2,j}) = \#G_{y_j}$. Therefore,

$$D \cdot F_1 = \#G \left[\deg K_{\overline{C}_2} + \sum_{j \in J} \left(1 - \frac{1}{\#G_{y_j}} \right) \right] = 2g(C_2) - 2$$

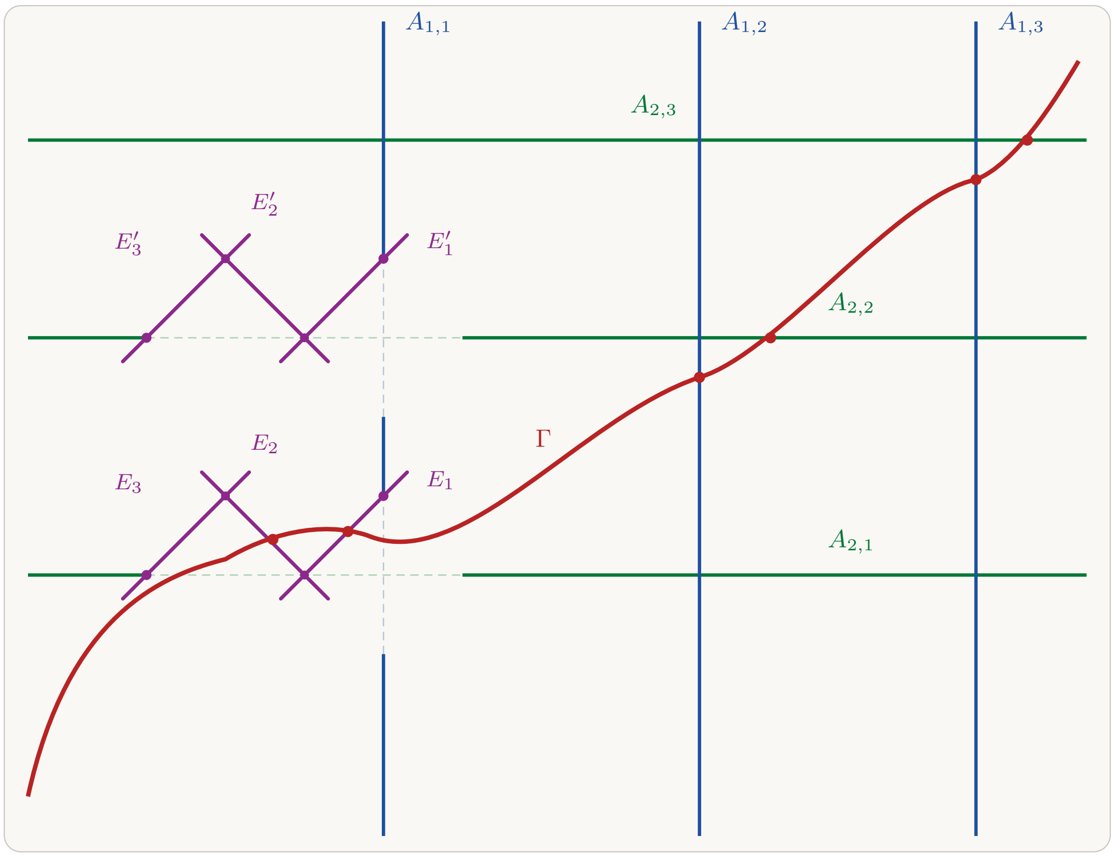


Figure 3.2: A schematic of the nefness argument. Blue curves $A_{1,i}$ are central components of σ_1-fibers; green curves $A_{2,j}$ are central components of σ_2-fibers. At the singular fiber containing $A_{1,1}$, two disjoint Hirzebruch–Jung strings (violet) resolve the quotient singularities at $A_{1,1} \cap A_{2,1}$ and $A_{1,1} \cap A_{2,2}$; the remaining fiber intersections are clean. Near one singularity, the test curve Γ (red) avoids all central components and instead passes through the exceptional chains, so $\Gamma \cdot F \geq 0$ is achieved entirely through exceptional components; at fibers without exceptional components, Γ is forced to cross the central component directly.

by Riemann–Hurwitz. We always assume $g(C_i) \geq 2$ so the fiber classes F_1, and symmetrically F_2, intersect D positively.

The strategy is to partition the sets of main components $A_{1,i}$ and $A_{2,j}$ into those which lie in a fiber containing an exceptional component $E_i \subset E$ and those that do not. Let $I^e \subset I$ and $J^e \subset J$ be the collections of singular fibers containing an exceptional component. The idea is that if $C \subset X$ is a curve not contracted by either σ_1 or σ_2 then C must intersect each fiber of σ_1 and σ_2 at the same number of points counted with multiplicity. Let m_i and n_j be the orders of G_{x_i} and G_{y_j} respectively.

Theorem 3.5.8. *Let X be a product-quotient surface defined by the data written in the above notation. Assume that $\overline{C}_i \cong \mathbb{P}^1$ and*

$$\sum_{i \in I \setminus I^e} \frac{m_i - 1}{m_i} \geq \max\left\{2, \max_{j \in J} \frac{\#G_{y_j}}{\#G}\left\{2\#G_{y_j} + \#Y^{\mathrm{sing}} \cap A_{2,j}\right\}\right\}$$
$$\sum_{j \in J \setminus J^e} \frac{n_j - 1}{n_j} \geq \max\left\{2, \max_{i \in I} \frac{\#G_{x_i}}{\#G}\left\{2\#G_{x_i} + \#Y^{\mathrm{sing}} \cap A_{1,i}\right\}\right\}$$

then each partial twist of the product-quotient surface X has $K_X - E$ nef and hence contains finitely many rational curves. If additionally each X satisfies $(K_X - E)^2 > 0$ then X is algebraically quasi-hyperbolic.

Proof. The second statement follows from the Nakai–Moishezon criterion applied to $K_X - E$. Thus it suffices to show that the inequalities imply $D = K_X - E$ is nef. We already showed that exceptional curves and smooth fibers intersect positively against D. There are two remaining types of curves: those not contracted by either σ_1 or σ_2, and central components of singular fibers.

First, let $\Gamma \subset X$ be a curve of the first type and let $d_i := \deg(\Gamma \to \overline{C}_i)$ be the degrees of the projections via σ_i. These are positive integers by assumption. Since Γ intersects each component of the singular fibers properly, they give nonnegative contributions to its intersection with D. Therefore, dropping the contributions of the exceptionals,

$$\Gamma \cdot D \geq \Gamma \cdot \sigma_1^* K_{\overline{C}_1} + \Gamma \cdot \sigma_2^* K_{\overline{C}_2} + \sum_{i \in I}(m(A_{1,i}) - 1)\Gamma \cdot A_{1,i} + \sum_{j \in J}(m(A_{2,j}) - 1)\Gamma \cdot A_{2,j}.$$

We see that $\Gamma \cdot \sigma_i^* K_{\overline{C}_1} = -2d_i$ are the only negative terms. Now Γ must intersect every fiber of σ_i at exactly d_i points counted with multiplicity. Equivalently, since the fiber class $F_1 \sim m(A_{1,i})A_{1,i}$ for $i \in I \setminus I^e$, so that there is no contribution from exceptionals, we have

$$d_1 = \Gamma \cdot F_1 = m(A_{1,i})\Gamma \cdot A_{1,i} = m_i \Gamma \cdot A_{1,i}$$

and likewise $d_2 = n_j\Gamma \cdot A_{2,j}$ for each $j \in J \setminus J^e$. At the fibers indexed by $i \in I^e$ however the curve Γ might only intersect the fiber along exceptionals and therefore will not contribute any intersections to this count. Putting this together we conclude

$$\Gamma \cdot D \geq -2d_1 - 2d_2 + d_1 \sum_{i \in I \setminus I^e} \frac{m_i - 1}{m_i} + d_2 \sum_{j \in J \setminus J^e} \frac{n_j - 1}{n_j} \geq 0$$

from our assumptions (really just using the first term in the maximum).

Now we need to consider the case of central components $A_{1,i} \cdot D$ and $A_{2,j} \cdot D$. Since $A_{1,i}$ is contracted by σ_1 it intersects the fiber F_1 trivially. Furthermore,

$$F_1 \sim m(A_{1,i})A_{1,i} + \sum_k m(E_{1,i,k})E_{1,i,k}$$

so intersecting with $A_{1,i}$ we find

$$A_{1,i}^2 = -\sum_{\substack{k \\ A_{1,i} \cap E_{1,i,k} \neq \emptyset}} \frac{m(E_{1,i,k})}{m(A_{1,i})}.$$

Next we compute each term appearing in $A_{1,i} \cdot D$. The first two terms are easy:

1. $A_{1,i} \cdot \sigma_1^* K_{\overline{C}_1} = 0$ because $A_{1,i} \cdot F_1 = 0$
2. $A_{1,i} \cdot \sigma_2^* K_{\overline{C}_2} = -2 \deg{(A_{1,i} \to \overline{C}_2)} = -2\frac{\#G}{\#G_{x_i}}$.

Next we have,

$$\begin{aligned} A_{1,i} \cdot &\sum_{i \in I} \left((m(A_{1,i}) - 1)A_{1,i} + \sum_{k \in \mathrm{Exc}_{1,i}} (m(E_{1,i,k}) - 1)E_{1,i,k} \right) \\ &= \sum_{\substack{k \\ A_{1,i} \cap E_{1,i,k} \neq \emptyset}} \left[(m(E_{1,i,k}) - 1) - (m(A_{1,i}) - 1)\frac{m(E_{1,i,k})}{m(A_{1,i})} \right] = \sum_{\substack{k \\ A_{1,i} \cap E_{1,i,k} \neq \emptyset}} \left[\frac{m(E_{1,i,k})}{m(A_{1,i})} - 1 \right]. \end{aligned}$$

Now $A_{1,i}$ intersects every element of the last sum properly so each gives a non-negative contribution. Hence we may drop all of them except for $A_{2,j}$ indexed by $j \in J \setminus J^e$ to get,

$$\begin{aligned} A_{1,i} \cdot \sum_{j \in J} \left((m(A_{2,j}) - 1)A_{2,j} + \sum_{k \in \mathrm{Exc}_{1,i}} (m(E_{2,j,k}) - 1)E_{2,j,k} \right) &\geq \sum_{j \in J \setminus J^e} (n_j - 1)A_{1,i} \cdot A_{2,j} \\ &= \sum_{j \in J \setminus J^e} \frac{n_j - 1}{n_j} \frac{\#G}{\#G_{x_i}}. \end{aligned}$$

To see the last equality note that $F_2 \sim m(A_{2,j})A_{2,j}$ for $j \in J \setminus J^e$ and therefore

$$A_{1,i} \cdot A_{2,j} = \frac{1}{n_j} \deg\left(A_{1,i} \to \overline{C}_2\right) = \frac{1}{n_j} \cdot \frac{\#G}{\#G_{x_i}}.$$

Putting everything together,

$$A_{1,i} \cdot D \geq \frac{\#G}{\#G_{x_i}} \left[\sum_{j \in J \setminus J^e} \frac{n_j - 1}{n_j} - 2 \right] + \sum_{\substack{k \\ A_{1,i} \cap E_{1,i,k} \neq \varnothing}} \left[\frac{m(E_{1,i,k})}{m(A_{1,i})} - 1 \right]$$

The last term satisfies,

$$\sum_{\substack{k \\ A_{1,i} \cap E_{1,i,k} \neq \varnothing}} \left[\frac{m(E_{1,i,k})}{m(A_{1,i})} - 1 \right] \geq -\#Y^{\text{sing}} \cap A_{1,i}$$

since the H–J strings are attached at the singular points of the surface that lie on $A_{1,i}$. Therefore, the second in the maximum of the second inequality implies $A_{1,i} \cdot D \geq 0$. The first inequality gives, symmetrically, $A_{2,j} \cdot D \geq 0$. □

Example 3.5.9. For an explicit example, we form a modified version of the $K_X^2 = 6$ examples considered by [GRVR18] (cf. [BP12, Table 1] and [BCGP12, Table 2]). For concreteness we choose to modify the example [BCGP12, §6.20] with $G := A_6$ and signatures $(2, 5, 5)$ and $(3, 3, 4)$ with generating vectors of monodromy

$$g_1 := (1\,6)(3\,4), \quad g_2 := (2\,5\,5\,3\,6), \quad g_3 := (1\,6\,4\,5\,2)$$

and

$$h_1 := (1\,5\,6), \quad h_2 := (1\,4\,6)(2\,3\,5), \quad h_3 := (1\,5\,3\,2)(4\,6).$$

These curves produce a singular quotient model $(C_1 \times C_2)/G$ with a basket of singularities:

$$\mathscr{B}(Y) := \{2 \times A_1\}.$$

The point is, if we modify the monodromy representation by adding additional generators to the first vector of order 5 and additional generators to the second list of order 3 the basket of singularities is unchanged because these new monodromy stabilizers cannot "interact" since they are coprime to each element of the signature of the other representation. We choose lists of generators,

$$(g_1, g_2, g_3, g_3^{-1}, g_3) \quad \text{and} \quad (h_1, h_1^{-1}, h_1, h_2, h_3)$$

which are still spherical generators of G. Then the index set contains $I \setminus I^e \supset \{2,3,4,5\}$ with no singularities in the corresponding fiber on the quotient model and hence meeting no exceptional curves on X. Likewise, $J \setminus J^e \supset \{1,2,3,4\}$. The two singularities appear only in the fiber over $\bar{x}_5 \times \bar{y}_1$. Furthermore, each stabilizer G_{x_i} or G_{y_i} has size bounded by 5 and since there are only two singularities total there are only two singularities on any fiber. Therefore, the inequalities,

$$\sum_{i \in I \setminus I^e} \frac{m_i - 1}{m_i} = 4 \cdot \frac{4}{5} \geq \max\left\{2, \frac{5}{360}(10+2)\right\}$$
$$\sum_{j \in J \setminus J^e} \frac{n_j - 1}{n_j} = 4 \cdot \frac{2}{3} \geq \max\left\{2, \frac{5}{360}(10+2)\right\}$$

are true. It is also easy to compute that $(K_X - E)^2 > 0$ proving that X is algebraically quasi-hyperbolic.

Non-quasi-hyperbolic examples

Here we present examples of general type product-quotient surfaces that we can prove are not covered by genus 0 or 1 curves yet contain infinitely many rational curves. In the previous section, we made the two curves have very different monodromy actions so that there was little "interaction" between the two curves which forced nefness of $(K_X - E)$ in a very coarse way. Here we are at the opposite extreme where we take the two curves C_1 and C_2 with their respective G-actions to be equal, as will also be done for the main counterexamples in the following section §3.6 for different reasons related primarily to controlling the Galois representation.

For us, the advantage of taking $C_1 = C_2$ is that the diagonal $\Delta_C \subset C \times C$ provides a natural G-invariant curve (since the actions are also diagonal). Assuming that $\overline{C} \cong \mathbb{P}^1$ then the quotient $\overline{\Delta} := \Delta_C / G \subset Y := (C \times C)/G$ is also rational and provides a nontrivial choice of a rational curve on the singular model. Then its strict transform $R \subset X$ is a smooth rational curve not living in the exceptional locus nor a fiber since it projects nontrivially to each factor. In fact, R is a section of both σ_1 and σ_2.

Now the partial Frobenius iterates of R form an infinite discrete set of rational curves on X. Assume for simplicity that C is defined over $\mathbb{F}_p$. These new curves can be easily described by starting with the graph of the n-th iterate $F^n : C \to C$ of Frobenius: $\Gamma_{F^n} \subset C \times C$. The cycle Γ_{F^n} is also G-invariant and hence defines a curve $R_n \subset X$. The curves R_n are also rational because $\Gamma_{F^n} \cong C$ equivariantly via the first projection. However, R_n has degrees $(1, p^n)$ with respect to the projections σ_1, σ_2. Hence we get an infinite set of rational curves.

This construction applies immediately whenever the curves have a G-invariant morphism. In all the examples constructed in §3.6, we will have the diagonal at our disposal. These examples will also have $K_X - E$ big (as verified in the D_7, D_6, and A_4 cases below) and therefore are not covered by genus 0 or genus 1 curves by Theorem 3.2.3 and Theorem 3.2.7. Finally, we show how to compute R^2 and show that this infinite proliferation of rational curves can start from a (-1)-curve on X even when X is not uniruled.

Example 3.5.10. The following example was suggested by ChatGPT-5.5 Pro. Consider the following spherical generators of $G = A_7$ consisting of three 5-cycles:

$$g_1 = (1\,5\,2\,4\,6), \quad g_2 = (2\,7\,3\,6\,4), \quad g_3 = (1\,3\,7\,2\,5).$$

satisfying $g_1 g_2 g_3 = 1$. Let $C \to \mathbb{P}^1$ be the corresponding G-cover which has signature $(5, 5, 5)$ and hence is uniformized by the triangle group $\Delta(5, 5, 5)$. This curve has genus,

$$g(C) = 1 + \frac{\#A_7}{2}\left[-2 + 3 \cdot \left(1 - \frac{1}{5}\right)\right] = 505.$$

For the singular model $Y = (C \times C)/G$, the total basket of singularities is

$$\mathscr{B}(Y) := \{9 \times \tfrac{1}{5}(1, 1), 18 \times \tfrac{1}{5}(1, 2), 9 \times A_4\}.$$

The formula of Corollary 3.4.6 gives

$$(K_X - E)^2 = 783 - 9 \cdot 11 - 18 \cdot 5 - 9 \cdot 2 = 576 > 0.$$

Therefore $\omega_X(-E)$ is big. Since all 5-cycles are conjugate in A_7, the same is true of all asymmetric Frobenius twists of X (see §3.6.2). Therefore X is not covered by genus 0 or 1 curves. Finally, we consider the curve $R \subset X$. First $\Delta_C^2 = 2 - 2g = -1008$ and therefore

$$\overline{\Delta}^2 = \frac{\Delta_C^2}{\#G} = -\frac{1008}{2520} = -\frac{2}{5}.$$

This is fractional because $\overline{\Delta}$ passes through non-canonical singularities of Y. It passes through exactly the three $\frac{1}{5}(1, 1)$-singularities given by the diagonal of the branch points of

$C \to \mathbb{P}^1$. To compute R^2, we use that a $\frac{1}{5}(1,1)$-singularity is resolved by a single blowup. Since $\overline{\Delta}$ is a smooth rational curve meeting the main component of the fiber transversely (their tangent lines differ in the tangent cone of the singularity) the blowup separates these curves so R only intersects the exceptional curve E in those fibers. Moreover, E enters with multiplicity 1 because if $F_1 \sim 5A_{1,i} + aE$ and $E^2 = -5$ then $E \cdot F_1 = 0$ implies $a = 1$. The same holds for the multiplicity of E viewed as a fiber of σ_2. Therefore, by Serrano's formula

$$R \cdot K_X = -2 \deg (R \to \overline{C}_1) - 2 \deg (R \to \overline{C}_2) + 3R \cdot E = -2 - 2 + 3 = -1$$

so R is a (-1)-curve since it is smooth and rational. Alternatively, the map of pairs $\pi : (X, R) \to (Y, \overline{\Delta})$ is a log resolution. Write,

$$R = \pi^* \overline{\Delta} + \sum b_i E_i$$

but $R \cdot E_i = 1$ so $b_i = -\frac{1}{5}$. Therefore,

$$R^2 = \overline{\Delta}^2 - \frac{3}{5} = -1.$$

Moreover, $K_X \sim \pi^* K_Y - \frac{3}{5} E$ hence $(K_X + E) \cdot E = -2$. Therefore, writing

$$K_X + R = \pi^*(K_Y + \overline{\Delta}) + \sum a_i E_i$$

we see that $a_i = -\frac{4}{5}$ for the three diagonal $\frac{1}{5}(1,1)$ singularities. Hence the pair $(X, \overline{\Delta})$ is dlt.

3.6 Constructions and examples

In this section, we prove Theorem 3.1.1 by checking that the example described in the introduction satisfies the necessary conditions to apply Theorem 3.1.2.

3.6.1 Hurwitz curves of genus 14

A *Hurwitz curve* is any hyperbolic smooth projective curve C over $\mathbb{C}$ realizing equality in the Hurwitz bound

$$\# \operatorname{Aut}(C) = 84(g(C) - 1).$$

Any such curve is a Belyĭ curve and hence is defined over $\overline{\mathbb{Q}}$. As a complex curve, C can be realized as a quotient of the upper half-plane $\mathbb{H}$ by a Fuchsian group of finite index in the von Dyck group $\mathbb{T}(2,3,7)$. This result arises from $2,3,7$ being the unique (up to reordering) sequence of positive integers $e_1, \ldots, e_r$ minimizing the sum

$$\sum_i \left(1 - \frac{1}{e_i}\right)$$

over all sequences such that the sum is > 2 and obtaining the minimum value of $2 + \frac{1}{42}$. This means we can always present a Hurwitz curve as a Belyĭ map $C \to \mathbb{P}^1$ ramified over $0, 1, \infty$ to order $2, 3, 7$. Because the ramification orders are distinct, given C, this map is unique up to isomorphism. Therefore, the data to specify a Hurwitz curve with automorphism group G is a list of spherical generators $g_1, g_2, g_3 \in G$ of orders $2, 3, 7$ up to simultaneous conjugacy (which allows for isomorphisms not respecting a marked base-point in C). A group admitting such generators is called a *Hurwitz group*. Actually, what we have specified is a G-Belyĭ cover of $\mathbb{P}^1$ meaning along with a particular choice of isomorphism $G \xrightarrow{\sim} \mathrm{Aut}(C)$. To remove this additional choice, we can consider tuples (g_1, g_2, g_3) equivalent up to overall automorphism of G rather than only inner automorphisms. When G has trivial center, the choice of an isomorphism $\varphi : G \to \mathrm{Aut}(C)$ rigidifies the pair (C, φ), giving a point on a certain Hurwitz scheme.

For $g = 14$, there exist exactly three isomorphism classes of Hurwitz curves which are exchanged by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In fact, as we will see in §4.2, these three curves arise from a curve C/K_+ under the three complex embeddings $K_+ \hookrightarrow \mathbb{C}$ of the field $K_+ = \mathbb{Q}(\eta)$ for $\eta = \zeta_7 + \zeta_7^{-1}$. These curves have $\mathrm{Aut}(C) \cong \mathrm{PSL}_2(\mathbb{F}_{13})$ which is a simple group of order 1092. However, there are 6 isomorphism classes of $\mathrm{PSL}_2(\mathbb{F}_{13})$-Belyĭ $(2,3,7)$-ramified covers of $\mathbb{P}^1$ corresponding to the 6 equivalence classes of $(2,3,7)$-spherical generators up to simultaneous conjugation. Under the unique outer automorphism of $\mathrm{PSL}_2(\mathbb{F}_{13})$, these 6 classes form 3 orbits corresponding to the three Hurwitz curves of genus 14, each presented as a $\mathrm{PSL}_2(\mathbb{F}_{13})$-cover of $\mathbb{P}^1$ in two distinct ways up to $\mathrm{PSL}_2(\mathbb{F}_{13})$-equivariant isomorphism. In fact, in §4.2, we show that the stack $\mathscr{H}_{14,\mathrm{PSL}_2(\mathbb{F}_{13})}$ of pairs (C, φ) consisting of a curve C of genus 14 and an isomorphism $\varphi : \mathrm{PSL}_2(\mathbb{F}_{13}) \to \mathrm{Aut}(C)$ is, over $\mathbb{Q}$, isomorphic to $\mathrm{Spec}\, K$ where $K = K_+(\sqrt{-3\eta - 2})$ is the ray class field of a prime in $\mathscr{O}_{K_+}$ lying over 13. This is the degree 6 number field recorded as [LMF25p] in the LMFDB, and the various embeddings $K \hookrightarrow \mathbb{C}$ correspond to the 6 isomorphism classes of $\mathrm{PSL}_2(\mathbb{F}_{13})$-Belyĭ curves considered above. Note that $K_+ \subset K$ is the subfield consisting of totally real elements, justifying the notation.

For example, we can choose a particular set of spherical generators (g_1, g_2, g_3) defining a surjection $\mathbb{T}(2,3,7) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{F}_{13})$,

$$g_1 = \begin{pmatrix} 5 & 3 \\ 0 & 8 \end{pmatrix} \qquad g_2 = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix} \qquad g_3 = \begin{pmatrix} 0 & 2 \\ 6 & 6 \end{pmatrix}.$$

This corresponds to the "refined passport" [LMF25g] with conjugacy classes $(2, 3, 5)$ using the labeling scheme of the LMFDB. We call this curve C_{ν_0}, or to be precise $(C, \varphi)_{\nu_0}$, with its $\mathrm{PSL}_2(\mathbb{F}_{13})$-action, for a certain fixed embedding $\nu_0 : K \hookrightarrow \mathbb{C}$. Because in Theorem 3.2.3 it is also necessary to understand partial Frobenius twists, it is insufficient to consider only C_{ν_0}. We also look at the 6 Galois conjugates corresponding to different embeddings $\nu : K \hookrightarrow \mathbb{C}$. These are permuted by $\mathrm{Gal}(K'/\mathbb{Q}) \cong C_2 \times A_4$ where K' is the Galois closure of K. Generators for all 6 isomorphism classes of $\mathrm{PSL}_2(\mathbb{F}_{13})$-Belyĭ covers can be found in [LMF25f].

The following computations will be based on the subgroups of $\mathrm{PSL}_2(\mathbb{F}_{13})$ and the curves appearing as intermediate quotients of C. The subgroup lattice of $\mathrm{PSL}_2(\mathbb{F}_{13})$ is recorded in Figure 3.3. $\mathrm{Out}(\mathrm{PSL}_2(\mathbb{F}_{13})) \cong C_2$ acts trivially on this diagram except that it exchanges the two copies of S_3. For each subgroup $H \subset G := \mathrm{PSL}_2(\mathbb{F}_{13})$ we can compute the intermediate curve $C \to C/H \to \mathbb{P}^1$ as follows. The curve $C/H \to \mathbb{P}^1$ is a Belyĭ curve whose monodromy is given by the action of g_1, g_2, g_3 on the fiber G/H. The cycle decomposition of this action determines the ramification structure over $0, 1, \infty$. Using Riemann–Hurwitz, we get a diagram of quotient curves whose genera are labeled in Figure 3.4. Notice that the genus 1 quotients are via the two copies of S_3 and are hence exchanged under the outer automorphism. This suggests, as we will see is indeed the case in §4.2, that the elliptic curve factor is not defined over K_+ but rather over the field of definition K of $\mathrm{Aut}(C)$ whose relative Galois group $\mathrm{Gal}(K/K_+) \cong C_2$ acts on conjugacy classes of subgroups of $\mathrm{Aut}(C)$ by the full group of outer automorphisms. Additionally, as is verified in the Magma script [Chu25, `subgroup_schemes.m`], $\mathrm{Aut}(C)$, as a form of $\mathrm{PSL}_2(\mathbb{F}_{13})$ over K_+, admits subgroups $H \subset \mathrm{Aut}(C)$ which recover a copy of each of D_7, D_6, A_4 geometrically. Hence, the surfaces in Theorem 3.1.1 are defined over K_+ (we can also use Proposition 3.7.3 to obtain the Hirzebruch–Jung resolution over K_+).

In order for the product-quotient surface $X \to (C_{\nu_0} \times C_{\nu_0})/H$ to have zero irregularity, we must choose $H \subset G$ so that C/H has genus 0. Among these, the minimal groups having rational quotients are most promising since the larger $\#H$ the more difficult it becomes for $(K_X - E)^2 > 0$ to be satisfied. From these figures, the minimal groups with genus 0 quotient are D_7, D_6, A_4, and $C_6 \rtimes C_3$. It turns out, the first three, but not the last, will produce simply-connected examples with $(K_X - E)^2 > 0$.

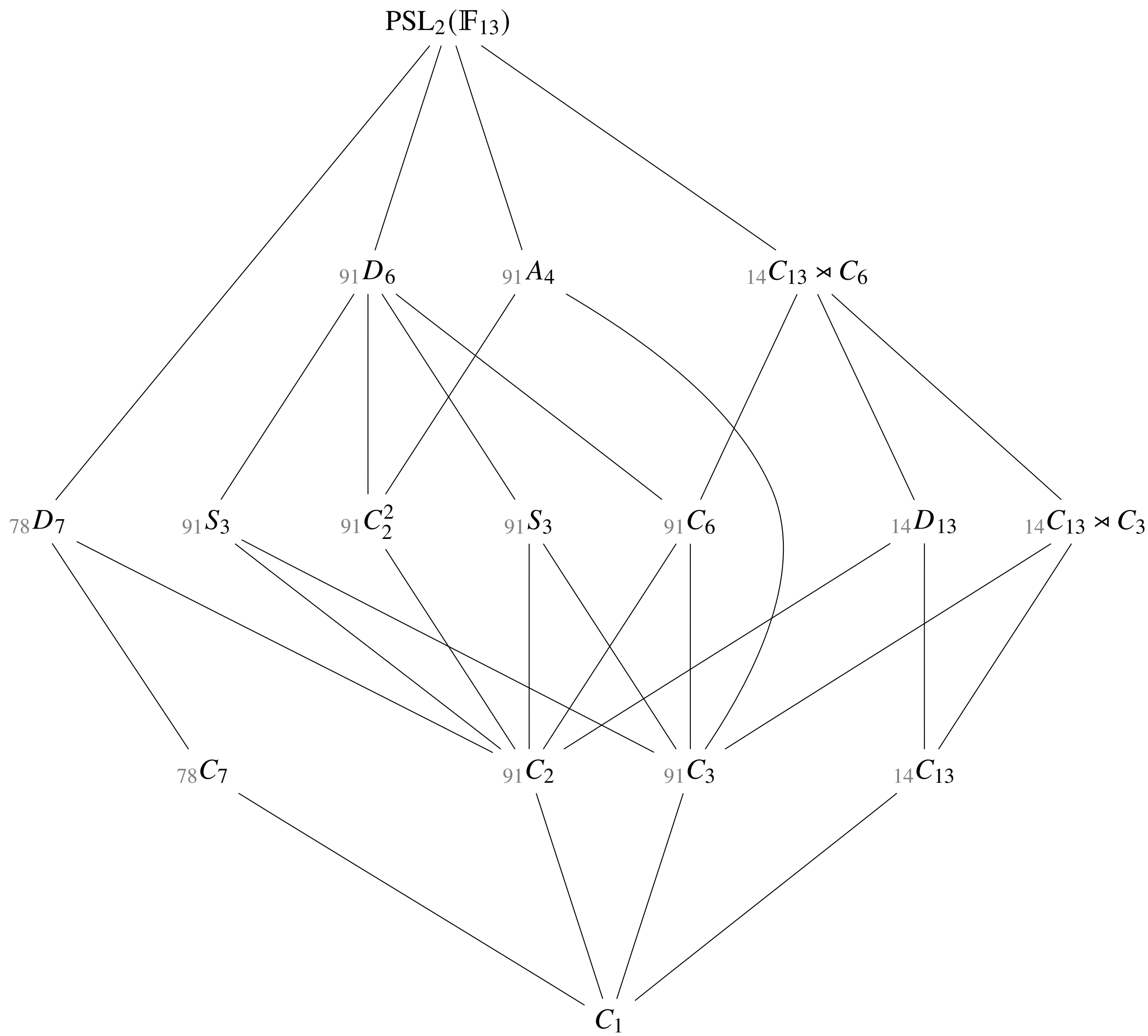


Figure 3.3: Subgroup lattice of $\mathrm{PSL}_2(\mathbb{F}_{13})$ up to conjugacy. The small numbers in gray represent the number of conjugate copies of a subgroup.

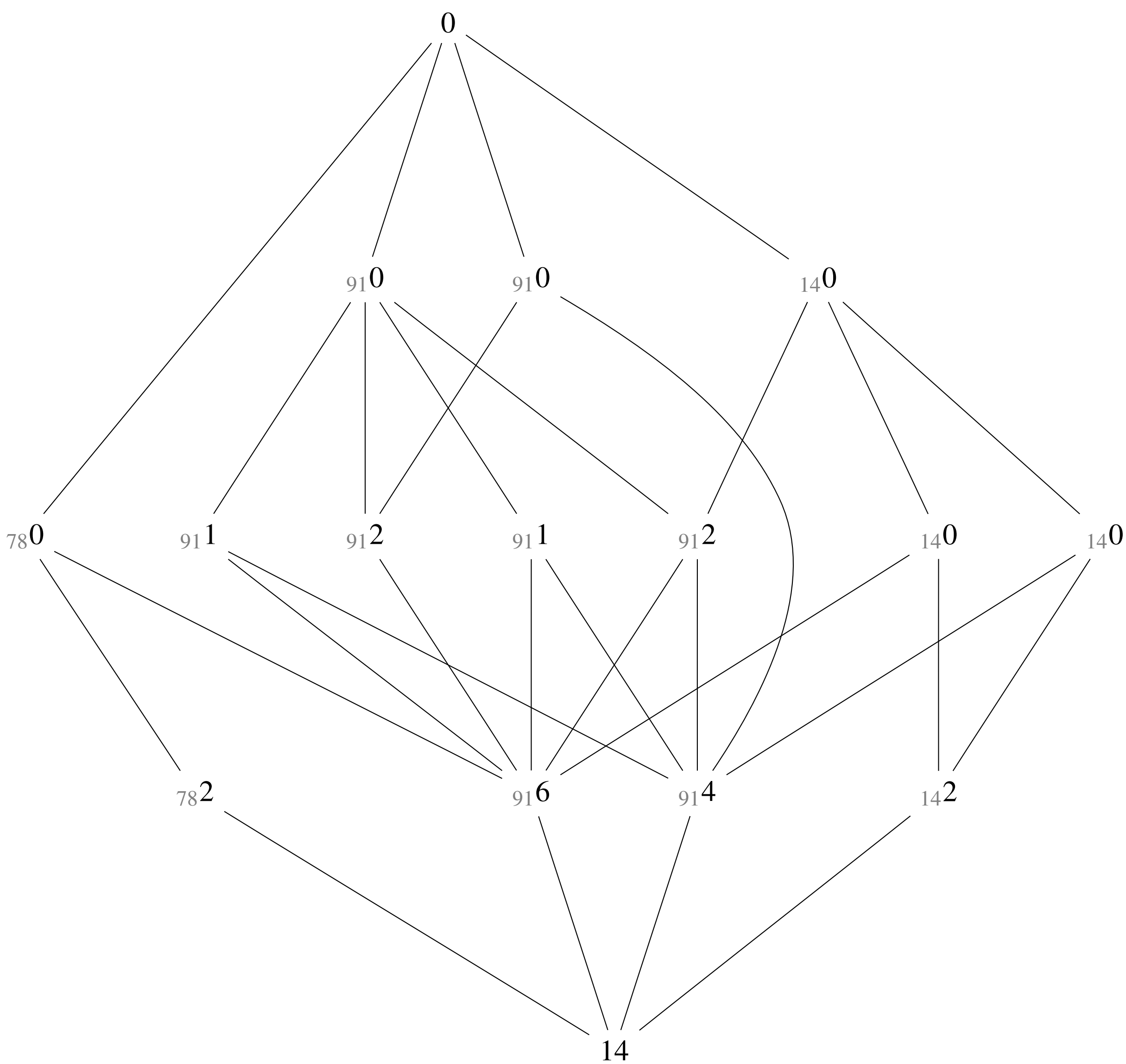


Figure 3.4: Lattice of intermediate curves of the cover $C \to \mathbb{P}^1$ up to conjugacy with their genera marked. The small numbers in gray represent the number of isomorphic conjugate copies of a curve represented as a particular quotient.

Supersingularity

The Hurwitz curve C of genus 14 has the amiable property that its Jacobian is isogenous to 14 copies of some elliptic curve E. Moreover, this curve E is defined over the field $K = (K_+)^{(\mathfrak{p}_{13}\infty)} = K_+(\sqrt{-3\eta - 2})$ by Theorem 4.2.5 which we postpone until §4.2 in the subsequent chapter. Since K has a real place, we may apply Elkies' theorem [Elk89] to conclude that there are infinitely many places of K at which E has supersingular reduction. Since supersingularity is a geometric property, at each prime $\mathfrak{p} \subset \mathcal{O}_{K_+}$ lying under a prime in $\mathcal{O}_K$ of supersingular reduction for E where $X_\mathfrak{p}$ has good reduction, we conclude that $X_\mathfrak{p}$ is supersingular by Corollary 3.3.19.

3.6.2 Proof of Theorem 3.1.1

For any product-quotient surface X dominated by a product of Hurwitz curves of genus 14, we saw in the previous section that X has infinitely many primes of supersingular reduction. In order for X to violate Shioda's conjecture, we must show that $\pi_1(X) = 1$ and, to apply Theorem 3.2.3, that its asymmetric Frobenius twists all carry diagonal symmetric forms.

For the minimal resolution $\pi : X \to Y := (C \times C)/H$ of each surface defined by a subgroup $H \subset \mathrm{PSL}_2(\mathbb{F}_{13})$ for H among the copies of D_7, D_6, A_4 in $\mathrm{PSL}_2(\mathbb{F}_{13})$, we will verify these claims. Fix the following notation: for $\sigma, \tau \in \mathrm{Gal}(K'/\mathbb{Q})$, denote by $\pi^{\sigma,\tau} : X^{\sigma,\tau} \to ({}^\sigma(C_{v_0}) \times {}^\tau(C_{v_0}))/H$ the minimal resolution, where H acts diagonally using the fixed inclusion $H \subset G$ and the induced actions under the Galois twists of G. We will need to check

1. $\pi_1(X_\mathbb{C}) = 1$
2. for all $\sigma, \tau \in \mathrm{Gal}(K'/\mathbb{Q})$ we have $(K_{X^{\sigma,\tau}} - \mathrm{Exc}\,\pi^{\sigma,\tau})^2 > 0$

Let us first see why this is sufficient to prove Theorem 3.1.1 by an application of Theorem 3.2.3 and Theorem 3.2.7. Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_{K',S}$ be a spreading out as a smooth projective morphism over the ring of S-integers of K' away from some finite set of primes S. An explicit set S is computed in §4.2. For any prime $\mathfrak{P} \subset \mathcal{O}_{K',S}$, by Artin comparison [Gro71, Exposé XII, Cor. 5.2] and surjectivity of Grothendieck's specialization maps [Gro71, Exposé X, Cor. 2.3], the geometric fiber of $\mathcal{X}$ is simply-connected, $\pi_1^{\text{ét}}(\mathcal{X}_{\overline{\mathfrak{P}}}) = 1$. By Corollary 3.4.6, $X^{\sigma,\tau}$ carries a diagonal symmetric form whose zero locus contains a big divisor. Likewise, spread out $X^{\sigma,\tau}$ to $f^{\sigma,\tau} : \mathcal{X}^{\sigma,\tau} \to \operatorname{Spec} \mathcal{O}_{K',S}$. For any prime $\mathfrak{P} \subset \mathcal{O}_{K',S}$ there exists $\mathrm{Frob}_\mathfrak{P} \in \mathrm{Gal}(K'/\mathbb{Q})$ in the decomposition group of $\mathfrak{P}$ lifting Frobenius of the residue field $\mathcal{O}_{K'}/\mathfrak{P}$. Therefore, setting $\sigma := (\mathrm{Frob}_\mathfrak{P})^a$ and $\tau := (\mathrm{Frob}_\mathfrak{P})^b$ we obtain that the fiber $(\mathcal{X}^{\sigma,\tau})_\mathfrak{P}$ is naturally isomorphic to $(\mathcal{X}_\mathfrak{P})^{a,b}$ in the notation of §3.2. By upper semi-continuity, $(\mathscr{X}^{\sigma,\tau})_{\mathfrak{P}}$ also carries a diagonal symmetric form whose zero locus contains a big divisor. Hence, Theorem 3.2.3 and Theorem 3.2.7 apply to the geometric fibers $\mathscr{X}_{\overline{\mathfrak{P}}}$.

It remains to check the numerical claims, which are computable over any field extension. For the remainder of this section, everything is base-changed to $\mathbb{C}$ along $\nu_0 : K \hookrightarrow \mathbb{C}$ without comment. To perform the calculations in each case, we need to know the spherical generators for the monodromy of an intermediate cover $C \to C/H \cong \mathbb{P}^1$. This is computed as follows. Consider the factorization $C \to C/H \to C/G \cong \mathbb{P}^1$. Above each ramification point $p_i \in \mathbb{P}^1$ of $f : C \to \mathbb{P}^1$, the fiber of $C/H \to \mathbb{P}^1$ is in bijection with the cycle decomposition of the monodromy g_i action on G/H. There is a loop $\gamma_{i,j}$ around each point of the fiber corresponding to a cycle $\sigma_{i,j}$ in this decomposition. To compute the monodromy of $C \to C/H$, we need to know the monodromy around each $\gamma_{i,j}$. Let ℓ be the order of $\sigma_{i,j}$; by definition g_i^ℓ fixes any coset Hr appearing in the cycle $\sigma_{i,j}$. This means rg_i^ℓ is conjugate to r, i.e., $rg_i^\ell r^{-1} = h_{i,j}$, for some element $h_{i,j} \in H$ well-defined up to H-conjugacy. The conjugacy class of $h_{i,j}$ is the conjugacy class of the monodromy action of $\gamma_{i,j}$ on the fiber H of $C \to C/H$. Note that this procedure only determines each monodromy element up to conjugacy and additional care must be taken to extract spherical generators, i.e., a well-defined map $\pi_1((C/H)^\circ) \to H$ since there may be multiple inequivalent covers defined by spherical generators with the same conjugacy classes. Luckily for the purposes of computing baskets of singularities, any two choices are equivalent (see Remark 3.3.17).

From here, we can verify $\pi_1(Y) = 1$ and compute the basket of singularities of Y. The explicit computation of the basket $\mathscr{B}$ is done as in §3.3.4. Using Lemma 3.3.9, it suffices to show that $\pi_1(Y^\circ)$ is normally generated by torsion. For each example, we will apply Proposition 3.3.13 to conclude that $\pi_1(X) = 1$. Finally, we will directly verify $(K_{X^{\sigma,\tau}} - \operatorname{Exc}\pi^{\sigma,\tau})^2 > 0$ by applying Proposition 3.4.5.

The Galois action on monodromy classes

By definition $X^{\sigma,\tau}$ is the product-quotient surface associated to the Galois twisted curves ${}^\sigma C_{\nu_0}$ and ${}^\tau C_{\nu_0}$. Therefore, we would like to understand how the absolute Galois group G_K acts on G-covers $C \to \mathbb{P}^1$ in terms of their monodromy information. In totality, this action is indescribably complex since the outer action of G_K on $\pi_1^{\text{ét}}(\mathbb{P}^1_K \setminus \{0, 1, \infty\})$ is faithful. However, we can understand quite directly how the G_K-action alters the conjugacy classes of monodromy $g_i \in G$. We will, for simplicity, assume that the branch points $p_1, \ldots, p_r \in \mathbb{P}^1(K)$ are K-rational and hence fixed by the G_K-action. Consider a surjection, $\varphi : \mathbb{T} = \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \ldots, p_r\}) \to G$ to a finite group G. Since this factors through the profinite completion $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \ldots, p_r\})$ of $\mathbb{T}$ we obtain an outer action of G_K on $\operatorname{Surj}(\mathbb{T}, G)$. Since it preserves orders of ramification, it naturally restricts to a G_K outer action on $\mathrm{Surj}(\mathbb{T}(m_1, \ldots, m_r), G)$.

Theorem 3.6.1. *[MM18, Theorem 2.6] Let $C_i \subset G$ be the conjugacy class of $\varphi(c_i)$, the monodromy class around p_i. Then for $\sigma \in G_K$, the monodromy class of $(\sigma \cdot \varphi)(c_i)$ is ${}^{\sigma}C_i = C_i^{\epsilon_{m_i}(\sigma)}$ where $\epsilon_m : G_K \to (\mathbb{Z}/m_i\mathbb{Z})^\times$ is the cyclotomic character and m_i is the order of $g_i = \varphi(c_i)$ or equivalently of any element of C_i.*

In particular, by Remark 3.3.17, if the conjugacy classes of monodromy are *rational* meaning $C_i^k = C_i$ for all k coprime to m_i, then the basket of singularities and hence the numerical invariants of X are the same as $X^{\sigma,\tau}$ for all σ, τ.

Case: $H \cong D_7$

Let $Y = (C \times C)/H$ with H acting diagonally and $\pi : X \to Y$ the minimal resolution. Using the method described in the preceding section, we compute that $g : C \to C/H$ is ramified at 7 points with conjugacy classes

$$\mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(r).$$

This means $g : C \to C/H \cong \mathbb{P}^1$ corresponds to the refined passport [LMF25j] with conjugacy classes $(2, 2, 2, 2, 2, 2, 3)$ using the labeling scheme of the LMFDB. The spherical generators of $g : C \to \mathbb{P}^1$ viewed as a D_7-cover fit the description of Example 3.3.15. Therefore, applying Proposition 3.3.13, we see that $\pi_1(X) = 1$.

Using the method described in §3.3.4, Y has the following basket of singularities

$$\mathscr{B}(Y) := \{36 \times A_1, 1 \times A_6, 1 \times \tfrac{1}{7}(1,1)\}.$$

Plugging into Proposition 3.3.5 gives Chern numbers:

$$K_X^2 = 93 \qquad c_2(X) = 111.$$

Using Corollary 3.4.6:

$$(K_X - E)^2 = 93 - 36 \cdot 2 - 1 \cdot 2 - 1 \cdot 17 = 2$$

is positive and hence $\omega_X(-E)$ is big. This surface has Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 16 & & 77 & & 16 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and is of general type. It is a routine matter to repeat these calculations for $X^{\sigma,\tau}$. The MAGMA script `verify_examples.m`, available on `church:github` at [Chu25], performs the necessary checks.

Example 3.6.2. A surprising phenomenon appears in this example: X and $X^{\sigma,\tau}$ do not even have the same numerical invariants. For example, among the $X^{\sigma,\tau}$ for $H \cong D_7$ is the product-quotient surface X' whose pair of spherical generators for D_7 defining the two curves C_1 and C_2 have conjugacy classes,

$$\mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(r).$$

and

$$\mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(r^2)$$

respectively. This leads to a basket of singularities for X':

$$\mathscr{B}(Y') := \{36 \times A_1, 1 \times \frac{1}{7}(1,3), 1 \times \tfrac{1}{7}(1,4)\}.$$

which is different from X, replacing the $\frac{1}{7}(1,1)$ and $\frac{1}{7}(1,6)$ singularities by the pair $\frac{1}{7}(1,3)$ and $\frac{1}{7}(1,4)$. This leads to different numerics. It has Chern numbers,

$$K_{X'}^2 = 95 \qquad c_2(X') = 109$$

and thus a Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 17 & & 73 & & 17 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Finally, using Corollary 3.4.6,

$$(K_{X'} - E')^2 = 95 - 36 \cdot 2 - 1 \cdot 5 - 1 \cdot 8 = 10.$$

Case: $H \cong D_6$

Let $Y = (C \times C)/H$ with H acting diagonally and $\pi : X \to Y$ the minimal resolution. Using the method described in the preceding section, we compute that $g : C \to C/H$ is ramified at 8 points with conjugacy classes

$$\mathrm{cl}(r^3), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(s), \mathrm{cl}(rs), \mathrm{cl}(rs), \mathrm{cl}(rs), \mathrm{cl}(r^2).$$

This means $g : C \to C/H \cong \mathbb{P}^1$ corresponds to the refined passport [LMF25i] with conjugacy classes $(2, 3, 3, 3, 4, 4, 4, 5)$ using the labeling scheme of the LMFDB. We can use Magma to enumerate the possible spherical generators with these prescribed conjugacy classes and for each check explicitly that they extend a good presentation. Therefore, applying Proposition 3.3.13, we see that $\pi_1(X) = 1$.

Using the method described in §3.3.4, Y has the following basket of singularities

$$\mathscr{B}(Y) := \{42 \times A_1, 2 \times A_2, 2 \times \tfrac{1}{3}(1, 1)\}.$$

Plugging into Proposition 3.3.5 gives Chern numbers:

$$K_X^2 = 112 \qquad c_2(X) = 128$$

and using Corollary 3.4.6:

$$(K_X - E)^2 = 112 - 42 \cdot 2 - 2 \cdot 2 - 2 \cdot 5 = 14$$

is positive and hence $\omega_X(-E)$ is big. This surface has Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 19 & & 88 & & 19 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and is of general type. Unlike the D_7 case, all partial twists have the same Hodge diamond and $(K_X - E)^2$, checked in `verify_examples.m`.

Case: $H \cong A_4$

Let $Y = (C \times C)/H$ with H acting diagonally and $\pi : X \to Y$ the minimal resolution. Fix notation $a = (1\,2)(3\,4)$ and $b = (1\,2\,3)$ and $c = (1\,3\,2)$. Using the method described in the preceding section, we compute that $g : C \to C/H$ is ramified at 7 points with conjugacy classes,

$$\mathrm{cl}(a), \mathrm{cl}(a), \mathrm{cl}(a), \mathrm{cl}(b), \mathrm{cl}(b), \mathrm{cl}(c), \mathrm{cl}(c).$$

This means $g : C \to C/H \cong \mathbb{P}^1$ corresponds to the refined passport [LMF25h] with conjugacy classes $(2, 2, 2, 3, 3, 4, 4)$ using the labeling scheme of the LMFDB. We can use Magma to enumerate the possible spherical generators with these prescribed conjugacy classes and for each check explicitly that they extend a good presentation. Therefore, applying Proposition 3.3.13, we see that $\pi_1(X) = 1$.

Using the method described in §3.3.4, Y has the following basket of singularities:

$$\mathscr{B}(Y) := \{18 \times A_1, 8 \times A_2, 8 \times \tfrac{1}{3}(1, 1)\}.$$

Plugging into Proposition 3.3.5 gives Chern numbers,

$$K_X^2 = 110 \qquad c_2(X) = 118.$$

and using Corollary 3.4.6:

$$(K_X - E)^2 = 110 - 18 \cdot 2 - 8 \cdot 2 - 8 \cdot 5 = 18$$

is positive and hence $\omega_X(-E)$ is big. This surface has Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 18 & & 80 & & 18 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and is of general type. Unlike the D_7 case, all partial twists have the same Hodge diamond and $(K_X - E)^2$, checked in `verify_examples.m`.

3.7 Cyclic quotient singularities in mixed characteristic

In this section, we show that the product-quotient surfaces X have good reduction away from the order of G and the bad primes of the curves C_1 and C_2. Of course, X has only finitely many primes of bad reduction so this section is only necessary to get an explicit set of primes in Theorem 3.1.1. To do this, we study cyclic quotient singularities and their resolutions in mixed characteristic. For simplicity, we only study the case of *tame* group actions, meaning those for which $\#G$ is coprime to the residue characteristic.

Lemma 3.7.1. *Let $\mathscr{C} \to S$ be a smooth proper relative curve of genus $g \geq 2$. Let G be a finite group acting on $\mathscr{C}$ as an S-scheme whose action is faithful on the generic fiber. If $\#G$ is coprime to all residue characteristics of S, then for any $g \in G$ the fixed point scheme* $\mathrm{Fix}_g(\mathscr{C})$ *is finite étale over S.*

Proof. First we show that $\mathrm{Aut}(\mathscr{C}/S) \to S$ is finite and unramified. Since $g \geq 2$, the infinitesimal automorphisms $H^0(\mathscr{C}_s, \mathscr{T}_{\mathscr{C}_s}) = 0$ vanish on each fiber. Therefore, infinitesimal deformation theory shows that $\mathrm{Aut}(\mathscr{C}/S)$ is formally unramified. The theory of regular proper models over a DVR shows that $\mathrm{Aut}(\mathscr{C}/S)$ satisfies the valuative criterion of properness.

Properness of $\mathrm{Fix}_g(\mathscr{C}) \to S$ is automatic from the definition. It suffices to show it is étale. This locus is the intersection of two flat divisors on $\mathscr{C} \times_S \mathscr{C}$, namely the diagonal Δ and the graph Γ_g of g. Their intersection is étale over S if and only if they are transverse on each geometric fiber. To prove this, we must show that if $\varphi \in \mathrm{Aut}(C)$ is a nontrivial automorphism of a smooth proper curve C over an algebraically closed field k of characteristic p and φ has finite order coprime to p then $\mathrm{d}\varphi$ is nonzero at every fixed point. Since φ is nontrivial, $\mathrm{Fix}_\varphi(C)$ is a finite collection of points. For $x \in \mathrm{Fix}_\varphi(C)$, we will show that if $\mathrm{d}\varphi|_x = 0$ then φ acts trivially on the formal local ring $\widehat{\mathscr{O}}_{C,x}$ which implies it is trivial on an open $x \in U \subset C$ and hence trivial on C since C is separated. Using induction and the sequences

$$0 \to \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1} \to \mathscr{O}_{C,x}/\mathfrak{m}_x^{n+1} \to \mathscr{O}_{C,x}/\mathfrak{m}_x^n \to 0$$

we will show $\varphi : \widehat{\mathscr{O}}_{C,x} \to \widehat{\mathscr{O}}_{C,x}$ is trivial. The case $n = 1$ is by assumption. Regularity gives $\mathfrak{m}_x^n/\mathfrak{m}_x^{n+1} = \mathrm{Sym}^n(\mathfrak{m}_x/\mathfrak{m}_x^2)$ and we assume φ is the identity on $\mathfrak{m}_x/\mathfrak{m}_x^2$. Therefore, assuming towards induction that $\varphi_n := \varphi \otimes \mathscr{O}/\mathfrak{m}_x^n = \mathrm{id}$ then $\varphi_{n+1} - \mathrm{id}$ defines a k-linear map,

$$\delta : \mathscr{O}_{C,x}/\mathfrak{m}_x^n \to \mathrm{Sym}^n(\mathfrak{m}_x/\mathfrak{m}_x^2).$$

Now if we iterate the automorphism, $\varphi_{n+1}^k = (\mathrm{id} + \delta)^k = \mathrm{id} + k\delta$ since $\delta^2 = 0$ because it lands in a square zero ideal. But φ has finite order m, so we must have $m\delta = 0$ and we assumed m is prime to p so $\delta = 0$. Hence φ is trivial by induction. □

Lemma 3.7.2. *Let n be coprime to the characteristic of $\bar{k}$ and $C_n \circlearrowright X$ be the action of a cyclic group on a smooth surface over $\bar{k}$ with isolated fixed points. Then each singularity of X/C_n is a $\frac{1}{n'}(1, a)$ singularity for some $n' \mid n$ and a coprime to n'.*

Proof. We need to show this singularity is formally equivalent to

$$\mathbb{A}^2 / \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^a \end{pmatrix} \right\rangle$$

for some choice of n-th root of unity $\zeta \in \bar{k}$. This follows from the combination of [Kir10, Proposition 2.4.8] and [Kir10, Proposition 2.3.6] to pass to the completion. □

Proposition 3.7.3. *Let $\mathscr{C}_1 \to S$ and $\mathscr{C}_2 \to S$ be smooth proper relative curves equipped with an action, over S, of a finite group G satisfying the assumptions of Lemma 3.7.1. Then there exists a smooth proper S-scheme $\mathscr{X} \to S$ equipped with a S-morphism $\pi : \mathscr{X} \to (\mathscr{C}_1 \times_S \mathscr{C}_2)/G$ which is identified with the Hirzebruch–Jung resolution on each geometric fiber.*

Proof. This is a consequence of the fact that the algorithm to produce Hirzebruch–Jung resolutions can be run globally and that (weighted) blowups along finite étale multi-sections (indeed along any subscheme flat over S) are compatible with base change. This follows from the ideal sheaf sequence for a closed subscheme $Z \subset X$,

$$0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$$

being compatible with base change along $S' \to S$ when $\mathscr{O}_Z$ is flat over $\mathscr{O}_S$.

Over an algebraically closed field $\bar{k}$, the Hirzebruch–Jung resolution of a cyclic $\frac{1}{n}(1, a)$ singularity is obtained by a sequence of weighted blowups constructed from the ideals of the two torus invariant divisors (see [QR21]). This sequence of weighted blowups is determined entirely from the Hirzebruch–Jung continued fraction representation $\frac{n}{a} = [\![b_1, \ldots, b_\ell]\!]$.

The singular locus of $\mathscr{Y} = (\mathscr{C}_1 \times_S \mathscr{C}_2)/G$ consists of a collection of finite étale multi-sections whose geometric fibers are isolated cyclic quotient singularities. Furthermore, the images in $\mathscr{Y}$ of $\mathscr{C}_1 \times_S \mathrm{Fix}_g(\mathscr{C}_2)$ and $\mathrm{Fix}_g(\mathscr{C}_1) \times_S \mathscr{C}_2$ as g ranges over nontrivial elements of G produce a collection of Weil divisors flat over S whose complement is a toroidal embedding. In particular, near each singularity on a geometric fiber of $\mathscr{Y}$, there is a pair of such divisors formally equivalent to the torus-invariant coordinate axes in the standard quotient form of the singularity. Hence running the weighted-blowups with the Hirzebruch–Jung weights along these divisors produces a relative flat proper surface $\mathscr{X} \to \mathscr{Y}$ over S. Since the blowups commute with base change, on each fiber $\mathscr{X}$ is the Hirzebruch–Jung resolution. In particular, $\mathscr{X} \to S$ is smooth. □

Remark 3.7.4. It is not necessary to use weighted blowups if we only ask that some smooth projective birational model of X_η has good reduction. Indeed, the previous results show that the singular locus of $(\mathscr{C}_1 \times_S \mathscr{C}_2)/G$ is finite étale over S. Blowing up these loci repeatedly does give a simultaneous resolution (this runs Lipman's algorithm fiberwise as no non-normal singularities are introduced as can be seen by considering the subdivision of the fan associated to blowing up a cyclic quotient singularity) but it is not minimal. Since the existence of diagonal symmetric forms is a birational invariant of smooth proper models, this is sufficient for our purposes. We can check existence on the minimal resolution and pull back these forms to any other model before specializing by upper semi-continuity.

Now we use this to justify the claims made in the main text about the places of good reduction of product-quotient surfaces. It follows from the existence of integral canonical models of Shimura varieties of hyperspecial level [Car86, Kis10] that Shimura curves have good reduction away from the level and the discriminant of B. In particular, $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})$ has good reduction away from $\mathfrak{p}_{13}$. Therefore, the product-quotient surfaces in Theorem 3.1.1 have good reduction away from primes over 13 and $\#G$, which is divisible only by $2, 3, 7, 13$. In particular, the supersingular primes of E, which are all > 1000, are primes of good supersingular reduction for the surfaces X in Theorem 3.1.1.

Chapter 4

Supersingular Reduction of Hurwitz Curves

Out of this stony rubbish? Son of man,
You cannot say, or guess, for you know only
A heap of broken images, where the sun beats,
And the dead tree gives no shelter, the cricket no relief,
And the dry stone no sound of water.

T.S. Eliot

4.1 Introduction

Let C/K be a smooth projective curve over a field of characteristic zero. Hurwitz proved a uniform bound on the number of automorphisms of C depending on the genus g,

$$\# \operatorname{Aut}(C) \leq 84(g-1).$$

We call a curve achieving this bound a *Hurwitz curve*. Every Hurwitz curve is defined over a number field because the quotient by the automorphism group $C \to C/\operatorname{Aut}(C) \cong \mathbb{P}^1$ canonically presents C as a G-Belyĭ map with orders of monodromy $(2,3,7)$ over $(0,1,\infty)$. The arithmetic of Hurwitz curves is quite rich, for example González-Diez and Jaikin-Zapirain [GDJZ15] showed that the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of isomorphism classes of Hurwitz curves is faithful. However, Hurwitz curves only appear in a sparse set of genera. Their geometric classification can be entirely reduced to group theory. Indeed, over $\mathbb{C}$, the set of isomorphism classes of pairs $(C, \varphi : \operatorname{Aut}(C) \xrightarrow{\sim} G)$ where C is a Hurwitz curve and

g	G	N	$\mathrm{Out}(G)$
3	$\mathrm{PSL}_2(\mathbb{F}_7)$	1	$\mathbb{Z}/2\mathbb{Z}$
7	$\mathrm{SL}_2(\mathbb{F}_8)$	1	$\mathbb{Z}/3\mathbb{Z}$
14	$\mathrm{PSL}_2(\mathbb{F}_{13})$	3	$\mathbb{Z}/2\mathbb{Z}$
17	G_{1344}	2	$\mathbb{Z}/2\mathbb{Z}$

Table 4.1: The Hurwitz curves of genus $g \leq 100$. Here G is the automorphism group (of which there is a unique isomorphism class for each genus in the table), N is the number of non-isomorphic geometric curves of genus g, and $\mathrm{Out}(G)$ is the outer automorphism group of G [Con90, Con10].

$\varphi : \mathrm{Aut}(C) \xrightarrow{\sim} G$ is an identification of its automorphism group with G is in canonical bijection with the set of surjections

$$\Delta(2,3,7) \twoheadrightarrow G$$

up to conjugacy where $\Delta(a,b,c)$ is the triangle group

$$\Delta(a,b,c) := \left\langle x, y, z \mid x^a, y^b, z^c, xyz \right\rangle$$

identified with the orientation-preserving isometries of the hyperbolic tiling by $(\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c})$-hyperbolic triangles. Then $\mathrm{Out}(G)$ acts freely on the set of conjugacy classes of surjections whose orbits correspond to unmarked Hurwitz curves.

Hurwitz curves of genus ≤ 100 have been tabulated by Conder [Con90, Con10] and the results are summarized in table 4.1. Here G_{1344} denotes the particular group of order 1344 arising from Macbeath's homology cover construction [Mac99, Mac61] applied to the Klein quartic. It is a non-split extension

$$0 \to (\mathbb{Z}/2\mathbb{Z})^3 \to G_{1344} \to \mathrm{PSL}_2(\mathbb{F}_7) \to 0$$

corresponding to an irreducible 3-dimensional $\mathbb{F}_2$-representation of $\mathrm{PSL}_2(\mathbb{F}_7)$. In the GAP Small Groups library, this group is `SmallGroup(1344,814)`. The fact that (P)SL_2 of a finite field shows up as the automorphism group for $g \leq 14$ is not a coincidence. Indeed, those curves are all *Shimura curves* for a particular quaternion algebra over the totally real field $F = \mathbb{Q}(\zeta_7)^+$ (see §4.2). However, the curves of genus 17 are uniformized by a *non-congruence* normal subgroup of $\Delta(2,3,7)$ and hence do not arise as Shimura curves. In this chapter, we also study other genus 17 curves with large automorphism groups. In particular, we show that a genus 17 curve related to the Fermat quartic, uniformized by $\Delta(2,3,8)$ rather than $\Delta(2,3,7)$, has infinitely many primes of supersingular reduction (see

§4.7.6).

We say an abelian variety A/k over a field of characteristic p is *supersingular* if $A_{\bar{k}}$ is isogenous to E^g where E is a supersingular elliptic curve over $\bar{k}$ [Oor74]. Over a finite field $k = \mathbb{F}_q$, this is equivalent to requiring that the eigenvalues of Frob_q on $H^1_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell)$ are all of the form $\zeta \cdot q^{1/2}$ for ζ a root of unity [Tat66]. We say a smooth curve C/k is *supersingular* if $\mathrm{Jac}(C)$ is supersingular. It is unknown for which pairs (p, g) there is a supersingular curve of genus g in characteristic p.

Given a curve C/K over a number field, a natural question concerns the set of primes of K for which the reduction of C is supersingular. This set is often sparse and, in large genus, expected to be finite. Elkies [Elk89] proved that if E/K is an elliptic curve and K has a real place then E has infinitely many primes of supersingular reduction. Our arguments reduce the supersingularity problem for these curves to elliptic curves arising as isogeny factors of Jacobians or Prym varieties.

Theorem 4.1.1. *Let C/K be a Hurwitz curve of genus $g = 3, 7, 14$. Then there are infinitely many primes of K at which C has supersingular reduction.*

It seems interesting and difficult to determine if Hurwitz curves of larger genera can have infinitely many primes of supersingular reduction. In genus 17, the isogeny structure of the Jacobian, which decomposes as $E^{14} \times E^3_{\mathbb{Q}(\sqrt{-7})}$ where $E_{\mathbb{Q}(\sqrt{-7})}$ is CM, suggests the answer is yes.

Conjecture 4.1.2. *The two Hurwitz curves of genus* 17 *have infinitely many primes of supersingular reduction.*

Although we are unable to answer this question, we show that a similar genus 17 curve we call $X_{17,\mathrm{Fermat}}$ which is uniformized by $\Delta(2, 3, 8)$ does have infinitely many primes of supersingular reduction.

Theorem 4.1.3. *The curve $X_{17,\mathrm{Fermat}}$ is defined over $\mathbb{Q}$ and has infinitely many primes of supersingular reduction.*

However, for higher genera, the infinitude of supersingular primes becomes more unlikely. For example, the Hurwitz curve with automorphism group $\mathrm{PSL}_2(\mathbb{F}_{3^3})$ arising as a Shimura curve (see §4.2.3) has in its Jacobian a 3-dimensional isogeny factor [FLP16, §6]. It would be interesting to understand the supersingular primes of curves uniformized by triangle groups $\Delta(a, b, c)$ of large genus. The proofs of these results are organized by genus. Section 4.2 supplies the Shimura-curve input needed for genus 14. Sections 4.4-4.5 treat the genera 3 and 7 Hurwitz curves by explicit equations, while the following section on the first Hurwitz triplet handles genus 14. Section 4.6 develops Macbeath's construction for the genus 17 curves, and the later Prym computations culminate in the auxiliary Fermat congruence argument of §4.7.6. One motivation for this study is the author's construction of counterexamples to Shioda's conjecture on unirationality. To construct further examples, one needs a source of high genus curves with large automorphism groups admitting many primes of supersingular reduction. The Hurwitz curves therefore provide natural candidates. Theorem 4.1.1 for $g = 14$ was used in the previous chapter to conclude the constructed surfaces had supersingular reduction infinitely often. Similarly, Theorem 4.1.3 can be used to produce another infinite list of primes at which Shioda's conjecture is known to fail (see §4.7.7).

The lists of characteristics at which we have so far constructed a counterexample to Shioda's conjecture are ultimately just the lists of supersingular primes of certain elliptic curves. The number of such primes up to x is expected to satisfy the Lang–Trotter estimates $\sim x^{1/2-o(1)}$ [LT76].

4.2 Shimura curves

We survey Shimura's groundbreaking work [Shi67] constructing canonical models of Shimura curves with an eye toward the arithmetic of certain Hurwitz curves. The main purpose of this section is to prove that the elliptic curve E appearing as an isogeny factor in the Jacobian of a genus 14 Hurwitz curve C is defined over a number field with a real place; see Theorem 4.2.5. This is necessary to apply Elkies' theorem [Elk89] to show that C has infinitely many primes of supersingular reduction.

4.2.1 Canonical models of Shimura curves

The exposition follows §2–3 of [Shi67]. Let F be a totally real number field and B a quaternion algebra over F. Write $g = [F : \mathbb{Q}]$ and $\mathfrak{D}(B/F)$ for the product of the non-archimedean primes of F at which B is *non-split* (or *ramified*), meaning $B \otimes_F F_\mathfrak{p}$ is a division algebra over $F_\mathfrak{p}$. At all other finite places B is *split* meaning $B \otimes_F F_\mathfrak{p} \cong M_2(F_\mathfrak{p})$. Let $\nu_1, \ldots, \nu_g : F \hookrightarrow \mathbb{R}$ be the archimedean places ordered such that B is split at $\nu_1, \ldots, \nu_r$ and non-split at $\nu_{r+1}, \ldots, \nu_g$. We also choose an isomorphism compatible with these fixed places

$$B \otimes_\mathbb{Q} \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R})^r \times \mathbb{H}^{g-r}$$

where $\mathbb{H}$ are the Hamiltonian quaternions. Assuming $r > 0$, this defines an embedding

$$\iota_\infty : B \hookrightarrow M_2(\mathbb{R})^r.$$

Let B^+ denote the elements of B with totally positive reduced norm (note that Shimura writes $N_{B/F} : B \to F$ for the reduced norm whereas often this denotes the full norm $(\mathrm{Nm}_{B/F})^2$ where $\mathrm{Nm}_{B/F}$, often written as $\mathrm{nrd}_{B/F}$, is the reduced norm). Equivalently, since the definite quaternion algebra $\mathbb{H}$ has only elements of positive norm, these are the elements $\gamma \in B$ so that $\iota_\infty(\gamma)$ is a sequence of real matrices with positive determinant.

Via the embedding $\iota_\infty : B^+ \to \mathrm{GL}_2^+(\mathbb{R})^r$ we get an action $B^+ \circlearrowright \mathfrak{H}^r$ where $\mathfrak{H} \subset \mathbb{C}$ is the upper half-plane. Quotients by special subgroups of B^+ will lead to Shimura varieties. The associated Shimura datum is for the algebraic group $G := \mathrm{Res}_{F/\mathbb{Q}} B^\times$ which represents the functor $A \mapsto (B \times_{\mathbb{Q}} A)^\times$ on $\mathbb{Q}$-algebras, with the conjugacy class X generated by the map $h : \mathbb{S} \to G_{\mathbb{R}}$ defined by

$$\mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})^r \times \mathbb{H}^{g-r} \qquad x + iy \mapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \ldots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, 1, \ldots, 1 \right).$$

In general, the Shimura datum (G, X) is of abelian type but not necessarily of Hodge type [Mil05a]. Fix a maximal order $\mathcal{Q} \subset B$ and set $\Gamma(\mathcal{Q}, 1) := \mathcal{Q}^\times \cap B^+$. The action $\Gamma(\mathcal{Q}, 1) \circlearrowright \mathfrak{H}^r$ is totally discontinuous and cocompact as long as B is not split over F. For a finite index subgroup $\Gamma \subset \Gamma(\mathcal{Q}, 1)$ the quotients $Y = \mathfrak{H}^r/\Gamma$ are the complex points of the associated Shimura variety. For example:

1. If $B = M_2(F)$ then $\mathcal{Q} = M_2(\mathcal{O}_F)$ and Y is a Hilbert–Blumenthal modular variety which parametrizes abelian g-folds with real multiplication by $\mathcal{O}_F$. See chapter 6.

2. If $F = \mathbb{Q}$ and B is a non-split indefinite quaternion algebra with discriminant d, then Y is a Shimura curve parameterizing abelian surfaces with QM by B.

3. If B is split at every real place, so $r = g$, then Y is one of the higher-dimensional quaternionic Shimura varieties that admit honest moduli interpretations.

4. The case relevant for this chapter is $r = 1$. These one-dimensional quotients Y are the *Shimura curves*; when $g > 1$ they are no longer honest moduli spaces of abelian varieties. However, Shimura's theory still provides canonical models over number fields.

These moduli interpretations make sense over a number field, leading to *canonical models* for the modular Shimura varieties over this field. As in Shimura's original article, we focus here on the case $r = 1$. Even when $g > 1$ and the Shimura curve is not an honest moduli space, Shimura was able to construct canonical models over number fields by using isogenies to other moduli spaces. For an ideal $\mathfrak{a} \subset \mathcal{O}_F$, Shimura defines *congruence subgroups*

$$\Gamma(\mathcal{Q}, \mathfrak{a}) := \{\gamma \in \mathcal{Q}^\times \cap B^+ \mid \gamma - 1 \in \mathfrak{a}\mathcal{Q}\}$$

and he produces a canonical model for $\mathfrak{H}/\Gamma(\mathcal{Q}, \mathfrak{a})$ as follows.

Theorem 4.2.1. *[Shi67, (3.2) Main Theorem I and (3.3)] Let $F^{(\mathfrak{a}\infty)}$ be the narrow ray class field of $\mathfrak{a}$. Fix an embedding $F^{(\mathfrak{a}\infty)} \hookrightarrow \mathbb{C}$ compatible with $v_1 : F \hookrightarrow \mathbb{R}$. Then there exists a curve C defined over $F^{(\mathfrak{a}\infty)}$ and a holomorphic map $\varphi : \mathfrak{H} \to C_\mathbb{C}$ such that*

1. *φ descends to a biholomorphism $\mathfrak{H}/\Gamma(\mathcal{Q}, \mathfrak{a}) \xrightarrow{\sim} C_\mathbb{C}$*

2. *For any K/F totally imaginary quadratic extension so that $B \otimes_F K \cong M_2(K)$ and embedding $K \hookrightarrow \mathbb{C}$ compatible with $v_1 : F \hookrightarrow \mathbb{R}$ and any F-linear embedding $f : K \hookrightarrow B$ so that $f(\mathcal{O}_K) \subset \mathcal{Q}$, let z be the fixed point of $f(K) \mathrel{\subset\!\!\!\cdot} \mathfrak{H}$; then $K^{(\mathfrak{a}'\infty)}$ equals the compositum $K \cdot F^{(\mathfrak{a}\infty)}(\varphi(z)) = K^{(\mathfrak{a}'\infty)}$ where $\mathfrak{a}' := \mathfrak{a}\mathcal{O}_K$ and $F^{(\mathfrak{a}\infty)}(\varphi(z))$ is the field of definition of the point $\varphi(z) \in C(\overline{\mathbb{Q}})$.*

Furthermore, let (C, φ) and (C', φ') be two such pairs. Then there exists an isomorphism $j : C \to C'$ defined over $F^{(\mathfrak{a}\infty)}$ such that $\varphi' = j_\mathbb{C} \circ \varphi$.

Due to the uniqueness property, we call such a pair (C, φ) a *canonical model* and denote C by $\mathcal{X}(\mathcal{Q}, \mathfrak{a})$. This uniqueness property has the following consequence for automorphisms of $\mathcal{X}(\mathcal{Q}, \mathfrak{a})$. Note that $\Gamma(\mathcal{Q}, \mathfrak{a}) \triangleleft \Gamma(\mathcal{Q}, 1)$ so there is a well-defined action $\Gamma(\mathcal{Q}, 1) \mathrel{\subset\!\!\!\cdot} \mathfrak{H}/\Gamma(\mathcal{Q}, \mathfrak{a})$ which induces an action on $\mathcal{X}(\mathcal{Q}, \mathfrak{a})_\mathbb{C}$ via φ.

Lemma 4.2.2. $\Gamma(\mathcal{Q}, 1)$ *acts on* $\mathcal{X}(\mathcal{Q}, \mathfrak{a})$ *via automorphisms defined over* $F^{(\mathfrak{a}\infty)}$.

Proof. For any $\alpha \in \mathcal{Q}^\times \cap B^+$, we can conjugate $f : K \to B$ to get $f^\alpha : K \to B$ with image $\alpha f(K)\alpha^{-1}$ and $f^\alpha(\mathcal{O}_K) = \alpha f(\mathcal{O}_K)\alpha^{-1} \subset \mathcal{Q}$ since $\alpha \in \mathcal{Q}^\times$ and $f(\mathcal{O}_K) \subset \mathcal{Q}$. Thus, if $z \in \mathfrak{H}$ is the fixed point of $f(K)$ then $\alpha \cdot z$ is fixed by $f^\alpha(K)$, so the hypothesis says that $K \cdot F^{(\mathfrak{a}\infty)}(\varphi(\alpha \cdot z)) = K^{(\mathfrak{a}'\infty)}$. Let $\varphi' := \varphi \circ \alpha$ viewing α as its action on $\mathfrak{H}$. Then the pair (C, φ') is a canonical model because $\varphi'(z) = \varphi(\alpha \cdot z)$ also satisfies $K \cdot F^{(\mathfrak{a}\infty)}(\varphi(z)) = K^{(\mathfrak{a}'\infty)}$ for any choice of K/F. Therefore, there exists $j : C \to C$ defined over $F^{(\mathfrak{a}\infty)}$ so that $\varphi' = j_\mathbb{C} \circ \varphi$, meaning exactly that the automorphism of $\mathcal{X}(\mathcal{Q}, \mathfrak{a})$ induced by α is defined over $F^{(\mathfrak{a}\infty)}$. ∎

4.2.2 Congruence subgroups

At split places of B we can also define other congruence subgroups in analogy with the modular curves $X_0(p)$ and $X_1(p)$. Then we can fix an isomorphism:

$$\iota_{\mathfrak{p}} : B \otimes_F F_{\mathfrak{p}} \xrightarrow{\sim} M_2(F_{\mathfrak{p}})$$

with the property that it restricts to an isomorphism $\iota_{\mathfrak{p}} : \mathcal{Q} \otimes_{\mathcal{O}_F} \mathcal{O}_{F_{\mathfrak{p}}} \xrightarrow{\sim} M_2(\mathcal{O}_{\mathfrak{p}})$ (see [Voi21, Lemma 10.4.3]). Hence, reduction mod $\mathfrak{p}$ gives an isomorphism

$$\iota_{\mathfrak{p}} : \mathcal{Q}/\mathfrak{p}\mathcal{Q} \xrightarrow{\sim} M_2(\kappa_{\mathfrak{p}})$$

where $\kappa_{\mathfrak{p}} = \mathcal{O}_F/\mathfrak{p}$ is the residue field. This defines a reduction map

$$\iota_{\mathfrak{p}} : \Gamma(\mathcal{Q}, 1) \to \mathrm{GL}_2(\kappa_{\mathfrak{p}})$$

whose kernel is exactly $\Gamma(\mathcal{Q}, \mathfrak{p})$. More generally, if $\mathfrak{a} \subset \mathcal{O}_F$ is any ideal whose prime factorization consists entirely of primes splitting B, then we can fix a reduction map

$$\iota_{\mathfrak{a}} : \Gamma(\mathcal{Q}, 1) \to \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{a}).$$

Taking the preimage of the upper Borel (resp. its unipotent radical, the upper triangular matrices) gives congruence subgroups $\Gamma_0(\mathcal{Q}, \mathfrak{a})$ (resp. $\Gamma_1(\mathcal{Q}, \mathfrak{a})$). Beware: this is not consistent with Shimura's use of the notation Γ_1. The groups $\Gamma_0(\mathcal{Q}, \mathfrak{a})$ are the totally positive units in an *Eichler order* $\mathcal{Q}_0(\mathfrak{a}) \subset \mathcal{Q}$ of level $\mathfrak{a}$ cf. [DV13, §3]. Vitally, using Eichler approximation (see Lemma 4.3.3), the map

$$\iota_{\mathfrak{a}} : \Gamma^1(\mathcal{Q}, 1) \to \mathrm{SL}_2(\mathcal{O}_F/\mathfrak{a})$$

is surjective for any nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$.

However, there are two caveats with these definitions: first $\iota_{\mathfrak{a}} : \Gamma(\mathcal{Q}, 1) \to \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{a})$ may not be surjective even when $\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{a})^\times$ is surjective because of the restriction that elements of the source live in B^+. Second, ι_∞ does not give a *faithful* action of $\Gamma(\mathcal{Q}, 1)$ on $\mathfrak{H}$; the action factors through $\mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{PGL}_2^+(\mathbb{R})$. In order to understand the relationship between quotients by the various congruence subgroups, we need to precisely understand these failings.

For any $\Gamma \subset \Gamma(\mathcal{Q}, 1)$ denote by $\mathrm{P}\Gamma$ the quotient

$$\mathrm{P}\Gamma := \frac{\Gamma}{\mathcal{O}_F^\times \cap \Gamma} = \frac{\mathcal{O}_F^\times \cdot \Gamma}{\mathcal{O}_F^\times}$$

where $\mathcal{O}_F^\times \hookrightarrow \Gamma(\mathcal{Q}, 1)$ are the scalar matrices. Let

$$\Gamma^1(\mathcal{Q}, \mathfrak{a}) := \Gamma(\mathcal{Q}, \mathfrak{a})^{\mathrm{Nm}=1} = \{\gamma \in \mathcal{Q}^{\mathrm{Nm}=1} \mid \gamma - 1 \in \mathfrak{a}\mathcal{Q}\}$$

be the subgroup consisting of elements of $\Gamma(\mathcal{Q}, \mathfrak{a})$ with reduced norm 1. This is the kernel of $\iota_{\mathfrak{a}}$ restricted to $\Gamma^1(\mathcal{Q}, 1) := \mathcal{Q}^{\mathrm{Nm}=1}$. We get a diagram of exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, hook] & \Gamma^1(\mathcal{Q}, \mathfrak{a}) \arrow[r] \arrow[d, hook] & \mathrm{P}\Gamma(\mathcal{Q}, \mathfrak{a}) \arrow[r, "\mathrm{Nm}"] \arrow[d, hook] & U_{\mathfrak{a}}^+/U_{\mathfrak{a}}^2 \arrow[d, hook] & \\
0 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d] & \Gamma^1(\mathcal{Q}, 1) \arrow[r] \arrow[d, "\iota_{\mathfrak{a}}"] & \mathrm{P}\Gamma(\mathcal{Q}, 1) \arrow[r, "\mathrm{Nm}"] \arrow[d] & (\mathcal{O}_F^\times)^+/(\mathcal{O}_F^\times)^2 \arrow[d] & \\
0 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{SL}_2(\mathcal{O}_F/\mathfrak{a}) \arrow[r] \arrow[d] & \mathrm{PGL}_2(\mathcal{O}_F/\mathfrak{a}) \arrow[r, "\det"] & \frac{(\mathcal{O}_F/\mathfrak{a})^\times}{(\mathcal{O}_F/\mathfrak{a})^{\times 2}} \arrow[r] & 0 \\
 & 0 & 0 & & &
\end{tikzcd}$$

where $U_{\mathfrak{a}} \subset \mathcal{O}_F^\times$ is the group of $u \in \mathcal{O}_F^\times$ such that $u \equiv 1 \mod \mathfrak{a}$ and L is $\{\pm 1\}$ if $2 \in \mathfrak{a}$ and trivial otherwise. Therefore, when $(\mathcal{O}_F^\times)^+ = (\mathcal{O}_F^\times)^2$, for example, when the narrow class number of F equals its class number, then the diagram shows that:

$$\Gamma^1(\mathcal{Q}, \mathfrak{a})/L = \mathrm{P}\Gamma(\mathcal{Q}, \mathfrak{a}) \text{ and } \Gamma^1(\mathcal{Q}, 1)/\{\pm 1\} = \mathrm{P}\Gamma(\mathcal{Q}, 1) \text{ and } \operatorname{im} \iota_{\mathfrak{a}} = \mathrm{PSL}_2(\mathcal{O}_F/\mathfrak{a}).$$

Therefore, in this case, $\mathcal{X}(\mathcal{Q}, \mathfrak{a}) \to \mathcal{X}(\mathcal{Q}, 1)$ is a $\mathrm{PSL}_2(\mathcal{O}_F/\mathfrak{a})$-cover. It has intermediate covers

$$\mathcal{X}(\mathcal{Q}, \mathfrak{a}) \to \mathcal{X}_1(\mathcal{Q}, \mathfrak{a}) \to \mathcal{X}_0(\mathcal{Q}, \mathfrak{a}) \to \mathcal{X}(\mathcal{Q}, 1)$$

which are quotients of $\mathcal{X}(\mathcal{Q}, \mathfrak{a})$ by the Borel and its unipotent radical respectively.

Let $H \subset \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ be an open subgroup for a prime $\mathfrak{p}$ at which B is split. The *level* of H is the smallest ℓ such that H is the preimage of a subgroup $\overline{H} \subset \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}^\ell)$. Then we can define the Shimura curve $\mathcal{X}_H(\mathcal{Q})$ with H-level structure as the quotient of $\mathcal{X}(\mathcal{Q}, \mathfrak{p}^\ell)$ by the subgroup $\Gamma_H(\mathcal{Q}) := \Gamma(\mathcal{Q}, 1) \cap H = \iota_{\mathfrak{p}^\ell}^{-1}(\overline{H})$. By Lemma 4.2.2, $\mathcal{X}_H(\mathcal{Q})$ also has a canonical model over $F^{(\mathfrak{a}\infty)}$.

4.2.3 Certain Hurwitz curves as Shimura curves

Much of the favorable arithmetic of the genus 14 Hurwitz curves is due to their realization as Shimura curves for a particular quaternion algebra B over the totally real field $K_+ = \mathbb{Q}(\eta)$ where $\eta = \zeta_7 + \zeta_7^{-1} = 2\cos\frac{2\pi}{7}$. The exposition here closely follows §3.19 of [Shi67] and Elkies' article [Elk99, §4.4]. Choose as B the following quaternion algebra:

$$B = \left(\frac{\eta, \eta}{K_+}\right) := \frac{K_+\langle i, j\rangle}{(i^2 - \eta, j^2 - \eta, ij + ji)}.$$

This is the unique quaternion algebra over K_+ that is non-split exactly over the two real places of K_+ where η embeds negatively and split over all non-archimedean places. Let $v_\infty : K_+ \hookrightarrow \mathbb{R}$ denote the obvious real embedding sending $\eta \mapsto 2\cos\frac{2\pi}{7}$ which splits B. We can thus fix an embedding $\iota_\infty : B \hookrightarrow M_2(\mathbb{R})$ compatible with v_∞ arising from a choice of isomorphism $B \otimes_{K_+, v_\infty} \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R})$. We apply Shimura's construction from the previous section to this quaternion algebra. Define the following maximal order:

$$\mathcal{Q}_{\mathrm{Hur}} = \mathcal{O}_{K_+}[i, j, j'] \subset B$$

where $j' := \frac{1}{2}(1 + \eta i + \tau j)$ and $\tau := 1 + \eta + \eta^2$. Since K_+ has narrow class number 1, the previous section shows $\Gamma(\mathcal{Q}, 1) \twoheadrightarrow \mathrm{PSL}_2(\mathcal{O}_{K_+}/\mathfrak{a}) \subset \mathrm{PGL}_2(\mathcal{O}_{K_+}/\mathfrak{a})$.

Miraculously,

$$\mathrm{P}\Gamma(\mathcal{Q}_{\mathrm{Hur}}, 1) = (\mathcal{Q}_{\mathrm{Hur}})^{\mathrm{Nm}=1}/\{\pm 1\} \cong \Delta(2,3,7) = \left\langle g_1, g_2, g_3 \mid g_1^2, g_2^3, g_3^7, g_1 g_2 g_3 \right\rangle$$

recovers the Hurwitz triangle group. Therefore, any finite-index normal subgroup $\Gamma \subset \Gamma(\mathcal{Q}_{\mathrm{Hur}}, 1)$ without elliptic elements produces a cover $\mathcal{X}_\Gamma \to \mathcal{X}(\mathcal{Q}_{\mathrm{Hur}}, 1) \cong \mathbb{P}^1$ ramified at exactly three points with orders $2, 3, 7$ and hence a Hurwitz curve with automorphism group $\mathrm{P}\Gamma(\mathcal{Q}_{\mathrm{Hur}}, 1)/\mathrm{P}\Gamma$. Conversely, any Hurwitz curve uniformized by $\mathfrak{H}$ with deck transformation group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ (i.e., Γ is a Fuchsian subgroup so that $\mathfrak{H}/\Gamma$ has $84(g-1)$ automorphisms) is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to a normal subgroup of $\Gamma(\mathcal{Q}_{\mathrm{Hur}}, 1)$ modulo the center.

Because B is non-split only at ∞ we know that $\mathcal{Q}_{\mathrm{Hur}}/\mathfrak{p}\mathcal{Q}_{\mathrm{Hur}} \cong M_2(\kappa_\mathfrak{p})$ for all primes $\mathfrak{p} \subset \mathcal{O}_{K_+}$. Therefore, the Shimura curves $\mathcal{X}(\mathcal{Q}_{\mathrm{Hur}}, \mathfrak{p})$ are Hurwitz curves with automorphism groups $\mathrm{PSL}_2(\kappa_\mathfrak{p})$. By Theorem 4.2.1, this curve is defined over the ray class field $K_+^{(\mathfrak{p}\infty)}$. For any prime $\mathfrak{p}_{13}$ lying over 13 this field $(K_+)^{(\mathfrak{p}_{13}\infty)}$ is isomorphic to the field K described in the main text ([LMF25p]). In fact, when p is inert ($p \equiv \pm 2, \pm 3 \mod 7$) or ramified ($p = 7$) in K_+, the curves $\mathcal{X}^{\mathrm{Hur}}(\mathfrak{p})$ for the unique prime $\mathfrak{p}$ over p are defined over $\mathbb{Q}$ and when p splits ($p \equiv \pm 1 \mod 7$) in K_+ the three curves $\mathcal{X}^{\mathrm{Hur}}(\mathfrak{p})$ for $\mathfrak{p} \mid p$ are defined over

$N(\mathfrak{p})$	$\mathfrak{p}$	m	g	G
7	$\mathfrak{p}_7$	1	3	$\mathrm{PSL}_2(\mathbb{F}_7)$
8	(2)	1	7	$\mathrm{SL}_2(\mathbb{F}_8)$
13	$\mathfrak{p}_{13}$	3	14	$\mathrm{PSL}_2(\mathbb{F}_{13})$
27	(3)	1	118	$\mathrm{PSL}_2(\mathbb{F}_{3^3})$
29	$\mathfrak{p}_{29}$	3	146	$\mathrm{PSL}_2(\mathbb{F}_{29})$

Table 4.2: Prime ideals of $\mathcal{O}_{K_+}$ ordered by norm. Here m is the number of Galois conjugate prime ideals which all give distinct Hurwitz curves (see [Shi67, 3.19.2]) each having genus g and automorphism group G.

K_+ and arise from the three embeddings $K_+ \hookrightarrow \mathbb{C}$ by [CV19, Theorem B] (cf. [Voi05, Proposition 5.1.2]). From now on, we make the abbreviation $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{a}) := \mathscr{X}(\mathcal{O}_{\mathrm{Hur}}, \mathfrak{a})$.

Theorem 4.2.3. *All three Hurwitz curves of genus* 14 *occur as* $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})$ *for the three primes* $\mathfrak{p}_{13}$ *lying over* $p = 13$. *In particular, these three curves are Galois conjugate.*

Proof. As tabulated in [LMF25f], there are exactly three isomorphism classes of curves over $\mathbb{C}$ of genus 14 with automorphism group $\mathrm{PSL}_2(\mathbb{F}_{13})$ and no other Hurwitz groups with order 1092 exist. Furthermore, by [Shi67, 3.19.2], no two $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})$ are isomorphic over $\mathbb{C}$ (base changed along the fixed embedding $K_+ \hookrightarrow \mathbb{R}$) for different choices of the prime $\mathfrak{p}_{13}$ over p. Therefore, all three isomorphism classes of Hurwitz curves over $\mathbb{C}$ appear. □

Remark 4.2.4. Transitivity of the Galois action also follows from [CV19, Theorem B (b)] where it is shown that K_+ is the field of moduli for $\mathrm{PSL}_2(\mathbb{F}_{13})$-Galois $(2,3,7)$ Belyĭ maps forgetting the isomorphism $\mathrm{PSL}_2(\mathbb{F}_{13}) \xrightarrow{\sim} \mathrm{Aut}(C)$. Note: given a Hurwitz curve, its $(2,3,7)$ Belyĭ map is unique and canonically determined as the group scheme quotient $C \to C/\mathrm{Aut}(C)$.

Theorem 4.2.5. *Let* $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})$ *be a genus* 14 *Hurwitz curve. Then there is an elliptic curve* E *defined over* $K = (K_+)^{(\mathfrak{p}_{13}\infty)}$ *such that* $\mathrm{Jac}(\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13}))$ *is isogenous to* E^{14} *over* K.

Proof. Let $C = \mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})$. Computing fixed points and comparing to genera of curves in the quotient lattice, we determine $H^1(C(\mathbb{C}), \mathbb{Q})$ as a G-representation to be $V_{14} \oplus V_{14}$ where V_{14} is the 14-dimensional irreducible $\mathbb{Q}$-representation of G corresponding to the character [LMF25q]. Moreover, the action of $\mathbb{Q}[G]$ on V_{14} gives a surjection $\mathbb{Q}[G] \twoheadrightarrow M_{14}(\mathbb{Q})$ through which $\mathbb{Q}[G] \subset H^1(C(\mathbb{C}), \mathbb{Q})$ factors. Since $G \subset C$ is defined over K, the group algebra $\mathbb{Q}[G]$ acts by K-isogenies on $\mathrm{Jac}(C)$ so the idempotents in $M_{14}(\mathbb{Q})$ give an isogeny decomposition

$$\mathrm{Jac}(C) \sim E^{14}$$

defined over K. □

Corollary 4.2.6. $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})$ *has supersingular reduction at infinitely many primes of* K_+.

Proof. Since K has a real place, E has infinitely many primes of supersingular reduction by [Elk89]. Therefore, for infinitely many primes $\mathfrak{p} \subset \mathscr{O}_{K_+}$ there is a prime $\mathfrak{P} \subset \mathscr{O}_K$ over $\mathfrak{p}$ such that $E_{\mathfrak{P}}$ is supersingular. Therefore, whenever C and E have good reduction, we conclude that $C_{\mathfrak{P}}$ is supersingular since supersingularity is a property of the Tate module of $\mathrm{Jac}(C)_{\mathfrak{P}}$ which is isomorphic to the Tate module of $E_{\mathfrak{P}}^{14}$. However, supersingularity is a geometric property, so $C_{\mathfrak{p}}$ is supersingular as well. □

4.3 Arithmetic of the first Hurwitz triplet

The goal of this section is to explicitly determine the isogeny factors of the Jacobian of a genus 14 Hurwitz curve. To do this, we need to use computations of quaternionic modular forms (or Hilbert modular forms via the Jacquet–Langlands correspondence) that are recorded in the LMFDB. Here we give a brief overview of the theory of quaternionic modular forms in the general setting of a totally real field F and a quaternion algebra B over F and their relationship to the étale cohomology of Shimura curves in the case $r = 1$. We will then specialize to the case of the Hurwitz quaternion algebra B over K_+ and finally inspect our specific level $\mathfrak{p}_{13}$ congruence subgroup $\Gamma(\mathscr{Q}_{\mathrm{Hur}}, \mathfrak{p}_{13})$.

4.3.1 Quaternionic modular forms

In this section, we discuss modular forms invariant under action of congruence subgroups for a quaternion order $\mathscr{Q}$. The relevance for us is that the Eichler–Shimura relation computes Frobenius traces on étale cohomology in terms of Hecke eigenvalues.

Definition 4.3.1. For a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$$

and $\tau \in \mathfrak{H}$ define the *factor of automorphy*

$$j(\gamma, \tau) := (c\tau + d) \cdot (\det \gamma)^{-1/2} \in \mathbb{C}.$$

For a *weight* vector $k = (k_1, \ldots, k_r) \in (\mathbb{Z}^+)^r$, we define a right *weight-k action* of $\Gamma(\mathscr{Q}, 1)$

on functions $f : \mathfrak{H}^r \to \mathbb{C}$ via

$$(f|_k\gamma)(\tau) := f(\gamma \cdot \tau) \prod_{i=1}^{r} j(\nu_i(\gamma), \tau_i)^{-k_i}.$$

For $\Gamma \subset \Gamma(\mathcal{Q}, 1)$ the space $M_k^B(\Gamma)$ of *weight-k modular forms at level* Γ is the $\mathbb{C}$-vector space of holomorphic functions $f : \mathfrak{H}^r \to \mathbb{C}$ invariant under the weight-k action of Γ and bounded at the cusps. The space $S_k^B(\Gamma)$ of *weight-k cusp forms at level* Γ is the subspace of functions vanishing at the cusps of $\mathfrak{H}^r/\Gamma$.

In the case $B = M_2(F)$, $\mathcal{Q} = M_2(\mathcal{O}_F)$, and $r = g$, we recover the spaces of Hilbert modular forms for the totally real field F. In that example, we drop the quaternion algebra "B" from the notation. We focus on forms of level $\Gamma_0(\mathcal{Q}, \mathfrak{a})$ because these spaces have been computed in [LMF25a] for ideals $\mathfrak{a}$ of small norm. For fields of general narrow class number, the natural Hecke action, defined in terms of the adelic description of shimura varieties, may permute the connected components indexed by $\mathcal{C}\ell^+(F)$, so one cannot view every $T_\mathfrak{p}$ as acting on the cohomology of a single quotient $\mathfrak{H}^r/\Gamma_0(\mathcal{Q}, \mathfrak{a})$. In our application, will not need this generality since the Hurwitz field K_+ has narrow class number 1. From now on in this discussion, assume F has narrow class number 1 so there are well-defined Hecke operators on the classical quaternionic modular forms (see the parallel discussion of classical vs adelic quarternionic modular forms in §3 and §7 of [DV13]). Let $\mathfrak{D} = \mathfrak{D}(B/F)$ be the discriminant of B. Then we write $S_k^B(\mathfrak{a}) := S_k^B(\Gamma_0(\mathcal{Q}, \mathfrak{a}))$ and similarly for $M_k^B(\mathfrak{a})$. The good Hecke operators $T_\mathfrak{p}$, for primes $\mathfrak{p} \nmid \mathfrak{D}\mathfrak{a}$, act on this space by the usual double-coset correspondences. In the case when B is non-split over F it turns out there are no cusps (the corresponding subgroup is cocompact) so the condition at cusps is vacuous. In this case we say $S_k^B(\mathfrak{a}) = M_k^B(\mathfrak{a})$ is the space of *quaternionic cusp forms*.

The case of parallel weight 2, meaning $k = (2, \dots, 2)$, is particularly important because such cusp forms define holomorphic symmetric r-forms on the Shimura variety $\mathfrak{H}^r/\Gamma$ (really the associated analytic quotient stack). In the case $r = 1$, this induces the *Eichler–Shimura isomorphism*

$$S_2^B(\mathfrak{a}) \oplus \overline{S_2^B(\mathfrak{a})} \xrightarrow{\sim} H^1(\mathcal{X}_0(\mathfrak{a}), \mathbb{C}).$$

Let $\mathbb{T}_0^B(\mathfrak{a}) := \mathbb{Z}[T_\mathfrak{p} : \mathfrak{p} \nmid \mathfrak{D}\mathfrak{a}] \subset \mathrm{End}_\mathbb{C}(S_2^B(\mathfrak{a}))$ denote the faithful image of the good Hecke algebra; when no confusion is possible we write $\mathbb{T}_0(\mathfrak{a})$. Then $H^1(\mathcal{X}_0(\mathfrak{a}), \mathbb{C})$ becomes a free $\mathbb{T}_0(\mathfrak{a})_\mathbb{C}$-module of rank 2. A Galois orbit of *eigenforms* $f \in S_2^B(\mathfrak{a})$ for the Hecke operators is equivalent to an evaluation map $\lambda_f : \mathbb{T}_0^B(\mathfrak{a}) \to K_f$ sending $T_\mathfrak{p} \mapsto a_\mathfrak{p}(f)$ where $a_\mathfrak{p}(f)$ is the eigenvalue of $T_\mathfrak{p} f = a_\mathfrak{p}(f) f$ and K_f is the trace field generated by the $a_\mathfrak{p}(f)$. The kernel defines a prime $\mathfrak{p}_f = \ker \lambda_f$ of the Hecke algebra. Since $\mathbb{T}_0(\mathfrak{a})$ acts by correspondences, it acts on $\mathrm{Jac}(\mathscr{X}_0(\mathfrak{a}))$ by isogenies so there is a quotient A_f by $\mathfrak{p}_f$ well-defined up to isogeny. Here A_f is an abelian variety of dimension $[K_f : \mathbb{Q}]$ defined over F [Zha01]. By above, its Tate module $V_\ell(A_f)$ is a $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} K_f$-module (whose action commutes with the Galois action) of rank 2. The *Eichler–Shimura relation* then says that for the Galois representation

$$\rho_{A_f,\ell} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{Q}_\ell \otimes_{\mathbb{Q}} K_f)$$

and any prime $\mathfrak{p} \subset \mathscr{O}_F$ with $\mathfrak{p} \nmid \ell\mathfrak{a}$, the characteristic polynomial of $\rho_{A_f,\ell}(\mathrm{Frob}_\mathfrak{p})$ is

$$T^2 - a_\mathfrak{p}(f)\,T + N(\mathfrak{p})$$

viewed as a polynomial over $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} K_f$ [Car86, §10.3].

The Jacquet–Langlands correspondence now relates the Hecke module structure for parallel weight modular forms in the quaternionic and Hilbert cases.

Theorem 4.3.2 (Eichler–Shimizu–Jacquet–Langlands correspondence)**.** *Let $\mathfrak{D} \subset \mathscr{O}_F$ be the discriminant of B and $\mathfrak{a} \subset \mathscr{O}_F$ an ideal coprime to $\mathfrak{D}$. Then there is a Hecke module embedding*

$$S_2^B(\mathfrak{a}) \hookrightarrow S_2(\mathfrak{D}\mathfrak{a})$$

whose image consists of Hilbert cusp forms which are new at all $\mathfrak{p} \mid \mathfrak{D}$.

See [Hid81, Proposition 2.12] (cf. [GV11, Theorem 2.9] and [DV13, Theorem 3.9]). In particular, when $\mathfrak{D} = (1)$, Theorem 4.3.2 yields an isomorphism $S_2^B(\mathfrak{a}) \cong S_2(\mathfrak{a})$ as Hecke modules.

4.3.2 Conjugation isomorphisms between Shimura curves

Using conjugacy of certain Fuchsian groups, we can relate different Shimura curves at various levels. This will be important for explicit computations using the Jacquet–Langlands correspondence. The exposition closely follows the case of modular curves recorded in the appendix to [RSZB22] by John Voight.

Let $H \subset \mathrm{GL}_2(\mathscr{O}_\mathfrak{p})$ be an open subgroup for a prime $\mathfrak{p}$ at which B is split. Assume H is contained in the preimage of the upper Borel of $\mathrm{GL}_2(\kappa_\mathfrak{p})$. Also assume that $\mathfrak{p}$ has a totally positive generator $\pi \in \mathfrak{p}$ (such as occurs when F has narrow class number 1). Consider the matrix

$$\varpi := \begin{pmatrix} 0 & -1 \\ \pi & 0 \end{pmatrix} \in \mathrm{GL}_2^+(\mathscr{O}_F)$$

This acts by conjugation in $\mathrm{GL}_2(F_{\mathfrak{p}})$ as follows:

$$\varpi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varpi^{-1} = \begin{pmatrix} d & -\pi^{-1}c \\ -\pi b & a \end{pmatrix}.$$

Hence, if $c \in \mathfrak{p}$ this produces a matrix in $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$. Therefore, there is a conjugation operation on the subgroups H as follows:

$$H^{\varpi} := \varpi H \varpi^{-1} = \left\{ \begin{pmatrix} d & -c \\ -\pi b & a \end{pmatrix} \middle| \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix} \in H \right\}$$

This conjugation action on subgroups induces nontrivial isomorphisms between Shimura curves at different levels. To prove this, we need an important result of Eichler.

Lemma 4.3.3 (Eichler Approximation)**.** *[Shi67, Lemma 2.15] Let $\mathcal{Q} \subset B$ be a maximal order in a quaternion algebra over F and $\mathfrak{I} \subset \mathcal{Q}$ a two-sided ideal. Let $\beta \in \mathcal{O}_F$ and $\gamma \in \mathcal{Q}$ such that β is positive at the non-split non-archimedean places of B and* $\mathrm{Nm}(\gamma) \equiv \beta \mod (\mathfrak{I} \cap \mathcal{O}_F)$. *Then there exists $\gamma' \in \mathcal{Q}$ such that $\gamma' \equiv \gamma \mod \mathfrak{I}$ and* $\mathrm{Nm}(\gamma') = \beta$.

Proposition 4.3.4. *Let $H \subset \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ be an open subgroup for a prime $\mathfrak{p}$ at which B is split contained in the preimage of the upper Borel of* $\mathrm{GL}_2(\kappa_{\mathfrak{p}})$. *Then the Shimura curves $\mathcal{X}_H(\mathcal{Q})$ and $\mathcal{X}_{H^{\varpi}}(\mathcal{Q})$ are isomorphic as curves over $\mathbb{C}$.*

Proof. We need to show that there is an automorphism of $\mathfrak{H}$ taking the action of $\Gamma_H(\mathcal{Q})$ to $\Gamma_{H^{\varpi}}(\mathcal{Q})$. A natural candidate is $\nu_1(\varpi)$ but it is not clear that this element actually does the job since conjugation by it may not preserve B. Instead, we will π-adically approximate ϖ by an element of $\mathcal{Q}$. Using the isomorphism $\iota_{\mathfrak{a}} : \mathcal{Q}/\mathfrak{p}^{\ell}\mathcal{Q} \xrightarrow{\sim} M_2(\mathcal{O}_F/\mathfrak{p}^{\ell})$ we can find an element $\varpi'_{\ell} \in \mathcal{Q}$ so that $\varpi'_{\ell} \equiv \varpi \mod \mathfrak{p}^{\ell}$. Now we use Eichler's approximation theorem to find $\varpi_{\ell} \in \mathcal{Q}$ such that $\mathrm{Nm}(\varpi_{\ell}) = \pi$ and $\varpi_{\ell} \equiv \varpi \mod \mathfrak{p}^{\ell}$. It is clear that $(-)^{\varpi_{\ell}}$ preserves B^{+}. Since π is totally positive, $\iota_{\infty}(\varpi_{\ell})$ acts on $\mathfrak{H}$. Now, if H has level $\mathfrak{p}^{\ell}$ then notice

$$H^{\varpi} = H^{\varpi_{\ell}}$$

because $\varpi_{\ell}\gamma\varpi_{\ell}^{-1} \equiv \varpi\gamma\varpi^{-1} \mod \mathfrak{p}^{\ell}$ and H contains every element which is $1 \mod \mathfrak{p}^{\ell}$. Now, B^{+} is stable under $(-)^{\varpi_{\ell}}$ but $\mathcal{Q}^{\times}$ is not, so we need to be careful in checking that $(-)^{\varpi_{\ell}}$ takes $\Gamma_H(\mathcal{Q})$ to $\Gamma_{H^{\varpi}}(\mathcal{Q})$. We do this at each local place using that

$$\mathcal{Q} = \bigcap_{\mathfrak{p}' \subset \mathcal{O}_F} (\mathcal{Q}_{\mathfrak{p}'} \cap B)$$

with the intersection taking place in B. Since $\mathrm{Nm}(\varpi_\ell) = \pi$ is a unit for $\mathfrak{p}' \neq \mathfrak{p}$, it preserves $\mathcal{Q}_{\mathfrak{p}'}$. At $\mathfrak{p}$, an element of $\Gamma_H(\mathcal{Q})$ lands inside the maximal order $M_2(\mathcal{O}_\mathfrak{p})$. Therefore $(-)^{\varpi_\ell}$ sends $\Gamma_H(\mathcal{Q})$ inside $\mathcal{Q}$. Its image must land inside $\mathcal{Q}^\times$ since it is multiplicative and $\Gamma_H(\mathcal{Q})$ is a subgroup. □

For modular curves, Deligne–Rapoport prove that this isomorphism is defined over $\mathbb{Q}$ [DR73, Propositions IV-3.16 and IV-3.19]. It is likely that the above isomorphism is defined over the reflex field $F^{(\mathfrak{p}\infty)}$ but we will not need this.

Example 4.3.5. For example, let $\mathfrak{p} \subset \mathcal{O}_F$ be a prime ideal generated by a totally positive element π at which B is split. The full congruence subgroup $\Gamma(\mathcal{Q}, \mathfrak{p})$ is associated to the subgroup $H := \ker(\mathrm{GL}_2(\mathcal{O}_\mathfrak{p}) \to \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}))$ which satisfies

$$H^\varpi = \left\{ \begin{pmatrix} d & -c \\ -\pi b & a \end{pmatrix} \middle| \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix} \in H \right\} = \left\{ \begin{pmatrix} 1+\pi d & -c \\ -\pi^2 b & 1+\pi a \end{pmatrix} \middle| a, b, c, d \in \mathcal{O}_\mathfrak{p} \right\}$$

which is the preimage of the intersection of the Borel in $\mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}^2)$ and the unipotent radical in $\mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p})$. Call this subgroup H_{ub} and $\mathscr{X}_{\mathrm{ub}}(\mathcal{Q}, \mathfrak{p}^2)$ its associated Shimura curve. Likewise, we can ask what larger subgroup H' (hence also of level $\mathfrak{p}$) corresponds to the full Borel $B(\mathfrak{p}^2) \subset \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}^2)$. This turns out to be the subgroup

$$H_{\mathrm{sp}} = \left\{ \begin{pmatrix} a & \pi b \\ \pi c & d \end{pmatrix} \middle| a, b, c, d \in \mathcal{O}_\mathfrak{p} \right\}$$

which is the preimage of the split Cartan subgroup of diagonal matrices in $\mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p})$. Indeed,

$$H_{\mathrm{sp}}^\varpi = \left\{ \begin{pmatrix} d & -c \\ -\pi^2 b & a \end{pmatrix} \middle| a, b, c, d \in \mathcal{O}_\mathfrak{p} \right\}$$

is exactly the Borel in $\mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}^2)$. Therefore, Proposition 4.3.4 produces a diagram

$$\begin{array}{ccc} \mathscr{X}_{\mathrm{ub}}(\mathcal{Q}, \mathfrak{p}^2) & \xrightarrow{\sim} & \mathscr{X}(\mathcal{Q}, \mathfrak{p}) \\ \downarrow & & \downarrow \\ \mathscr{X}_0(\mathcal{Q}, \mathfrak{p}^2) & \xrightarrow{\sim} & \mathscr{X}_{\mathrm{sp}}(\mathcal{Q}, \mathfrak{p}) \end{array}$$

4.3.3 $\mathrm{Jac}(\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13}))$

Here we compute the j-invariant of the elliptic curve E appearing in the isogeny decomposition of $\mathrm{Jac}(\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13}))$. From Figure 3.4 we know that $\mathscr{X}^{\mathrm{Hur}}_{\mathrm{sp}}(\mathfrak{p}_{13})$, the quotient of $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})$ by the maximal torus C_6, is a genus 2 curve. Since $\mathrm{Jac}(\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_{13})) \sim E^{14}$ over

K it suffices to compute $\mathrm{Jac}(\mathscr{X}_{\mathrm{sp}}^{\mathrm{Hur}}(\mathfrak{p}_{13}))$. The conjugation isomorphism of Example 4.3.5 exhibits $\mathscr{X}_{\mathrm{sp}}^{\mathrm{Hur}}(\mathfrak{p}_{13}) \xrightarrow{\sim} \mathscr{X}_{0}^{\mathrm{Hur}}(\mathfrak{p}_{13}^2)$. The Eichler–Shimizu–Jacquet–Langlands correspondence,

$$S_2^B(\mathfrak{p}_{13}^2) \cong S_2(\mathfrak{p}_{13}^2)$$

expresses the Hecke action on $H^1(\mathscr{X}_0^{\mathrm{Hur}}(\mathfrak{p}_{13}^2)_{\mathbb{C}}, \mathbb{Q})$ in terms of Hilbert cusp forms of parallel weight 2 for level $\Gamma_0(\mathfrak{p}_{13}^2)$. Fixing the prime $\mathfrak{p}_{13}$ over 13, the space $S_2(\mathfrak{p}_{13}^2)$ consists of a unique Galois orbit f of Hilbert eigenforms [LMF25k]. The Hecke trace field is $K_f = \mathbb{Q}(\sqrt{3})$, hence f corresponds to the abelian surface $A_f \cong \mathrm{Jac}(\mathscr{X}_0^{\mathrm{Hur}}(\mathfrak{p}_{13}^2))$. We expect an elliptic curve E in the isogeny decomposition over K but the Hecke eigenvalues are not rational so the eigenform cannot correspond to an elliptic curve over K_+. Indeed, K is the field of definition of E and there is an isogeny $A_f \xrightarrow{\sim} \mathrm{Res}_{K_+}^{K} E$. To find E we analyze the Galois representation $\rho_{A_f,\ell}$ attached to the Tate module $V_\ell(A_f)$. Eichler–Shimura gives

$$\operatorname{tr}\rho_{A_f,\ell}(\mathrm{Frob}_{\mathfrak{p}}) = a_{\mathfrak{p}}(f)$$

for $\mathfrak{p} \subset \mathscr{O}_{K_+}$ not dividing $\ell\mathfrak{p}_{13}$. If $V_\ell(A_f)$ splits when restricted to $\mathrm{Gal}(\overline{K}/K)$, we can compute its traces of Frobenius directly. Consider $\mathfrak{P} \subset \mathscr{O}_K$ lying over $\mathfrak{p} \subset \mathscr{O}_{K_+}$. If $\mathfrak{p}$ is split or ramified then $\mathrm{Frob}_{\mathfrak{P}}$ and $\mathrm{Frob}_{\mathfrak{p}}$ are conjugate in $\mathrm{Gal}(\overline{K_+}/K_+)$, so $\operatorname{tr}\rho_{A_f,\ell}(\mathrm{Frob}_{\mathfrak{P}}) = a_{\mathfrak{p}}(f)$. If $\mathfrak{p}$ is inert, then $\mathrm{Frob}_{\mathfrak{P}}$ is conjugate to $\mathrm{Frob}_{\mathfrak{p}}^2$ in $\mathrm{Gal}(\overline{K_+}/K_+)$ and therefore, by Cayley–Hamilton,

$$\operatorname{tr}\rho_{A_f,\ell}(\mathrm{Frob}_{\mathfrak{P}}) = (\operatorname{tr}\rho_{A_f,\ell}(\mathrm{Frob}_{\mathfrak{p}}))^2 - 2\det\rho_{A_f,\ell}(\mathrm{Frob}_{\mathfrak{p}}) = a_{\mathfrak{p}}(f)^2 - 2N(\mathfrak{p}).$$

As expected, every irrational value of $a_{\mathfrak{p}}(f)$ appears for $\mathfrak{p}$ inert in K and $a_{\mathfrak{p}}(f)^2 \in \mathbb{Z}$ meaning, in particular, the traces are the same for the two eigenforms exchanged by Galois. Hence, we get a unique list of rational traces $\operatorname{tr}\rho_{A_f,\ell}(\mathrm{Frob}_{\mathfrak{P}})$ for the primes $\mathfrak{P} \subset \mathscr{O}_K$ corresponding to the decomposition $(A_f)_K \sim E^2$ also showing that E is isogenous to its conjugate under $\mathrm{Gal}(K/K_+)$ (which also follows from the fact that A_f has an isogeny action by $K_f = \mathbb{Q}(\sqrt{3})$). Since we know E is defined over K and has conductor dividing $\mathfrak{p}_{13}^2$, since $\mathscr{X}_0^{\mathrm{Hur}}(\mathfrak{p}_{13})$ has good reduction away from $\mathfrak{p}_{13}$ by [Car86, Kis10], there are finitely many potential elliptic curves. The built-in Magma function `EllipticCurveSearch` identifies E and its conjugate from finitely many $\mathrm{Frob}_{\mathfrak{P}}$ traces (see the script `hilbert_modular_forms.m`). The minimal polynomial of $j(E) \in K$ is

$$x^6 - 14475x^5 + 66476374x^4 + 86360537888x^3 - 1766214565832855x^2 + 5634850735711464943x - 5057260487992776789167.$$

This elliptic curve has bad reduction exactly at the unique prime of K lying over $\mathfrak{p}_{13}$ which is also where K/K_+ is ramified. The first few supersingular primes of E have norms $1091, 2339, 6551, 7643, \ldots$.

4.4 Genus 3

Let us first consider which Hurwitz curves may be produced via the Shimura construction. Since 7 is ramified in $K_+ = \mathbb{Q}(\cos\frac{2\pi}{7})$, there is a unique prime $\mathfrak{p}_7$ over 7 leading to a Hurwitz curve $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_7)$ of genus 3. It turns out that, up to isomorphism over $\overline{\mathbb{Q}}$, there is a unique such Hurwitz curve X_3. This follows from an elementary search over surjections $\Delta(2,3,7) \twoheadrightarrow G$ onto a group of size $84\cdot(3-1)$ up to isomorphism. Since it is a non-hyperelliptic curve of genus 3, this Hurwitz curve arises as a plane quartic. Classically known as the Klein quartic, it has the following Klein model over $\mathbb{Q}$:

$$C_{\mathrm{Klein}}: \quad x^3y + y^3z + z^3x = 0.$$

Elkies discusses various $\mathbb{Q}$-models of this curve at length in [Elk99]. Purely from the representation theory of $\mathrm{Aut}(C_{\mathrm{Klein},\overline{\mathbb{Q}}}) \cong \mathrm{PSL}_2(\mathbb{F}_7)$ acting on $H^1(C_{\mathrm{Klein}}(\mathbb{C}),\mathbb{Q})$, we can conclude that $\mathrm{Jac}(C_{\mathrm{Klein}})_{\overline{\mathbb{Q}}} \sim E^3_{\mathbb{Q}(\sqrt{-7})}$ where $E_{\mathbb{Q}(\sqrt{-7})}$ is an elliptic curve with CM by $\mathbb{Q}(\sqrt{-7})$. However, Elkies gives a more satisfying explanation of the Jacobian of C_{Klein}. He identifies C_{Klein} with the modular curve $X(7)$ over $\mathbb{Q}$ and therefore determines the Galois structure of

$$\mathrm{Aut}(C_{\mathrm{Klein}}) = \mathrm{Aut}(X(7)) = \mathrm{Aut}_{\wedge}(\mathbb{Z}/7\mathbb{Z} \oplus \mu_7)/\{\pm\mathrm{id}\},$$

where $\mathrm{Aut}_{\wedge}$ denotes automorphisms preserving the standard μ_7-valued symplectic pairing. The quotient by the diagonal maximal torus gives a degree 3 map $X(7) \to X_0(49)$ producing a surjection $\mathrm{Jac}(X(7)) \to \mathrm{Jac}(X_0(49)) = E_{\mathbb{Q}(\sqrt{-7})}$. Over $\mathbb{Q}(\zeta_7)$ there are additional maps $X(7) \to X_0(49)$ given by quotienting the other maximal tori, giving the isogeny decomposition

$$\mathrm{Jac}(X(7))_{\mathbb{Q}(\zeta_7)} \to (E^3_{\mathbb{Q}(\sqrt{-7})})_{\mathbb{Q}(\zeta_7)}.$$

There is no $\mathbb{Q}$-model of the Klein quartic with automorphism group defined over $\mathbb{Q}$. However, there is a model with all automorphisms defined over $\mathbb{Q}(\sqrt{-7})$. Since $\mathrm{PSL}_2(\mathbb{F}_7)$ is centerless, the moduli space $\mathscr{H}_{\mathrm{PSL}_2(\mathbb{F}_7)}$ of pairs (X,φ) consisting of a curve X of genus 3 and an isomorphism $\varphi : \mathrm{PSL}_2(\mathbb{F}_7) \xrightarrow{\sim} \mathrm{Aut}(X)$ is a scheme of degree 2 over $\mathbb{Q}$ because it has two geometric points corresponding to the outer automorphisms of $\mathrm{PSL}_2(\mathbb{F}_7)$. Thus, $\mathscr{H}_{\mathrm{PSL}_2(\mathbb{F}_7)} = \mathrm{Spec}\,\mathbb{Q}(\sqrt{-7})$, so this $\mathbb{Q}(\sqrt{-7})$-model is unique. In fact, this model is defined over $\mathbb{Q}$ as the quartic curve

$$C_{\text{twist}}: \quad x^4 + y^4 + z^4 + 6(xy^3 + yz^3 + zx^3) - 3(x^2y^2 + y^2z^2 + z^2x^2) + 3xyz(x + y + z)$$

or what Elkies calls the *rational S_3 model* because it has automorphism group S_3 over $\mathbb{Q}$ [Elk99, p.59, (1.22)].

Proposition 4.4.1. *C_{twist} is the unique curve of genus 3 over $\mathbb{Q}$ such that over $\mathbb{Q}(\sqrt{-7})$ it has automorphism group $\mathrm{PSL}_2(\mathbb{F}_7)$.*

Proof. We know that any such curve is a form of C_{twist} trivialized by $\mathbb{Q}(\sqrt{-7})$. Therefore, it suffices to compute

$$H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q}), \mathrm{Aut}(C_{\text{twist}})) = 1.$$

This is done in the Magma script [Chu25, `H1_crossed_homs.m`]. □

Since $\mathrm{Jac}(X_3)$ is CM, it is immediate that it has infinitely many primes of supersingular reduction; in fact, these primes have positive density.

4.5 Genus 7

Likewise, we first look at the Shimura curves. Since (2) is inert in K_+ we obtain a single Hurwitz curve $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_2)$ with genus 7 and automorphism group $\mathrm{SL}_2(\mathbb{F}_8)$. As before, an elementary search proves this is the unique Hurwitz curve of this genus. This curve $X_7 = \mathscr{X}^{\mathrm{Hur}}(\mathfrak{p}_2)$ is often called the *Fricke–Macbeath curve*. Again, a purely representation-theoretic argument shows that $\mathrm{Jac}(X_7)_{\overline{\mathbb{Q}}} \sim E^7$ for some elliptic curve E. This elliptic curve factor is defined by an idempotent in $\mathrm{End}(\mathrm{Jac}(X_7)_{\overline{\mathbb{Q}}})$ defined over the field of definition of $\mathrm{Aut}(X_7)$. The theory of canonical models for Shimura curves then implies (see Lemma 4.2.2) that the automorphisms are defined over $(K_+)^{(2\infty)} = K_+$. Hence E is defined over K_+, which is a totally real field. Therefore, E, and hence X_7, have infinitely many primes of supersingular reduction by Elkies' theorem [Elk89]. In fact, up to a choice of isogeny, the curve E is defined over $\mathbb{Q}$ and has a model

$$y^2 = x^3 + x^2 - 114x - 127$$

according to the explicit computations of [TV18, Theorem 3]. See [Wol02] for a nice overview of some of the techniques discussed in these sections.

4.6 Genus 17

4.6.1 Macbeath's construction

In [Mac61], Macbeath shows that any Hurwitz curve C of genus g admits a canonical finite étale cover $C_m \to C$ by another Hurwitz curve C_m of genus

$$g_m := m^{2g}(g-1)+1.$$

The construction proceeds topologically as follows. Let $\pi_1(C) \to H_1(C,\mathbb{Z})$ be the Hurewicz map. A characteristic subgroup of $\pi_1(C)$ will be a normal subgroup of $\pi_1(C) \subset \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$ and hence itself define a Hurwitz curve. One manifestly characteristic subgroup is $\Gamma_m \subset \pi_1(C)$ generated by all commutators and all m-th powers. This is the same as the kernel of $\pi_1(C) \to H_1(C,\mathbb{Z}/m\mathbb{Z})$. Hence $\Gamma_m \subset \pi_1(C)$ defines a $(\mathbb{Z}/m\mathbb{Z})^{2g}$-cover of C which is also a Hurwitz curve. The Riemann–Hurwitz formula then gives

$$g_m = g(C_m) = m^{2g}(g-1)+1.$$

Applying this construction to the Klein quartic, for $m=2$ we get a Hurwitz curve C_2 of genus 129. However, $C_2 \to C$ has intermediate quotients. In general, there is an extension,

$$0 \to (\mathbb{Z}/m\mathbb{Z})^{2g} \to \mathrm{Aut}(C_m) \to \mathrm{Aut}(C) \to 0$$

and intermediate Galois quotients of $C_m \to C$ correspond to subgroups of $(\mathbb{Z}/m\mathbb{Z})^{2g}$ stable under the $\mathrm{Aut}(C)$-action. We will see that, for the Klein quartic, $(\mathbb{Z}/2\mathbb{Z})^6$ splits as a direct sum of two irreducible $\mathrm{Aut}(X_3)$-representations over $\mathbb{F}_2$. These give intermediate curves $(X_3)_2 \to X_{17,i} \to X_3$ for $i \in \{1,2\}$ which are unramified $(\mathbb{Z}/2\mathbb{Z})^3$-covers of X_3 and hence have genus 17. These produce the only two Hurwitz curves of genus 17.

4.6.2 Arithmetic aspects of Macbeath's construction

First we consider the general problem of producing geometrically irreducible étale covers in arithmetic situations (or more generally over any non-algebraically closed field) from the geometric fundamental group. Let X be a geometrically integral variety over k, fix a geometric point $\bar{x}$: Spec $\bar{k} \to X$ and consider the homotopy exact sequence

$$1 \to \pi_1(X_{\bar{k}}) \to \pi_1(X) \to G_k \to 1$$

Suppose we are given a finite index open subgroup $\Gamma \subset \pi_1(X_{\bar{k}})$ defining a finite étale cover $\overline{Y} \to X_{\bar{k}}$. The question is when it can be extended to a map of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\bar{k}}) \arrow[r] & \pi_1(X) \arrow[r] & G_k \arrow[r] & 1 \\
1 \arrow[r] & \Gamma \arrow[r] \arrow[u, hook] & \widetilde{\Gamma} \arrow[r] \arrow[u, hook] & G_k \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

in which case $\widetilde{\Gamma} \subset \pi_1(X)$ defines a finite étale cover $Y \to X$ pulling back to $\overline{Y} \to X_{\bar{k}}$.

The existence of $\widetilde{\Gamma}$ is a purely group-theoretic question which we turn to next. We phrase the existence of $\widetilde{\Gamma}$ as a series of obstructions that must be overcome successively.

1. Let $N := N_{\pi_1(X)}(\Gamma)$ be the normalizer of Γ in the arithmetic fundamental group $\pi_1(X)$. If $N \to G_k$ is not surjective then no $\widetilde{\Gamma}$ exists since $\widetilde{\Gamma}$ must live inside N. If $N \to G_k$ is not surjective, this forms the primary obstruction.

2. Assuming $N \to G_k$ is surjective, now let $K := \ker(N \twoheadrightarrow G_k)$. Next we must construct an extension $\widetilde{\Gamma} \supset \Gamma$ fitting into a diagram,

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] & N \arrow[r] & G_k \arrow[r] & 1 \\
1 \arrow[r] & \Gamma \arrow[r] \arrow[u, hook] & \widetilde{\Gamma} \arrow[r] \arrow[u, hook] & G_k \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

This reduces the problem to the case that $\Gamma \subset \pi_1(X_{\bar{k}})$ is normal. To do this, consider the quotient $Q := K/\Gamma$ and the induced diagram of exact sequences of groups,

$$\begin{tikzcd}
1 \arrow[r] & Q \arrow[r] & E \arrow[r] & G_k \arrow[r] \arrow[l, dashed, bend right] & 1 \\
1 \arrow[r] & K \arrow[r] \arrow[u, two heads] & N \arrow[r] \arrow[u, two heads] & G_k \arrow[r] \arrow[u, equal] & 1 \\
& \Gamma \arrow[u, hook] & & &
\end{tikzcd}$$

Here E is constructed as the pushout of $K \twoheadrightarrow Q$ and $K \hookrightarrow N$ (in this case, $E = N/\Gamma$ and exactness follows from the third isomorphism theorem). Then the existence of $\widetilde{\Gamma}$ is equivalent to the existence of $\Gamma_0 \subset E$ so that $\Gamma_0 \to G_k$ is an isomorphism. Indeed, then we take $\widetilde{\Gamma}$ to be the preimage of $\Gamma_0 \subset E$. This is the condition that the extension

$$1 \to Q \to E \to G_k \to 1$$

is split on the right. In general the obstruction is its class in pointed non-abelian cohomology; when Q is abelian this obstruction is the cohomology class

$$\mathrm{ob} \in H^2(G_k, Q).$$

We first make an important observation about the secondary obstruction. Let $X_N \to X$ be the finite étale cover corresponding to N. If X_N has a k-point then

$$1 \to K \to N \to G_k \to 1$$

has a section inducing a section of

$$1 \to Q \to E \to G_k \to 1$$

and hence the obstruction vanishes.

Let us specialize to the particular case when $\Gamma \subset \pi_1(X_{\bar{k}})$ is a normal subgroup with abelian quotient preserved by the outer action of G_k (e.g. Γ is a characteristic subgroup with abelian quotient). Then $N = \pi_1(X)$ and the primary obstruction is trivial. Therefore, the obstruction to extending Γ to $\widetilde{\Gamma}$ is a class

$$\mathrm{ob} \in H^2(G_k, Q).$$

For Macbeath's construction, $Q = H_{1,\text{ét}}(X_{\bar{k}}, \mathbb{Z}/m\mathbb{Z}) := \pi_1(X_{\bar{k}})^{\mathrm{ab}} \otimes \mathbb{Z}/m\mathbb{Z}$ as a G_k-module and the obstruction lives in

$$H^2(G_k, H_{1,\text{ét}}(X_{\bar{k}}, \mathbb{Z}/m\mathbb{Z})).$$

Furthermore, we know that if X has a k-point then this obstruction vanishes.

4.6.3 The mod 2 representation of the Klein quartic

Let $K := \mathbb{Q}(\sqrt{-7})$ and let

$$G := \mathrm{Aut}(C_{\mathrm{twist},\overline{\mathbb{Q}}}) \cong \mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{GL}_3(\mathbb{F}_2).$$

Recall from §4.4 that C_{twist} is the unique $\mathbb{Q}$-model of the Klein quartic for which all automorphisms are defined over K. Let M denote the fundamental 3-dimensional $\mathbb{F}_2$-representation of $\mathrm{GL}_3(\mathbb{F}_2)$ and let $M^\vee$ denote its dual. Write $\bar{\rho}_{C_{\mathrm{twist}},2} : G_{\mathbb{Q}} \to \mathrm{GL}(H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2))$ for the Galois action on the mod 2 étale cohomology.

Elkies analyzes the Klein model $C_{\mathrm{Klein}} \cong X(7)$, and the resulting decomposition is a statement about the common geometric curve $C_{\mathrm{Klein},\overline{\mathbb{Q}}} \cong C_{\mathrm{twist},\overline{\mathbb{Q}}}$. He shows that the action of G on $H^0(C_{\mathrm{Klein},\mathbb{C}}, \omega_{C_{\mathrm{Klein}}})$ is the contragredient 3-dimensional complex representation and that the associated G-stable $\mathcal{O}_K$-lattice L has the property that reduction modulo either prime above 2 yields the fundamental representation of $\mathrm{GL}_3(\mathbb{F}_2)$, with the two reductions related by the outer automorphism of G [Elk99, §§1.4, 2.3]. Since $2 = \mathfrak{p}_2\bar{\mathfrak{p}}_2$ splits in $\mathcal{O}_K$, Elkies' description of the Jacobian as $\mathrm{Jac}(C_{\mathrm{Klein}})_{\overline{\mathbb{Q}}} \cong E_{\mathbb{Q}(\sqrt{-7})} \otimes_{\mathcal{O}_K} L$ (as abelian varieties with $\mathcal{O}_K$-action) yields

$$\mathrm{Jac}(C_{\mathrm{twist},\overline{\mathbb{Q}}})[2] \cong (\mathcal{O}_K/2) \otimes_{\mathcal{O}_K} L \cong L/\mathfrak{p}_2 L \oplus L/\bar{\mathfrak{p}}_2 L \cong M \oplus M^\vee$$

as G-modules, where the last identification uses Elkies' determination of the two reductions. The Kummer sequence identifies $H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ with $\mathrm{Pic}^0(C_{\mathrm{twist},\overline{\mathbb{Q}}})[2]$, which is dual to $\mathrm{Jac}(C_{\mathrm{twist},\overline{\mathbb{Q}}})[2]$. Since $M \oplus M^\vee$ is self-dual via the Weil pairing, we obtain

$$H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2) \cong M \oplus M^\vee.$$

In particular, the mod 2 étale cohomology is the direct sum of two nonisomorphic irreducible 3-dimensional $\mathbb{F}_2$-representations of G.

Proposition 4.6.1. *As an $\mathbb{F}_2[G]$-module, the mod 2 étale cohomology of the Klein quartic decomposes as*

$$H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2) \cong M \oplus M^\vee$$

where M and $M^\vee$ are the natural representation and its dual, which are nonisomorphic irreducible representations of dimension 3 over $\mathbb{F}_2$. Furthermore:

(i) *The Galois action satisfies $\bar{\rho}_{C_{\mathrm{twist}},2}|_{G_K} = \mathrm{id}$.*

(ii) *The nontrivial element of* $\mathrm{Gal}(K/\mathbb{Q})$ *exchanges the two summands M and $M^\vee$.*

Proof. The decomposition is the content of the preceding paragraph.

For (i), because all automorphisms of C_{twist} are defined over K, the Galois action of G_K on $H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ commutes with the action of G. Hence $\bar{\rho}_{C_{\mathrm{twist}},2}(G_K)$ is contained in the centralizer of G in $\mathrm{GL}(H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2))$. Since M and $M^\vee$ are nonisomorphic absolutely irreducible representations, Schur's lemma gives $\mathrm{Hom}_G(M, M^\vee) = 0$, so

$$\mathrm{End}_G(M \oplus M^\vee) \cong \mathrm{End}_G(M) \times \mathrm{End}_G(M^\vee) \cong \mathbb{F}_2 \times \mathbb{F}_2.$$

The only invertible element of $\mathbb{F}_2 \times \mathbb{F}_2$ is $(1, 1)$, so the centralizer is trivial and $\bar{\rho}_{C_{\text{twist}},2}|_{G_k} = \text{id}$.

For (ii), the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\mathrm{Aut}(C_{\text{twist},\overline{\mathbb{Q}}})$ through the nontrivial outer automorphism of $\mathrm{PSL}_2(\mathbb{F}_7)$. Elkies' description identifies this outer automorphism with the permutation of the two reductions $L/\mathfrak{p}_2L$ and $L/\bar{\mathfrak{p}}_2L$, so it exchanges the two summands M and $M^\vee$. □

Proposition 4.6.1 determines $\bar{\rho}_{C_{\text{twist}},2} : G_{\mathbb{Q}} \to \mathrm{GL}_6(\mathbb{F}_2)$: its restriction to G_K is trivial, and the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$ exchanges the two summands. This implies the following:

Theorem 4.6.2. *The two Hurwitz curves of genus* 17 *are defined over* $K = \mathbb{Q}(\sqrt{-7})$ *and are exchanged by the Galois action.*

Proof. The two irreducible summands $M, M^\vee \subset H^1_{\text{ét}}(C_{\text{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ from Proposition 4.6.1 define (dually) two G-stable quotients of $\pi_1(C_{\text{twist},\overline{\mathbb{Q}}})^{\text{ab}} \otimes \mathbb{F}_2$ and hence two normal subgroups of $\pi_1(C_{\text{twist},\overline{\mathbb{Q}}})$ giving unramified $(\mathbb{Z}/2\mathbb{Z})^3$-covers. These are the two geometric genus 17 Hurwitz curves produced by Macbeath's construction.

By Proposition 4.6.1(i), G_K acts trivially on $H^1_{\text{ét}}(C_{\text{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$, so each summand is G_K-stable. Therefore the corresponding subgroups of $\pi_1(C_{\text{twist},\overline{\mathbb{Q}}})$ are preserved by the outer G_K-action, and the primary obstruction of the previous subsection vanishes. The descent obstruction is thus a class in $H^2(G_K, (\mathbb{Z}/2\mathbb{Z})^3)$.

Over $L := \mathbb{Q}(\zeta_7)$ the twist model C_{twist} becomes isomorphic to the Klein model C_{Klein}, and the latter has the rational point $[1 : 0 : 0]$. Hence, after base change to L, the descent obstruction vanishes by the observation at the end of the previous subsection. Since $[L : K] = 3$, the composition of restriction and corestriction

$$H^2(G_K, (\mathbb{Z}/2\mathbb{Z})^3) \xrightarrow{\text{res}} H^2(G_L, (\mathbb{Z}/2\mathbb{Z})^3) \xrightarrow{\text{cor}} H^2(G_K, (\mathbb{Z}/2\mathbb{Z})^3)$$

is multiplication by 3, which is invertible on a module killed by 2. Therefore the restriction map is injective, and since the obstruction vanishes over L it already vanishes over K. Both covers thus descend to curves over K.

Finally, the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$ exchanges M and $M^\vee$ by Proposition 4.6.1(ii), so it exchanges the two isomorphism classes of genus 17 curves as well. □

4.6.4 The Jacobian of X_{17}

In this section, we let X_{17} denote either Hurwitz curve of genus 17 over $\overline{\mathbb{Q}}$. We will reduce the computation of the Jacobian of X_{17} to that of certain Prym varieties (see §4.7 for the definition and basic properties of Prym varieties). The map $X_{17} \to X_3$ gives an isogeny decomposition $\mathrm{Jac}(X_{17}) \sim \mathrm{Jac}(X_3) \times \mathrm{Prym}(X_{17}/X_3)$ and $\mathrm{Jac}(X_3)$ is a CM abelian 3-fold isogenous to $E^3_{\mathbb{Q}(\sqrt{-7})}$. Therefore, it suffices to compute $\mathrm{Prym}(X_{17}/X_3)$. Luckily, we can further exploit the symmetries to reduce the computation to a Prym of a more tractable low genus cover. From the description of $\mathrm{Aut}(X_3) \circlearrowright \mathbb{F}_2^3$ we can easily find a presentation for $G_{1344} = \mathrm{Aut}(X_{17})$ a group of order 1344. It is `SmallGroup(1344,814)` in the GAP library and an explicit presentation as a $(2,3,7)$-group can be extracted from this data [Mac99, Con90].

We can compute the decomposition into irreducible $\mathbb{Q}$-reps of $G_{1344} \circlearrowright H^1(X_{17}(\mathbb{C}), \mathbb{Q})$ in a number of ways. One way is to consider all intermediate curves whose genera can be computed via the Riemann–Hurwitz formula and compare these numbers to the space of fixed points for the representation. We find that $H^1(X_{17}(\mathbb{C}), \mathbb{Q}) \cong V_{14}^{\oplus 2} \oplus V_6$ where V_{14} is an absolutely irreducible representation of dimension 14 and V_6 is a representation of dimension 6 factoring through $G_{1344} \to \mathrm{Aut}(X_3) \cong \mathrm{PSL}_2(\mathbb{F}_7)$ which becomes a sum of two conjugate 3-dimensional representations over $\mathbb{Q}(\sqrt{-7})$. This implies the $\mathbb{Q}$-algebra map $\mathbb{Q}[G_{1344}] \to \operatorname{End} H^1(X_{17}(\mathbb{C}), \mathbb{Q})$ factors through an injection

$$M_{14}(\mathbb{Q}) \times M_3(\mathbb{Q}(\sqrt{-7})) \hookrightarrow \operatorname{End} H^1(X_{17}(\mathbb{C}), \mathbb{Q}).$$

Each idempotent projects onto a 2-dimensional subspace leading to an isogeny decomposition $\mathrm{Jac}(X_{17}) \sim E^{14} \times E^3_{\mathbb{Q}(\sqrt{-7})}$ where $E_{\mathbb{Q}(\sqrt{-7})} = \mathrm{Jac}(X_0(49))$ is an elliptic curve with CM by $\mathbb{Q}(\sqrt{-7})$. Then we can identify the $E^3_{\mathbb{Q}(\sqrt{-7})}$ factor with $\mathrm{Jac}(X_3)$ and the E^{14} factor with $\mathrm{Prym}(X_{17}/X_3)$. Note this decomposition occurs only geometrically (at least over a field where the full group of automorphisms is defined). To compute E up to isogeny, we take an intermediate quotient $X_{17} \to C_5 \to X_3$ with C_5 a curve of genus 5. Then $\mathrm{Prym}(C_5/X_3)$ is an abelian surface which is isogenous to E^2 since it maps to $\mathrm{Prym}(X_{17}/X_3)$. Since we have equations for X_3 we will be able to leverage explicit results on degree 2 unramified coverings to compute equations for a curve C_2 such that $\mathrm{Jac}(C_2) \cong \mathrm{Prym}(C_5/X_3)$ and thus determine E.

First, we choose C_5 by quotienting by one of the $7 = 2^3 - 1$ hyperplanes of $\mathbb{F}_2^3 \subset G_{1344}$ making up the kernel $\ker(\mathrm{Aut}(X_{17}) \to \mathrm{Aut}(X_3)) = \mathrm{Aut}(X_{17}/X_3)$. These are not $\mathrm{Aut}(X_3)$-invariant (since the representation is irreducible) so C_5 is not itself a Hurwitz curve. In fact, all 7 choices of C_5 are permuted by G_{1344}. Furthermore, $\mathrm{Prym}(C_5/X_3) \to \mathrm{Prym}(X_{17}/X_3)$ span $\mathrm{Prym}(X_{17}/X_3)$ as we take different choices of C_5 by Lemma 4.7.7. Unfortunately, we will not be able to prove that X_{17} has infinitely many primes of supersingular reduction (see §4.7.8) but we will be able to show that a similarly constructed curve $X_{17,\mathrm{Fermat}}$ uniformized by the triangle group $\Delta(2,3,8)$ does (see §4.7.6).

4.7 Prym varieties of degree 2 covers of genus 3 curves

Given an étale double cover $\pi : D \to C$, the *Prym variety*

$$\mathrm{Prym}(D/C) := (\ker \pi_* : \mathrm{Jac}(D) \to \mathrm{Jac}(C))^\circ = \mathrm{im}\,(\mathrm{id} - \iota_* : \mathrm{Jac}(D) \to \mathrm{Jac}(D))$$

for ι the involution preserving π, is a *principally polarized* abelian variety of dimension $g(D) - g(C)$. We investigate here the case that C is a non-hyperelliptic curve of genus 3 hence D is a curve of genus 5 and $\mathrm{Prym}(D/C)$ is a principally polarized abelian surface. Therefore, there is a uniquely determined – by the Torelli theorem – hyperelliptic curve H such that $\mathrm{Prym}(D/C) \cong \mathrm{Jac}(H)$. If $\pi : D \to C$ is defined over a number field K then H and the isomorphism $\mathrm{Prym}(D/C) \cong \mathrm{Jac}(H)$ are all defined over K (this holds because the map of stacks $\mathscr{M}_2 \to \mathscr{A}_2$ is a locally closed immersion [Lan21, Theorem 1.2] – a fact that is special to genus 2).

There is a well-known construction of H explicitly in terms of equations for π (see [Bru08]) which we review here. Suppose $\pi : D \to C$ is an étale double cover. Then $\pi_*\omega_D = \pi_*\pi^*\omega_C \cong \omega_C \otimes \pi_*\mathscr{O}_D$. In characteristic $\neq 2$ we can write $\pi_*\mathscr{O}_D = \mathscr{O}_C \oplus \mathscr{L}$ for some 2-torsion line bundle $\mathscr{L}$ which is the line bundle defining D in the sense that it is the unique $\mathscr{L} \in J(K)[2]$ such that $\pi^*\mathscr{L} \cong \mathscr{O}_D$ and we can write D as the cyclic cover "$z^2 = s$" where $s \in H^0(C, \mathscr{L}^{\otimes 2})$ trivializes $\mathscr{L}^{\otimes 2}$. Then

$$H^0(D, \omega_D) = \pi^* H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \mathscr{L})$$

and counting dimensions gives $h^0(C, \omega_C \otimes \mathscr{L}) = 2$. This splitting is exactly the eigenspace decomposition of ι^* on $H^0(D, \omega_D)$ with $H^0(C, \omega_C)$ the $+1$-eigenspace and $H^0(C, \omega_C \otimes \mathscr{L})$ the -1-eigenspace. Consider the diagram of canonical embeddings,

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}^4 \arrow[d, dashed] \\ C \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd}$$

Since D is a non-hyperelliptic genus 5 curve, its canonical model is a complete intersection of quadrics: elements in $\mathrm{Sym}^2 H^0(D, \omega_D)$. Since D is invariant under the linear action of ι^* on $\mathbb{P}^4$ we get an action of ι on the ideal sheaf defining D. Since D is a complete intersection of quadrics $H^1(\mathbb{P}^4, \mathscr{I}_D(2)) = 0$ giving an exact sequence of $\mathbb{Z}/2$-modules

$$0 \to H^0(\mathbb{P}^4, \mathscr{I}_D(2)) \to H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(2)) \to H^0(D, \omega_D^{\otimes 2}) \to 0$$

Furthermore, the odd subspace (-1-eigenspace) $H^0(C, \omega_C^{\otimes 2} \otimes \mathscr{L}) \subset H^0(D, \omega_D^{\otimes 2})$ has dimension 6 equal to the dimension of the odd subspace of $H^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(2)) = \operatorname{Sym}^2 H^0(D, \omega_D)$. Hence $H^0(\mathbb{P}^4, \mathscr{I}_D(2)) \subset [\operatorname{Sym}^2 H^0(D, \omega_D)]^+$ lies in the even part. Choosing coordinates $u, v, w \in H^0(C, \omega_C)$ and $r, s \in H^0(C, \omega_C \otimes \mathscr{L})$ we may choose a basis of the three quadrics cutting out D to isolate the three generators of $\operatorname{Sym}^2 H^0(C, \omega_C \otimes \mathscr{L})$ so that D is defined by equations

$$D: \quad \begin{cases} Q_1(u, v, w) = r^2 \\ Q_2(u, v, w) = rs \\ Q_3(u, v, w) = s^2 \end{cases}$$

where $Q_1, Q_2, Q_3 \in K[u, v, w]$ are quadratic forms. Since $r^2 \cdot s^2 = (rs)^2$ and C is cut out by a quartic equation $F(u, v, w) = 0$, the equations for D imply that F takes the form,

$$\lambda F(u, v, w) = Q_1(u, v, w) Q_3(u, v, w) - Q_2(u, v, w)^2$$

for some $\lambda \in K$. We refer to such a rewriting of F as a *quadric decomposition*. Conversely, given such a quadric decomposition, we may build $\pi : D \to C$ using the above formulas. In this case, H is determined by the hyperelliptic equation

$$H: \quad y^2 = -\det\left(Q_1 + 2xQ_2 + x^2 Q_3\right)$$

where we regard Q_i as the symmetric 3×3 matrices representing the corresponding quadratic form [Bru08]. Note that $\pi : D \to C$ is determined (up to $K^\times / K^{\times 2}$-twists replacing $Q_i \mapsto \alpha Q_i$) by the 2-torsion line bundle

$$\mathscr{L} \cong \mathscr{O}_C\left(\tfrac{1}{2} \operatorname{div} Q_1|_C - K_C\right) \cong \mathscr{O}_C\left(\tfrac{1}{2} \operatorname{div} Q_3|_C - K_C\right)$$

since on D the divisors $\frac{1}{2} \operatorname{div} Q_1|_C$ and $\frac{1}{2} \operatorname{div} Q_3|_C$ pull back to $\operatorname{div} r|_D$ and $\operatorname{div} s|_D$ respectively which are in the linear series $|K_D|$.

4.7.1 Setup for the arithmetic aspects of Prym varieties

Now the question becomes, if we define an étale cover of degree 2 via a cohomology class $\alpha \in H^1_{\text{ét}}(\overline{C}, \mathbb{F}_2)$ where $\overline{C}$ is the base change to the algebraic closure $\overline{K}$, how do we compute the equations for H? First, we should recall that even if α is G_K-invariant, we saw in the previous section that there is an obstruction to α corresponding to a cover defined over K. This obstruction comes from the 5-term sequence for the Leray/Hochschild–Serre spectral sequence,

$$E_2^{p,q} = H^p(G_K, H^q_{\text{ét}}(\overline{C}, \mathbb{F}_2)) \implies H^{p+q}_{\text{ét}}(C, \mathbb{F}_2)$$

computing absolute étale cohomology. The 5-term sequence has the form

$$0 \to H^1(G_K, \mathbb{F}_2) \to H^1_{\text{ét}}(C, \mathbb{F}_2) \to H^1_{\text{ét}}(\overline{C}, \mathbb{F}_2)^{G_K} \to H^2(G_K, \mathbb{F}_2) \to H^2_{\text{ét}}(C, \mathbb{F}_2)$$

and therefore the obstruction to lifting along $H^1_{\text{ét}}(C, \mathbb{F}_2) \to H^1_{\text{ét}}(\overline{C}, \mathbb{F}_2)^{G_K}$ lies in $H^2(G_K, \mathbb{F}_2) = \mathrm{Br}(K)[2]$. We can rephrase this in terms of our previous obstructions. View $\alpha : \pi_1(\overline{C}) \to \mathbb{F}_2$ and let $\Gamma := \ker \alpha$. If α is G_K-invariant then Γ is stable under G_K-conjugation and $\Gamma \subset \pi_1(\overline{C})$ is normal so $N_{\pi_1(C)}(\Gamma) = \pi_1(C)$. Thus the primary obstruction vanishes and the secondary obstruction is a class $\mathrm{ob} \in H^2(G_K, Q)$ where $Q = \ker(N \to G_K)/\Gamma = \pi_1(\overline{C})/\Gamma = \mathbb{F}_2$. Note also that the descent of the line bundle $\overline{\mathscr{L}}$ defining $\bar{\pi} : \overline{D} \to \overline{C}$ to a line bundle $\mathscr{L}$ on C defined over K is sufficient to build $\pi : D \to C$ over K using the cyclic cover construction. The bundle $\mathscr{L}$ is unique and the isomorphism class of π is well-defined up to the action of $K^\times/K^{\times 2}$ given by scaling $s \in H^0(C, \mathscr{L}^{\otimes 2})$. This construction realizes the map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^1(G_K, \mathbb{F}_2) \arrow[r] \arrow[d] & H^1_{\text{ét}}(C, \mathbb{F}_2) \arrow[r] \arrow[d] & H^1_{\text{ét}}(\overline{C}, \mathbb{F}_2)^{G_K} \arrow[r] \arrow[d, equal] & H^2(G_K, \mathbb{F}_2) \arrow[r] \arrow[d, equal] & H^2_{\text{ét}}(C, \mathbb{F}_2) \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Pic}(C)[2] \arrow[r] & \mathrm{Pic}_{C/K}[2](K) \arrow[r] & \mathrm{Br}(C)[2] \arrow[r] & H^2_{\text{ét}}(C, \mathbb{G}_m)[2]
\end{tikzcd}$$

induced by the map of 5-term Leray/Hochschild–Serre sequences for $\mathbb{F}_2 = \mu_2 \hookrightarrow \mathbb{G}_m$. Recall that when C has a K-point, then $\mathrm{Pic}_{C/K}(K) = \mathrm{Pic}(C)$ verifying that the Brauer obstruction vanishes. However, in general $\mathrm{Pic}_{C/K}(K) = \mathrm{Pic}_{C/K}(\overline{K})^{G_K}$ is larger.

Suppose we have chosen an extension K_α/K killing the obstruction $\mathrm{ob}(\alpha) \in \mathrm{Br}(K)[2]$. Then there exists an étale double cover $\pi : D \to C$ defined over K_α and hence a quadratic decomposition $Q_1Q_3 - Q_2^2 = \lambda F$ of the quartic polynomial F cutting out C. It is computationally very expensive to find quadratic decompositions by exhaustively searching over quadrics with algebraic coefficients until we find a complete set of 2-torsion points. However, we can exploit the structure of theta characteristics and their relation to bitangent lines of C to reduce the search to finding all bitangents and certain quadrics witnessing so-called syzygies of bitangents, a much more computationally feasible task.

4.7.2 Theta characteristics and bitangents

In this section let C be a smooth projective curve over a field K of characteristic not 2.

Definition 4.7.1. A *theta characteristic* θ on C is a line bundle such that $\theta^{\otimes 2} \cong \omega_C$.

Theta characteristics have a natural notion of parity meaning they come in two types *even* and *odd* determined by the parity of $h^0(C, \theta)$. We can give a nice interpretation of this parity in terms of a quadratic form on $J(\overline{K})[2]$ where $J = \mathrm{Jac}(C)$. It is clear that the set $\Theta(C)$ of isomorphism classes of (geometric) theta characteristics is a $J(\overline{K})[2]$-torsor. It also has a Galois action compatible with the Galois module structure on $J(\overline{K})[2]$. For each theta characteristic θ we can define a quadratic form on the $\mathbb{F}_2$-vector space $J(\overline{K})[2]$.

Lemma 4.7.2. *[Har82, Theorem 1.13] The assignment*

$$q_\theta : \mathscr{L} \mapsto h^0(C, \theta \otimes \mathscr{L}) - h^0(C, \theta) \mod 2$$

defines a quadratic form on $J(\overline{K})[2]$ *whose associated bilinear form is the Weil pairing composed with the unique isomorphism* $\mu_2 \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$.

In fact, the association $\theta \mapsto q_\theta$ defines a bijection $\Theta(C) \xrightarrow{\sim} Q(J(\overline{K})[2], \langle -, - \rangle)$ where $Q(V, \langle -, - \rangle)$ is the space of quadratic forms on V whose associated bilinear form is $\langle -, - \rangle$. Furthermore, $Q(V) := Q(V, \langle -, - \rangle)$ naturally decomposes into two subsets $Q(V)^+$ and $Q(V)^-$ of quadratic forms of Arf invariant 0 and 1 respectively. These are the two orbits of $Q(V)$ under the action of the symplectic group $\mathrm{Sp}(V)$. This decomposition gives the parity of a theta characteristic because

$$\mathrm{Arf}(q_\theta) = h^0(C, \theta) \mod 2$$

by [Joh80, Theorem 4]. Atiyah proved there are exactly $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta characteristics and that parity is a deformation invariant of θ (unlike the integer $h^0(C, \theta)$ which is only upper semicontinuous) [Ati71].

Farkas explains [Far12] that Frobenius' *theory of fundamental systems* of theta characteristics is captured in modern notation by the notion of *syzygetic triads* and *tetrads*.

Definition 4.7.3. We say that a system of three theta characteristics $\theta_1, \theta_2, \theta_3 \in \Theta(C)$ is *syzygetic* or that $\{\theta_1, \theta_2, \theta_3\}$ form a *syzygetic triad* if

$$\mathrm{Arf}(q_{\theta_1}) + \mathrm{Arf}(q_{\theta_2}) + \mathrm{Arf}(q_{\theta_3}) + \mathrm{Arf}(q_{\theta_1+\theta_2+\theta_3}) = 0$$

where we define $\theta_1 + \theta_2 + \theta_3$ to be the theta characteristic $\theta_1 \otimes \theta_2 \otimes \theta_3^{-1}$. This expression is symmetric under reordering justifying the notation. We say that $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ is a *syzygetic tetrad* if any three elements form a syzygetic triad and $\theta_1 \otimes \theta_2 \otimes \theta_3 \otimes \theta_4 \cong \omega_C^{\otimes 2}$.

Frobenius clarified the relationship between *period characteristics* (2-torsion points on the Jacobian) and *theta characteristics* (discrete parameters of theta functions that correspond to square roots of the canonical bundle) with respect to modularity properties of theta functions. In modern language, the theta characteristics form an affine space and transform naturally in a "non-linear" way so that their associated quadratic form transforms equivariantly. Thus, theta characteristics do not form a natural binary additive structure, rather they have the trinary addition $\theta_1 + \theta_2 + \theta_3$ and syzygetic relations.

4.7.3 Plane quartics

In the particular case of $g = 3$, there are 36 even and 28 odd theta characteristics. From now on we specialize to the case of nonhyperelliptic genus 3 curves. The 28 odd theta characteristics correspond exactly to the 28 bitangents of the canonical embedding $C \hookrightarrow \mathbb{P}^2$. Indeed, an odd theta characteristic is an effective divisor D of degree 2 such that $2D \sim K_C$ thus $2D \in |\ell|$ is in the linear series of lines. Since C is non-hyperelliptic, $h^0(C, \mathcal{O}_C(D)) \leq \deg D - 1 = 1$ and hence there is a unique representation, $D = P + Q$ for points $P, Q \in C$. Then $2P + 2Q = C \cdot \ell$ for some line $\ell \subset \mathbb{P}^2$ meaning exactly that the line through P, Q (resp. the tangent line through $P = Q$) is a bitangent (resp. hyperflex). We will often abuse notation by using ℓ to refer to a linear form on $\mathbb{P}^2$ defining the line ℓ. Thus we will write $C \cdot \ell$ and $\operatorname{div} \ell|_C$ interchangeably.

Lemma 4.7.4. *Let $\ell_1, \ell_2, \ell_3, \ell_4$ be bitangent lines of C and let $\theta_1, \theta_2, \theta_3, \theta_4$ be their corresponding odd theta characteristics. Then the following are equivalent:*

(1) $\frac{1}{2}(\operatorname{div} \ell_1|_C - \operatorname{div} \ell_2|_C) \sim \frac{1}{2}(\operatorname{div} \ell_3|_C - \operatorname{div} \ell_4|_C)$ *are linearly equivalent 2-torsion divisors on C*

(2) the 8 intersection points (counted with half multiplicity) of $\ell_1, \ell_2, \ell_3, \ell_4$ with C lie on a conic

(3) $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ *form a syzygetic tetrad.*

Proof. (1) is equivalent to the condition that

$$\frac{1}{2}(\operatorname{div} \ell_1|_C + \operatorname{div} \ell_2|_C + \operatorname{div} \ell_3|_C + \operatorname{div} \ell_4|_C) \sim \operatorname{div} \ell_2|_C + \operatorname{div} \ell_3|_C \in |2H|$$

where H is the hyperplane class on C. This exactly says that the 8 intersection points lie in the complete linear series $|2H|$ but $H^0(\mathbb{P}^2, \mathcal{O}(2)) \to H^0(C, \mathcal{O}_C(2))$ is surjective because $H^1(\mathbb{P}^2, \mathcal{O}(2-3)) = 0$. Thus the linear equivalence exactly says that the eight points lie on a conic giving (1) $\iff$ (2). Thus it suffices to prove that (1) $\iff$ (3). Part of the definition of a syzygetic triad is that $\theta_1 \otimes \theta_2^\vee \cong \theta_3 \otimes \theta_4^\vee$ which is, by definition, (1). Because we may reorder freely, it suffices to show that $\{\theta_1, \theta_2, \theta_3\}$ is a syzygetic system. However, $\theta_1 + \theta_2 + \theta_3 \cong \theta_4$ so it suffices to show

$$\mathrm{Arf}(q_{\theta_1}) + \mathrm{Arf}(q_{\theta_2}) + \mathrm{Arf}(q_{\theta_3}) + \mathrm{Arf}(q_{\theta_4}) = 0.$$

Since these are all odd theta characteristics each Arf invariant equals 1 and hence the sum is 0 (mod 2). □

Every 2-torsion class on C is of the form $\frac{1}{2}(\operatorname{div} \ell_1|_C - \operatorname{div} \ell_2|_C)$ for some pair of bitangent lines [Dol12, Proposition 5.4.7]. Steiner studied the combinatorics of the 28 bitangents and showed that every 2-torsion class can be represented by exactly 6 pairs of lines; this collection of lines is called a *Steiner complex* of bitangents.

Definition 4.7.5. A *Steiner complex* is a set $\mathcal{S} = \{\{\ell_1, \ell_1'\}, \dots, \{\ell_6, \ell_6'\}\}$ of unordered pairs of distinct bitangent lines such that any pair of pairs $\{\ell_i, \ell_i'\}, \{\ell_j, \ell_j'\}$ is *syzygetic* in the sense of satisfying the equivalent conditions of Lemma 4.7.4.

The Steiner complexes are the sets of mutually syzygetic collections of pairs. Equivalently they are exactly the fibers of the map sending an unordered pair $\{\ell, \ell'\}$ of distinct bitangents (i.e. odd theta characteristics) to the nonzero 2-torsion class $\frac{1}{2}(\operatorname{div} \ell|_C - \operatorname{div} \ell'|_C)$ (i.e. the difference $\theta_1 \otimes \theta^{-1} \in J[2]$). There are 63 Steiner complexes corresponding to the $63 = 2^6 - 1$ nonzero 2-torsion classes on C. Since $6 \cdot 63 = \binom{28}{2}$ every pair of distinct bitangent lines appears in some Steiner complex. By definition, for any pair of pairs $\{\ell_i, \ell_i'\}, \{\ell_j, \ell_j'\}$ in a Steiner complex there exists a *witness conic* Q passing through the 8 intersection points. Hence, the rational function in $K(C)$

$$f = \frac{(\ell_1 \ell_2 \ell_3 \ell_4)|_C}{Q^2|_C}$$

has no poles and thus $f = a$ for some constant $a \in K$. This means $\ell_1 \ell_2 \ell_3 \ell_4 - aQ^2$ is zero in the coordinate ring

$$R(C) = K[u, v, w]/(F(u, v, w))$$

of C. By degree considerations, there is some scalar $\lambda \in K$ so that

$$\lambda F = \ell_1\ell_2\ell_3\ell_4 - aQ^2.$$

This gives a quadric decomposition of F, as needed to produce the Prym curve, of a particularly simple form. In the Magma script [Chu25, `steiner_computations.m`] we compute the 28 bitangents, the 63 Steiner complexes, and the 945 witness conics for any pair of pairs in a Steiner complex. Similar computations have also been described and implemented by the authors of [PSV11].

In Magma, we performed the full arithmetic and combinatorial analysis of the 28 bitangents for the following plane quartic curves defined over $\mathbb{Q}$

$$\begin{aligned} C_{\text{twist}}: &\quad x^4+y^4+z^4+6(xy^3+yz^3+zx^3)-3(x^2y^2+y^2z^2+z^2x^2)+3xyz(x+y+z) \\ C_{\text{Fermat}}: &\quad x^4+y^4+z^4 \\ C_{\text{Edge}}: &\quad 25(x^4+y^4+z^4)-34(x^2y^2+y^2z^2+x^2z^2) \end{aligned}$$

where C_{Edge} is the S_4-invariant *Edge quartic* which corresponds to $\gamma = \pm 2$ in Edge's family of determinantal quartics [Edg38, p.19]. This quartic is distinguished by the amazing property of having all 28 bitangents defined over $\mathbb{Q}$ and hence its Jacobian has full rational 2-torsion over $\mathbb{Q}$ [PSV11, p.2]. For each curve and each 2-torsion class $\alpha \in H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_2)$ we find the associated genus 2 curve computing the Prym of the cover $D_\alpha \to X_{\overline{K}}$. These data are recorded in [Chu25, `PRYM_INVARIANTS.md`] and [Chu25, `prym_table.md`].

4.7.4 Symmetric genus 2 curves

Any genus 2 curve C has a unique $\mathfrak{g}^1_2$ defining a degree 2 map $\pi : C \to \mathbb{P}^1$. This double cover is associated to an involution $\iota : C \to C$. Since ι is the unique involution with rational quotient, it must commute with any automorphism of C giving a central element of $\mathrm{Aut}(C)$. We call the quotient $\mathrm{RA}(C) := \mathrm{Aut}(C)/\langle\iota\rangle$ the *reduced automorphism group* of C. We say a genus 2 curve has *extra automorphisms* if $\mathrm{RA}(C)$ is nontrivial. Igusa [Igu60] classified all families of genus 2 curves with extra automorphisms. In the language of Ibukiyama–Katsura–Oort [IKO86] the following give a complete list of families of genus 2 curves with extra automorphisms in characteristic zero:

(1) $C: \quad y^2 = x(x-1)(x-\lambda)(x-\mu)(x-\lambda(1-\lambda)^{-1}(1-\mu))$ with $\mathrm{RA}(C) \cong \mathbb{Z}/2$ generically

(2) $C: \quad y^2 = x(x-1)(x-\lambda)(x-(\lambda-1)\lambda^{-1})(x-(1-\lambda)^{-1})$ with $\mathrm{RA}(C) = S_3$ generically

(3) $C: \quad y^2 = x(x-1)(x+1)(x-\lambda)(x-\lambda^{-1})$ with $\mathrm{RA}(C) \cong (\mathbb{Z}/2)^2$ generically

(4) $C: \quad y^2 = x(x-1)(x+1)(x-2)(x-1/2)$ with $\mathrm{RA}(C) \cong D_6$

(5) $C: \quad y^2 = x(x^2-1)(x^2+1)$ with $\mathrm{RA}(C) \cong S_4$

(6) $C: \quad y^2 = x(x-1)(x-1-\zeta)(x-1-\zeta-\zeta^2)(x-1-\zeta-\zeta^2-\zeta^3)$ with $\mathrm{RA}(C) \cong \mathbb{Z}/5$ where ζ is a primitive 5-th root of unity.

The curves of classes (1)-(5) have Jacobians isogenous to a pair of elliptic curves $E_\sigma \times E_\tau$ defined as $E_\sigma = C/\langle\sigma\rangle$ and $E_\tau = C/\langle\tau\rangle$ where

$$\sigma, \tau : \begin{cases} x \mapsto \lambda(x-\mu)(x-\lambda)^{-1} \\ y \mapsto \pm[\lambda^3(\lambda-\mu)^3]^{1/2} y(x-\lambda)^{-3} \end{cases}$$

where σ and τ correspond to the two square roots of $\lambda^3(\lambda-\mu)^3$ and hence $\tau = \sigma \circ \iota$. Ibukiyama–Katsura–Oort compute Legendre equations [IKO86, Equation (1.13)] for these elliptic curves:

$$u^2 = v(v-1)(v-\rho) \quad \rho := (1-\lambda)(\mu - 2\lambda \pm 2(\lambda^2 - \lambda\mu)^{1/2})(\mu-1)^{-1}$$

Recall the formula for the j-invariant of the Legendre family:

$$j = 2^8 \cdot \frac{(\rho^2-\rho+1)^3}{\rho^2(1-\rho)^2}.$$

Thus we can compute the j-invariant of E_σ and E_τ. Notice that, as members of the Legendre family, E_σ and E_τ have full rational 2-torsion over $\mathbb{Q}(\rho)$.

It is particularly nice to specialize the values of ρ to the families (2) and (3) which will appear for the genus 2 curves computed as Pryms. We tabulate these as follows:

(2) $\rho = (1-\lambda)(\lambda \mp (\lambda^2-\lambda+1)^{1/2})^2$

(3) $\rho = -(\lambda \mp (\lambda^2-1)^{1/2})^2$.

Generically, the elliptic curves E_σ and E_τ in class (1) are not isogenous. However, they are isogenous in classes (2)-(5). In case (2) this is forced by the representation theory of S_3 since the group ring contains a copy of $M_2(\mathbb{Q})$ which must act rationally on the Jacobian. For class (3), let

$$u_\pm = \lambda \mp (\lambda^2-1)^{1/2}$$

and notice that $u_+u_- = 1$. Therefore, the two curves

$$E_\pm : \quad y^2 = x(x-1)(x+u_\pm^2)$$

are isomorphic over $\mathbb{Q}(u_\pm^2, i)$ via the map

$$x \mapsto -u_\pm^2 x \quad y \mapsto iu_\pm^3 y.$$

Every genus 2 curve representing a Prym of a highly symmetric quartic will have extra automorphisms. Indeed, for a 2-torsion class $\alpha \in H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{F}_2)$ we consider the stabilizer $\operatorname{Aut}(C_{\overline{K}})_\alpha$ which acts on $C_{\overline{K}}$ in such a way that each automorphism lifts to $D_{\alpha,\overline{K}}$ giving an exact sequence

$$0 \to \langle\iota\rangle \to C_{\operatorname{Aut}(D_{\alpha,\overline{K}})}(\iota) \to \operatorname{Aut}(C_{\overline{K}})_\alpha \to 0$$

where ι is the deck transformation of $\pi : D_{\alpha,\overline{K}} \to C_{\overline{K}}$. Furthermore, $C_{\operatorname{Aut}(D_{\alpha,\overline{K}})}(\iota)$ acts on $\operatorname{Prym}(D_\alpha/C)$ geometrically and hence on H_α. Since $\#H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{F}_2) = 64$, as long as $\operatorname{Aut}(C_{\overline{K}})$ is reasonably large, any element of $H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{F}_2)$ will have a nontrivial stabilizer leading to extra automorphisms of H.

Lemma 4.7.6. *Let $\alpha \in H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{F}_2)$ be nonzero and let H_α be the genus 2 curve such that*

$$\operatorname{Jac}(H_\alpha) \cong \operatorname{Prym}(D_\alpha/C).$$

Then the exact sequence above induces a homomorphism

$$\operatorname{Aut}(C_{\overline{K}})_\alpha \to \operatorname{RA}(H_\alpha)$$

whose kernel has order dividing 4. In particular, if $\#\operatorname{Aut}(C_{\overline{K}})_\alpha > 4$ then H_α has, geometrically, extra automorphisms.

Proof. We first base change to $\overline{K}$. The action of $C_{\operatorname{Aut}(D_\alpha)}(\iota)$ on $\operatorname{Prym}(D_\alpha/C)$ induces an action on $\operatorname{Jac}(H_\alpha)$. Under this action, the deck involution ι acts by -1 on $\operatorname{Prym}(D_\alpha/C)$, which corresponds to the hyperelliptic involution on H_α. Therefore, the exact sequence above descends to a well-defined homomorphism

$$\operatorname{Aut}(C)_\alpha \to \operatorname{RA}(H_\alpha).$$

Let K denote its kernel. Since

$$T_0^* \operatorname{Prym}(D_\alpha/C) \cong H^0(C, \omega_C \otimes \alpha),$$

the Torelli isomorphism $\mathrm{Jac}(H_\alpha) \cong \mathrm{Prym}(D_\alpha/C)$ identifies

$$H^0(H_\alpha, \omega_{H_\alpha}) \cong H^0(C, \omega_C \otimes \alpha).$$

Set $\mathscr{L}_\alpha := \omega_C \otimes \alpha$. Since α is nontrivial of degree 0, Riemann–Roch gives

$$h^0(C, \mathscr{L}_\alpha) = \deg \mathscr{L}_\alpha + 1 - g(C) = 4 + 1 - 3 = 2.$$

Furthermore, $\mathscr{L}_\alpha$ is basepoint-free. Indeed, if $p \in C$ were a base point, then $h^0(C, \mathscr{L}_\alpha(-p)) = 2$ and Riemann–Roch would give $h^0(C, \alpha(p)) = 1$. Hence $\alpha \cong \mathscr{O}_C(q-p)$ for some $q \in C$, so $\alpha^{\otimes 2} \cong \mathscr{O}_C$ implies $2q \sim 2p$, producing a $\mathfrak{g}^1_2$ on C, contradicting that C is non-hyperelliptic.

Thus the complete linear series $|\mathscr{L}_\alpha|$ defines a morphism

$$f_\alpha : C \to \mathbb{P}^1$$

of degree $\deg \mathscr{L}_\alpha = 4$. Any element of K acts trivially on $\mathrm{RA}(H_\alpha)$, hence trivially on the projective space

$$\mathbb{P}(H^0(H_\alpha, \omega_{H_\alpha})) = \mathbb{P}(H^0(C, \mathscr{L}_\alpha)).$$

Therefore, f_α is K-invariant and factors through the quotient $C \to C/K$. Since $K \subset \mathrm{Aut}(C)$ acts faithfully on C, the quotient map has degree $\#K$, so

$$4 = \deg f_\alpha = \#K \cdot \deg\,(C/K \to \mathbb{P}^1).$$

Hence $\#K$ divides 4, as claimed. □

We use the above lemma to ensure that the curves produced have extra automorphisms. The relevance of computing these genus 2 curves in such detail comes from the isogeny decomposition into Pryms of étale 2-torsion covers. In the following, C is an arbitrary smooth projective curve over K.

Lemma 4.7.7. *Let $X \to C$ be an étale cover over K defined by some subgroup $M \subset H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{F}_2)$. Then the natural map*

$$\mathrm{Jac}(X) \to \mathrm{Jac}(C) \times \prod_{\alpha \in M \setminus 0} \mathrm{Prym}(D_\alpha/C)$$

is an isogeny.

Proof. It suffices to prove that the map is an isomorphism on $H^1_{\text{ét}}(-, \mathbb{Q}_\ell)$. The cover $X_{\overline{K}} \to C_{\overline{K}}$ is Galois with deck transformation group $M^\vee$. Furthermore, the map $\mathrm{Jac}(X_{\overline{K}}) \to$

$\mathrm{Jac}(D_{\alpha,\overline{K}})$ induces $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\alpha^\vee} \subset H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ on cohomology where $\alpha^\vee \subset M^\vee$ is the kernel of the map $\varphi \mapsto \varphi(\alpha)$. Consider the intersection of two such subspaces,

$$H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\alpha_1^\vee} \cap H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\alpha_2^\vee} = H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{M^\vee} = H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{Q}_\ell)$$

because $\alpha_1^\vee$ and $\alpha_2^\vee$ generate all of $M^\vee$ if $\alpha_1 \neq \alpha_2$. This implies the map is injective on $H^1_{\text{ét}}(-, \mathbb{Q}_\ell)$. Furthermore, Riemann–Roch gives

$$g(X) = (\#M)(g(C) - 1) + 1$$

and each Prym has dimension $g(C) - 1$ so the RHS has dimension

$$(\#M - 1) \cdot (g(C) - 1) + g(C) = (\#M) \cdot (g(C) - 1) + 1 = \dim \mathrm{Jac}(X)$$

proving that the surjection must be an isogeny. □

4.7.5 Results on the structure of Pryms

Here we record the automorphism groups, isogeny decomposition of the Jacobians, and the decomposition of $H^1_{\text{ét}}(-, \mathbb{F}_2)$ as a module for the automorphism groups. These calculations were performed in the Magma script [Chu25, `aut_j2_modules.m`]. We then go on to record the properties of interesting Prym varieties (in terms of their associated genus 2 curve) associated to certain automorphism orbits of 2-torsion classes. These latter computations are performed using the Magma script [Chu25, `steiner_computations.m`].

The twisted Klein quartic

$\mathrm{Aut}(C_{\text{twist},\overline{\mathbb{Q}}}) \cong \mathrm{PSL}_2(\mathbb{F}_7)$ and $\mathrm{Jac}(C_{\text{twist}})_{\overline{\mathbb{Q}}} \sim E^3_{\mathbb{Q}(\sqrt{-7})}$ where $E_{\mathbb{Q}(\sqrt{-7})}$ is an elliptic curve with CM by $\mathbb{Q}(\sqrt{-7})$. More canonically, we can choose the $\mathbb{Q}$-model $E_{\mathbb{Q}(\sqrt{-7})} = \mathrm{Jac}(X_0(49))$ so that the isogeny exists over $\mathbb{Q}$. As previously computed,

$$H^1_{\text{ét}}(C_{\text{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2) = M \oplus M^\vee$$

where M is an irreducible 3-dimensional $\mathrm{PSL}_2(\mathbb{F}_7)$-representation. Furthermore, as a set $H^1_{\text{ét}}(C_{\text{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2) \setminus \{0\}$ decomposes into 4 orbits. Each Prym is isogenous to the square of some elliptic curve. We record the minimal polynomials of the j-invariants (computed for the Legendre model as explained in the previous section) in the following table where orbits #3, 4 correspond to the nonzero elements of M and $M^\vee$ respectively. The orbits

orbit #	size	RA	minimal polynomial of j
1	28	S_3	$j^2 + 13856j - 26578688$
2	21	$(\mathbb{Z}/2)^2$	$j^2 + 7184j + 16777216$
3,4	7	S_3	$j^4 - 103439j^3 + 6670405329j^2 + 31128229267744j + 148381159092130048$

#1, #3, #4 produce Pryms whose genus two curves have RA $\cong S_3$ while orbit #2 produces RA $\cong (\mathbb{Z}/2)^2$. This explains why each Prym is isogenous to the square of an elliptic curve. Orbits #3 and #4 are exchanged under the Galois action of $\mathbb{Q}(\sqrt{-7})$ (in particular, the Pryms live over $\mathbb{Q}(\sqrt{-7})$ and are exchanged) so we record them once in the table. We label a choice of root as $j_1, j_2, j_{3,4}$ respectively. Notice that $\mathbb{Q}(j_1) = \mathbb{Q}(\sqrt{7})$ and $\mathbb{Q}(j_2) = \mathbb{Q}(\sqrt{-7})$ while $\mathbb{Q}(j_{3,4})$ is a degree 2 extension of $\mathbb{Q}(\sqrt{-7})$ which is not Galois over $\mathbb{Q}$ (labeled as [LMF25o] in the LMFDB).

There is a very clear explanation of the orbit structure of $H^1_{\text{ét}}(C_{\text{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ using the isomorphism $\mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{GL}_3(\mathbb{F}_2)$. The 3-dimensional module M becomes the standard representation of $\mathrm{GL}_3(\mathbb{F}_2)$ whose nonzero elements are the points of the *Fano plane* $\mathbb{P}^2(\mathbb{F}_2)$. We identify the nonzero elements of $M^\vee$ with *lines* in the Fano plane and the $\mathrm{GL}_3(\mathbb{F}_2)$-invariant pairing $M \otimes M^\vee \to \mathbb{F}_2$ with the incidence pairing. Note that addition is determined by the ternary relation $\alpha_1 + \alpha_2 + \alpha_3 = 0$. For three points of the Fano plane this corresponds to being *collinear*. For lines, this corresponds to being *concurrent* meaning they all pass through the same point. In each case, there is a uniquely determined dual object (three collinear define a distinguished line; three concurrent lines define a distinguished point) whose kernel defines the 2-dimensional $\mathbb{F}_2$-subspace $\langle \alpha_1, \alpha_2 \rangle$ whose nonzero points are $\alpha_1, \alpha_2, \alpha_3$. The orbits #3 and #4 are just the points and lines of the Fano plane respectively. Orbit #2 consists of *flags* (P, ℓ) with $P \in \ell$ i.e. $\langle \ell, P \rangle = 0$. Orbit #1 consists of *anti-flags* (P, ℓ) with $P \notin \ell$ i.e. $\langle \ell, P \rangle = 1$. This description will be used later to find collinear points in these orbits.

The Fermat quartic

The Fermat quartic has automorphism group $\mathrm{Aut}(C_{\text{Fermat},\overline{\mathbb{Q}}}) \cong (\mathbb{Z}/4)^2 \rtimes S_3$ given by permuting the variables and scaling the coordinates by 4-th roots of unity. The quotient by this group presents C_{Fermat} as a Galois Belyĭ curve for the triangle group $\Delta(2, 3, 8)$. The representation theory of its automorphism group forces $\mathrm{Jac}(C_{\text{Fermat}})_{\overline{\mathbb{Q}}} \sim E^3_{\mathbb{Q}(\sqrt{-1})}$ where $E_{\mathbb{Q}(\sqrt{-1})}$ is an elliptic curve with CM by $\mathbb{Q}(\sqrt{-1})$. In fact, the Fermat quartic is the modular curve $X_{\text{sp}}(8)$ and the map $X_{\text{sp}}(8) \to X^+_{\text{sp}}(8)$ (see [LMF25l]) produces an isogeny with the $\mathbb{Q}$-model $E_{\mathbb{Q}(\sqrt{-1})} = \mathrm{Jac}(X^+_{\text{sp}}(8))$ given by the elliptic curve with LMFDB label [LMF25e].

The module structure of $H^1_{\text{ét}}(C_{\text{Fermat},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ is not semisimple. It has a composition series,

$$M_1 \subset M_3 \subset M_5 \subset M_6$$

where $\dim M_i = i$. The quotients are irreducible modules of dimensions $1, 2, 2, 1$ over $\mathbb{F}_2$ (in particular the 1-dimensional subquotients are trivial modules) and the successive extensions are all non-split. As a set $H^1_{\text{ét}}(C_{\text{Fermat},\overline{\mathbb{Q}}}, \mathbb{F}_2) \setminus \{0\}$ decomposes into 5 orbits:

orbit #	size	RA	minimal polynomial of j
1	24	$(\mathbb{Z}/2)$	—-
2,3	16	S_3	$j^4+73216j^3+1533640704j^2+6119514701824j$ -15081210455785472
4	6	$(\mathbb{Z}/2)^2$	$j-128$
5	1	S_4	$j-8000$

where orbit #5 corresponds to the nonzero element of M_1 and orbit #4 corresponds to the elements of $M_3 \setminus M_1$. Then $M_5 \setminus M_3$ corresponds to orbit #1 and the remaining elements in $M_6 \setminus M_5$ split into the remaining two orbits #2, 3 exchanged by the Galois action on $\mathbb{Q}(i)$. Orbit #1 has its Prym decompose into two non-isogenous elliptic curves whose invariants we do not report here. The important orbits for us will be #4, 5. Here $j = 128$ is non-CM and $j = 8000$ corresponds to CM by $\mathbb{Q}(\sqrt{-2})$. We label these elliptic curves as E_{128} and $E_{\mathbb{Q}(\sqrt{-2})}$ respectively.

The S_4 Edge quartic

The Edge quartic C_{Edge} has automorphism group S_4 and a (geometric) isogeny decomposition $\text{Jac}(C_{\text{Edge}}) \sim E^3_{15a1}$ related to modular curves since $E_{15a1} = \text{Jac}(X_0(15))$ is the elliptic curve with LMFDB label [LMF25c]. The module structure of $H^1_{\text{ét}}(C_{\text{Edge},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ is not semisimple. It is a sum of two indecomposable dual 3-dimensional representations $M \oplus M^\vee$ where M has a composition series $M_1 \subset M$ with quotients of dimension $1, 2$. As a set, $H^1_{\text{ét}}(C_{\text{Edge},\overline{\mathbb{Q}}}, \mathbb{F}_2) \setminus \{0\}$ decomposes into 10 orbits:

orbit #	size	RA	minimal polynomial of j
1	12	$\mathbb{Z}/2$	—
2	12	$\mathbb{Z}/2$	—
3	12	$\mathbb{Z}/2$	—
4	6	$\mathbb{Z}/2$	—
5	6	$\mathbb{Z}/2$	—
6	4	D_6	$j - 54000$
7	4	S_3	$j^2 - (775180000/729)j + 40873252000000/6561$
8	3	$\mathbb{Z}/2$	—
9	3	$(\mathbb{Z}/2)^2$	$j + 11664/625$
10	1	S_3	$j^2 - (541157005431/15625)j + 25745806977673041/390625$

4.7.6 The 16-torsion congruence for $X_{17,\text{Fermat}}$

Let $\alpha \in H^1_{\text{ét}}(C_{\text{Fermat},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ be an element of the 6-orbit (label #4 in the Fermat-orbit table above). The corresponding $\text{Prym}(D_{\alpha,\overline{\mathbb{Q}}}/C_{\text{Fermat},\overline{\mathbb{Q}}})$ is only defined over $\mathbb{Q}(\sqrt{2})$ since the class α is only Galois-fixed over $\mathbb{Q}(\sqrt{2})$. The associated hyperelliptic curve H has $\text{RA}(H_{\overline{\mathbb{Q}}}) = (\mathbb{Z}/2)^2$ and its elliptic curve quotients both have $j = 128$. Furthermore, the Legendre parameter is $\rho = -i$ so the Legendre model has full rational 2-torsion over $\mathbb{Q}(i)$. We may choose a $\mathbb{Q}$-model $E_{128} = \{y^2 = x^3 + x^2 + x + 1\}$ (since we are only interested in supersingularity mod $\mathbb{F}_p$, a geometric property which at good primes is independent of the choice of model) which corresponds to LMFDB label [LMF25b]. This $\mathbb{Q}$-model does have full 2-torsion over $\mathbb{Q}(i)$. A remarkable coincidence forces the primes of supersingular reduction of E to satisfy exactly the congruence conditions necessary for other CM elliptic curves attached to the Fermat to be simultaneously supersingular. These congruences arise from the 16-torsion representation of E_{128} as follows.

Theorem 4.7.8. *Suppose that $p \in \mathbb{Z}$ is a supersingular prime of $E_{128}/\mathbb{Q}$. Then $p \equiv -1$* mod 16.

Proof. Let E denote E_{128} for the purpose of the proof. The LMFDB tabulates the images of the ℓ-adic Galois representations of E. These have full image $\text{GL}_2(\mathbb{Z}_\ell)$ except for $\ell = 2$ where the image of $\rho_{E,2} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{Z}_2)$ has index 96 with explicit generators presented in the database. In particular, the 16-torsion representation,

$$\bar{\rho}_{E,16} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{Z}/16\mathbb{Z})$$

has image $H \subset \mathrm{GL}_2(\mathbb{Z}/16\mathbb{Z})$ generated by matrices:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -3 & 6 \\ -6 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 4 \\ -6 & -3 \end{pmatrix} \quad \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}.$$

By a direct calculation, the subgroup H has the property that for all $\gamma \in H$ if $\operatorname{tr}\gamma = 0$ then $\det\gamma = -1$. In general, $\rho_{E,2}(\mathrm{Frob}_p)$ has characteristic polynomial

$$X^2 - a_p X + p$$

If p is a supersingular prime then $a_p = 0$ so $\rho_{E,2}(\mathrm{Frob}_p)$ has trace 0 and determinant p. Therefore, reducing mod 16, we see that $\bar{\rho}_{E,16}(\mathrm{Frob}_p)$ has trace 0 and hence its determinant must satisfy

$$\det\bar{\rho}_{E,16}(\mathrm{Frob}_p) = p \equiv -1 \mod 16.$$

□

This 2-torsion phenomenon has the following consequence. Let $X_{17,\mathrm{Fermat}}$ be the genus 17 curve obtained as the degree 8 étale cover $X_{17,\mathrm{Fermat}} \to C_{\mathrm{Fermat}}$ corresponding to the submodule $M_3 \subset H^1_{\text{ét}}(C_{\mathrm{Fermat},\overline{\mathbb{Q}}}, \mathbb{F}_2)$. Since C_{Fermat} has a rational point, the Brauer obstruction vanishes, and since M_3 is $G_{\mathbb{Q}}$-stable (being the unique $\mathrm{Aut}(C_{\mathrm{Fermat},\overline{\mathbb{Q}}})$-invariant 3-dimensional subspace), the cover $X_{17,\mathrm{Fermat}} \to C_{\mathrm{Fermat}}$ descends to $\mathbb{Q}$. Since M_3 is $\mathrm{Aut}(C_{\mathrm{Fermat},\overline{\mathbb{Q}}})$-invariant, there is an exact sequence

$$0 \to M_3 \to \mathrm{Aut}((X_{17,\mathrm{Fermat}})_{\overline{\mathbb{Q}}}) \to \mathrm{Aut}(C_{\mathrm{Fermat},\overline{\mathbb{Q}}}) \to 0.$$

Furthermore, since $X_{17,\mathrm{Fermat}} \to C_{\mathrm{Fermat}}$ is étale, the action $\mathrm{Aut}((X_{17,\mathrm{Fermat}})_{\overline{\mathbb{Q}}}) \circlearrowright (X_{17,\mathrm{Fermat}})_{\overline{\mathbb{Q}}}$ implies that $X_{17,\mathrm{Fermat}}$ is uniformized by the triangle group $\Delta(2,3,8)$.

Corollary 4.7.9. *The genus* 17 *curve* $X_{17,\mathrm{Fermat}}$ *has infinitely many primes of supersingular reduction.*

Proof. By Lemma 4.7.7, $\mathrm{Jac}(X_{17,\mathrm{Fermat}})_{\overline{\mathbb{Q}}}$ has an isogeny decomposition into $\mathrm{Prym}(D_\alpha/C_{\mathrm{Fermat}})_{\overline{\mathbb{Q}}}$ for $\alpha \in M_3$ and $\mathrm{Jac}(C_{\mathrm{Fermat}})_{\overline{\mathbb{Q}}} \sim E^3_{\mathbb{Q}(\sqrt{-1})}$. Furthermore, we know that for $\alpha \in M_1$ the unique nontrivial element, $\mathrm{Prym}(D_\alpha/C_{\mathrm{Fermat}})_{\overline{\mathbb{Q}}} \sim E^2_{\mathbb{Q}(\sqrt{-2})}$, and for $\alpha \in M_3 \setminus M_1$ we have $\mathrm{Prym}(D_\alpha/C_{\mathrm{Fermat}})_{\overline{\mathbb{Q}}} \sim E^2_{128}$. Since E_{128} is defined over $\mathbb{Q}$, Elkies' theorem applies to show that E_{128} has infinitely many primes of supersingular reduction. Furthermore, by Theorem 4.7.8 each of these infinitely many supersingular primes satisfies $p \equiv -1 \mod 16$.

Thus p is inert in $\mathbb{Q}(i)$ and in $\mathbb{Q}(\sqrt{-2})$ since $p \equiv -1 \mod 8$ implies that

$$\left(\frac{-2}{p}\right) = \left(\frac{2}{p}\right) \cdot \left(\frac{-1}{p}\right) = 1 \cdot (-1) = -1.$$

Hence $E_{\mathbb{Q}(\sqrt{-1})}$ and $E_{\mathbb{Q}(\sqrt{-2})}$ are both supersingular mod p and therefore $\mathrm{Jac}(X_{17,\mathrm{Fermat}})$ is supersingular mod p. □

Since Elkies' theorem gives no control on the residue classes of the supersingular primes it produces, usually one cannot force the curve to have infinitely many primes of supersingular reduction in the correct arithmetic progressions to force some other CM curves to concurrently also have supersingular reduction. The amazing coincidence that occurs for $X_{17,\mathrm{Fermat}}$ is that the 16-torsion representation forces the correct congruences to always hold. We will see a similar congruence coincidence for the Klein quartic but unfortunately not for the classes that form a subrepresentation of the automorphism group.

4.7.7 New counterexamples to the Shioda conjecture

Let $C := (X_{17,\mathrm{Fermat}})_{\overline{\mathbb{Q}}}$ be the genus 17 curve constructed in the previous section. Let $X \to (C \times C)/G$ be the product-quotient surface where $G \cong D_6$ via the unique embedding $D_6 \hookrightarrow G_{768} := \mathrm{Aut}(C)$ up to conjugacy. Here G_{768} is a non-split extension

$$0 \to (\mathbb{Z}/2\mathbb{Z})^3 \to G_{768} \to (\mathbb{Z}/4\mathbb{Z})^2 \rtimes S_3 \to 0.$$

In the GAP Small Groups library, this group is `SmallGroup(768, 1085341)`. It is easy to compute that $\pi_1(X) = 1$ directly. Using the method described in §3.3.4 of the previous chapter, $Y = (C \times C)/G$ has the following basket of singularities

$$\mathscr{B}(Y) := \{64 \times A_1, 2 \times A_2, 2 \times \tfrac{1}{3}(1,1)\}.$$

Plugging into Proposition 3.3.5 gives Chern numbers:

$$K_X^2 = 170 \qquad c_2(X) = 190$$

and using Corollary 3.4.6:

$$(K_X - E)^2 = 170 - 2 \cdot 64 - 2 \cdot 2 - 5 \cdot 2 = 28$$

is positive and hence $\omega_X(-E)$ is big. This surface has Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 29 & & 130 & & 29 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and is of general type. We check these facts and the positivity of $(K_X - E)^2$ for all partial twists in the Magma script [Chu25, `examples/group768.m`]. This gives a new counterexample to the Shioda conjecture at the infinitely many supersingular primes of C. The first few such primes are

$$31, 223, 383, 479, 2591, 5279, 8191, \ldots.$$

4.7.8 The 4-torsion obstruction for $X_{9,\mathrm{Klein}}$

The two Hurwitz curves X_{17} of genus 17 are constructed from the $\mathrm{Aut}(C_{\mathrm{twist},\overline{\mathbb{Q}}}) \cong \mathrm{PSL}_2(\mathbb{F}_7)$ submodules $M, M^\vee$ in the decomposition $H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2) = M \oplus M^\vee$. Therefore, $\mathrm{Jac}(X_{17})$ has an isogeny decomposition into $\mathrm{Jac}(C_{\mathrm{twist}})$ and copies of $\mathrm{Prym}(D_\alpha/C_{\mathrm{twist}})$ for $\alpha \in H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ in orbit #3 or #4 by applying Lemma 4.7.7. Unfortunately, since $j_{3,4}$ lives in a totally complex number field we don't have the techniques at our disposal to prove that $E_{j_{3,4}}$ has infinitely many primes of supersingular reduction. This curve does define a $\mathbb{Q}(\sqrt{-7})$-point of the modular curve $X_0^+(3) := X_0(3)/w_3$ due to the 3-isogeny between its conjugates over $\mathbb{Q}(\sqrt{-7})$ but unfortunately this is an imaginary quadratic point so we cannot apply the results of [Jao05].

However, orbit #1 corresponds to a Prym whose elliptic factors have real j-invariant so Elkies' theorem immediately applies. Moreover, there are three classes $\alpha_1, \alpha_2, \alpha_3 \in H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ lying in orbit #1 that are *collinear* in the sense that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Indeed, recall that we identified orbit #1 with anti-flags on the Fano plane. Thus α_i corresponds to a pair (P_i, ℓ_i) with $P_i \notin \ell_i$. The relation $\alpha_1 + \alpha_2 + \alpha_3 = 0$ says that P_1, P_2, P_3 are collinear and ℓ_1, ℓ_2, ℓ_3 are concurrent.

The practical consequence is that the $\mathbb{F}_2$-vector space

$$V := \langle \alpha_1, \alpha_2 \rangle \subset H^1_{\text{ét}}(C_{\mathrm{twist},\overline{\mathbb{Q}}}, \mathbb{F}_2)$$

consists entirely of classes of orbit #1 or the trivial orbit. These classes are all defined by bitangent lines over the field $\mathbb{Q}(\sqrt{-7})$. Therefore, we may define $X_{9,\mathrm{Klein}} \to C_{\mathrm{twist},\mathbb{Q}(\sqrt{-7})}$ to be the étale cover defined by V with deck transformation group $V^\vee$. Then Lemma 4.7.7

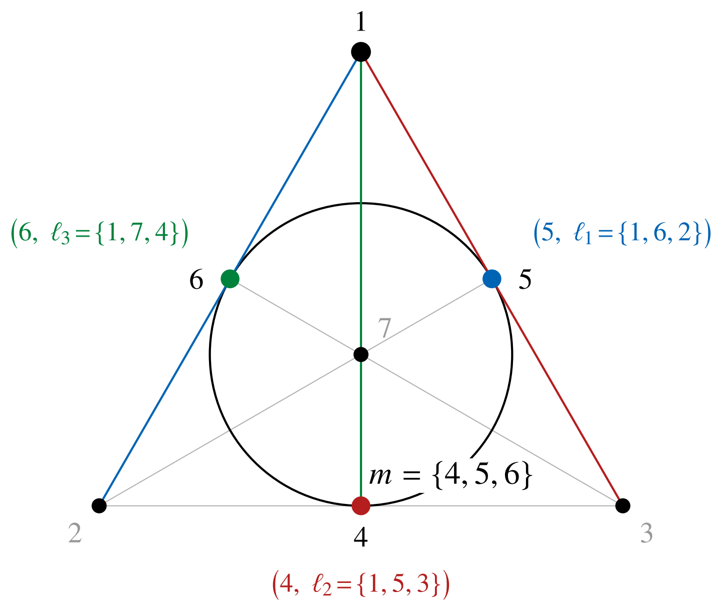


Figure 4.1: Collinear triple of anti-flags in $J[2] \cong M \oplus M^\vee$. Three points collinear on the line $m = \{4,5,6\}$ meaning $P_4 + P_5 + P_6 = 0$ in M. Three lines concurrent through P_1 meaning $\ell_1 + \ell_2 + \ell_3 = 0$ in $M^\vee$. Each point is colored to match its paired line ($P_i \notin \ell_i$): $4 \leftrightarrow \ell_2 = \{1,5,3\}$, $5 \leftrightarrow \ell_1 = \{1,6,2\}$, $6 \leftrightarrow \ell_3 = \{1,7,4\}$.

gives a decomposition

$$\mathrm{Jac}(X_{9,\mathrm{Klein}})_{\overline{\mathbb{Q}}} \sim \mathrm{Jac}(C_{\mathrm{twist}})_{\overline{\mathbb{Q}}} \times \prod_{\alpha \in V \setminus 0} \mathrm{Prym}(D_\alpha / C_{\mathrm{twist}})_{\overline{\mathbb{Q}}} \sim E_{j_1}^6 \times E^3_{\mathbb{Q}(\sqrt{-7})}.$$

Another remarkable coincidence allows us to conclude $X_{9,\mathrm{Klein}}$ has infinitely many primes of supersingular reduction. The coincidence concerns congruence classes of rational primes above which E_{j_1} is supersingular. Let E denote the particular $\mathbb{Q}(\sqrt{7})$-model with j-invariant j_1 labeled [LMF25d] in the LMFDB. This curve E is associated to the weight 2 newform $f \in S_2^{\mathrm{new}}(112, \chi_f)$ recorded in the LMFDB as [LMF25m] in the sense that the abelian surface $A = \mathrm{Res}_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{7})} E$ over $\mathbb{Q}$ satisfies $\rho_{A,\ell} \cong \rho_{f,\ell}$. The nebentypus character of f is the quadratic character $\chi_f(\bullet) = \left(\frac{28}{\bullet}\right)$. Furthermore, the coefficient field of f is $K_f = \mathbb{Q}(\zeta_3)$ in which 2 is inert. Therefore, since the $\mathbb{Q}(\zeta_3)$-multiplication implies that E is isogenous to its Galois conjugate, we see that E is supersingular at $\mathfrak{p} \mid p$ iff A is supersingular at p iff[1] $a_p(f) = 0$.

Theorem 4.7.10. *Let p be an odd prime split in $\mathbb{Q}(\sqrt{-7})$. Then $a_p(f) \neq 0$.*

[1]The equivalence is only for p sufficiently large, see [BG97, Proposition 5.2] and [Wan24, Lemma 4]. However the reverse implication $a_p(f) = 0$ implies A is supersingular at p holds for all primes of good reduction.

This congruence restriction on primes p such that $a_p(f) = 0$ gives our main result.

Corollary 4.7.11. $X_{9,\mathrm{Klein}}$ *has infinitely many primes of supersingular reduction.*

Proof. Since $E_{j_1}/\mathbb{Q}(\sqrt{7})$ has real j-invariant, Elkies' theorem implies it has infinitely many primes of supersingular reduction i.e. there are infinitely many rational primes for which $a_p(f) = 0$. By the theorem, each such p must be inert in $\mathbb{Q}(\sqrt{-7})$ and thus is automatically a prime of supersingular reduction of the elliptic curve $E_{\mathbb{Q}(\sqrt{-7})}/\mathbb{Q}$ since $E_{\mathbb{Q}(\sqrt{-7})}$ has CM by $\mathbb{Q}(\sqrt{-7})$. Since $\mathrm{Jac}(X_{9,\mathrm{Klein}})_{\overline{\mathbb{Q}}} \sim E_{j_1}^6 \times E^3_{\mathbb{Q}(\sqrt{-7})}$ geometrically, this implies that every prime of good reduction lying over the infinitely many rational primes with $a_p(f) = 0$ gives supersingular reduction for $X_{9,\mathrm{Klein}}$. □

To prove the theorem, we consider the 4-torsion Galois representation of A,

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r, "\rho_{A,2}"'] \arrow[rr, bend left, "\bar{\rho}_{A,4}"] & \mathrm{GSp}_4(\mathbb{Z}_2) \arrow[r] & \mathrm{GSp}_4(\mathbb{Z}/4\mathbb{Z}) \arrow[r] & \mathrm{GSp}_4(\mathbb{F}_2) \end{tikzcd}$$

We write $K = \mathbb{Q}(\sqrt{7})$ and $L = \mathbb{Q}(\sqrt{-7})$. The 4-torsion representation of $A = \mathrm{Res}_{K/\mathbb{Q}} E$ which equals

$$\bar{\rho}_{A,4} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \bar{\rho}_{E,4}$$

satisfies the identity:

$$\bar{\rho}_{A,4}|_{G_L} \cong \mathrm{Ind}_{G_{KL}}^{G_L} \left(\bar{\rho}_{E,4}|_{G_{KL}}\right).$$

This follows from applying the Mackey restriction formula

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G V \cong \bigoplus_{s \in K\backslash G/H} \mathrm{Ind}^K_{K\cap\sigma H\sigma^{-1}} \mathrm{Res}^{\sigma H\sigma^{-1}}_{K\cap\sigma H\sigma^{-1}} V^{\sigma}$$

and using $G_K \cdot G_L = G_{\mathbb{Q}}$ with G_K and G_L normal. We will see that $\bar{\rho}_{E,4}|_{G_{KL}}$ has exponent 2 meaning its image consists of elements of order dividing 2. In fact, we will need the stronger claim that $\bar{\rho}_{A,4}|_{G_L}$ has exponent 2, a fact that does not follow from the structure of the induced representation. Indeed, it requires some additional non-trivial information about how G_L interacts with $\bar{\rho}_{E,4}|_{G_{KL}}$.

Proof of Theorem 4.7.10. Recall that $\rho_{f,\ell}(\mathrm{Frob}_p)$ has characteristic polynomial

$$T^2 - a_p(f)T + p\,\chi_f(p).$$

Therefore, if $a_p(f) = 0$ then $\rho_{f,2}(\mathrm{Frob}_p^2)$ is the scalar matrix $-p\,\chi_f(p)$. In particular, it is also scalar under the isomorphism $\rho_{f,2} \cong \rho_{A,2}$ and hence on 4-torsion we see that $\bar{\rho}_{A,4}(\mathrm{Frob}_p^2)$ is also the scalar matrix $-p\,\chi_f(p) \mod 4$.

We compute the number field $K_4 = K(E[4])$ trivializing the 4-torsion $E[4]$ (computed in Magma from the splitting field of the 4-division polynomial of E). The number field K_4 is Galois over $\mathbb{Q}$ with Galois group $D_4 \times C_2$ and is recorded as [LMF25n] in the LMFDB. Some of this is easily explained from the structure of E. For example, E and its Galois conjugate are 3-isogenous so their 4-torsion representations are isomorphic. This implies that $G_{K_4} \subset G_{\mathbb{Q}}$ is normal i.e. $K_4/\mathbb{Q}$ is Galois. There are two important quadratic subfields of K_4

$$\begin{tikzcd} & K_4 \arrow[dl, dash, "D_4"'] \arrow[dr, dash, "C_2^3"] \arrow[dd, dash, "D_4\times C_2"'] & \\ \mathbb{Q}(\sqrt{7}) \arrow[dr, dash] & & \mathbb{Q}(\sqrt{-7}) \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

where we have marked the extensions by their Galois groups. In fact, the only essential information we need is that $\mathrm{Gal}(K_4/L)$ has exponent 2; a fact we will now justify directly. Recall that E, arising from a Legendre family, has full rational 2-torsion over $\mathbb{Q}(\rho) = \mathbb{Q}(\sqrt{7}, i) = KL$. Then the diagram

$$\begin{tikzcd} & G_{KL} \arrow[r, hook] \arrow[d, dashed] & G_K \arrow[d, "\bar{\rho}_{E,4}"] \arrow[dr, "\bar{\rho}_{E,2}"] & & \\ 0 \arrow[r] & (\mathbb{Z}/2\mathbb{Z})^4 \arrow[r] & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \arrow[r] & \mathrm{GL}_2(\mathbb{F}_2) \arrow[r] & 0 \end{tikzcd}$$

proves that the representation $\bar{\rho}_{E,4}|_{G_{KL}}$ has exponent 2. We get an extension,

$$1 \to \mathrm{Gal}(K_4/KL) \to \mathrm{Gal}(K_4/\mathbb{Q}) \to \mathrm{Gal}(KL/\mathbb{Q}) \to 1$$

which is an extension of C_2^2 by C_2^2. This gives an action

$$\mathrm{Gal}(KL/\mathbb{Q}) \to \mathrm{Aut}(\mathrm{Gal}(K_4/KL)) \cong \mathrm{GL}_2(\mathbb{F}_2) \cong S_3$$

but since $\mathrm{Gal}(KL/\mathbb{Q}) \cong C_2^2$ this has a nontrivial kernel. The kernel corresponds to a quadratic subfield $S \subset KL$ so that $\mathrm{Gal}(K_4/S) \cong C_2^3$. To show $S = L$, it suffices to exhibit order 4 elements of $\mathrm{Gal}(K_4/K)$ and $\mathrm{Gal}(K_4/\mathbb{Q}(i))$ since those fields are the only other possibilities. We will do this by inspecting the Fourier coefficients of f.

Let $T = \rho_{A,2}(\mathrm{Frob}_3)$. The characteristic polynomial of $\rho_{f,2}(\mathrm{Frob}_3)$ over K_f gives

$$T^2 - a_3(f)T + 3 \cdot \chi_f(3) = 0 \implies T^2 - 2T + 3 = 0.$$

Since this identity is conjugation invariant, it also holds for $\rho_{A,2}(\mathrm{Frob}_3)$ and thus must also be satisfied by $\bar{\rho}_{A,4}(\mathrm{Frob}_3)$ in $\mathrm{GL}_4(\mathbb{Z}/4\mathbb{Z})$. Such elements of $\mathrm{GL}_4(\mathbb{Z}/4\mathbb{Z})$ all have order 4 so $\bar{\rho}_{A,4}(\mathrm{Frob}_3)$ has order 4. Since Frob_3 generates $\mathrm{Gal}(KL/K)$ we rule out K. We can similarly rule out $\mathbb{Q}(i)$.

Thus $H := \mathrm{Gal}(K_4/K) \cong D_4$ and $N := \mathrm{Gal}(K_4/KL)$ cannot be central in H otherwise H would be abelian. Since $G = \mathrm{Gal}(K_4/\mathbb{Q})$ is a 2-group, there is a central involution $z \in Z(G) \setminus H$ and let $c \in H \setminus N$ be complex conjugation. These commuting involutions give a section of $\mathrm{Gal}(K_4/\mathbb{Q}) \to \mathrm{Gal}(KL/\mathbb{Q})$. In particular, the extension

$$0 \to \mathrm{Gal}(K_4/L) \to \mathrm{Gal}(K_4/\mathbb{Q}) \to \mathrm{Gal}(L/\mathbb{Q}) \to 0$$

is trivial so $\mathrm{Gal}(K_4/\mathbb{Q}) \cong C_2^3$.

There is a factorization,

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r, "\bar{\rho}_{A,4}"] \arrow[d, two heads] & \mathrm{GSp}_4(\mathbb{Z}/4\mathbb{Z}) \\ \mathrm{Gal}(K_4/\mathbb{Q}) \arrow[ur] & \end{tikzcd}$$

where $A[4]$ is trivial over K_4 because $K_4 \supset K$ and K_4 is Galois over $\mathbb{Q}$ so $A[4](K_4) = E[4](K_4)^{\oplus 2} = (\mathbb{Z}/4\mathbb{Z})^{\oplus 4}$ is full. Since $\mathrm{Gal}(K_4/L)$ has exponent 2 so does $\bar{\rho}_{A,4}|_{G_L}$.

Now suppose that p is split in $L = \mathbb{Q}(\sqrt{-7})$. This means $\mathrm{Frob}_p \in G_L \subset G_{\mathbb{Q}}$ identified with $\mathrm{Frob}_{\mathfrak{p}}$ for either $\mathfrak{p} \mid p$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ (these are conjugate in $G_{\mathbb{Q}}$ but not in G_L). However, $\mathrm{Gal}(K_4/L)$ has exponent 2 and therefore Frob_p^2 is trivial in $\mathrm{Gal}(K_4/\mathbb{Q})$. In particular $\bar{\rho}_{A,4}(\mathrm{Frob}_p^2) = I$ but we also showed that $\rho_{A,4}(\mathrm{Frob}_p^2)$ is the scalar matrix $-p\,\chi_f(p)$. Hence we conclude that

$$-p\,\chi_f(p) \equiv 1 \mod 4.$$

However, p is split in $\mathbb{Q}(\sqrt{-7})$ so $\left(\frac{-7}{p}\right) = 1$ and thus

$$\chi_f(p) = \left(\frac{28}{p}\right) = \left(\frac{4}{p}\right)\left(\frac{-1}{p}\right)\left(\frac{-7}{p}\right) = \left(\frac{-1}{p}\right)$$

because p is odd. Therefore,

$$p \equiv -\left(\frac{-1}{p}\right) \mod 4$$

which is impossible. □

Remark 4.7.12. Warning: the $\mathbb{Z}_2$-lattice arising from the Tate module of A does not agree with any lattice corresponding to an $\mathcal{O}_{K_f,2}$-integral structure of $\rho_{f,2}$. Indeed, otherwise

$\bar{\rho}_{A,2}$ would factor through $\mathrm{GL}_2(\mathbb{F}_4) \subset \mathrm{GL}_4(\mathbb{F}_2)$ since 2 is inert in $K_f = \mathbb{Q}(\zeta_3)$ but $A[2](\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})$ is not an $\mathbb{F}_4$-vector space. This is witnessing the fact that ζ_3 does not act on A, only the order $\mathbb{Z}[\sqrt{-3}]$ inside $\mathbb{Z}[\zeta_3]$ does.

4.8 Some interesting curves of genus 5

In the previous sections, we constructed interesting curves of genus 17 and 9 with infinitely many primes of supersingular reduction using the Fermat and Klein quartics respectively. Here we consider curves constructed from covers of the Edge quartic. The Edge quartic has the intriguing property that all its 2-torsion covers are defined over $\mathbb{Q}$ and thus it will be easier to apply results in the spirit of Elkies' theorem to conclude the infinitude of supersingular primes for the requisite elliptic curves. On the other hand, the Edge quartic has markedly less symmetry than the previous cases. This is manifest in having a non-CM Jacobian and far more different orbits to consider. Unfortunately, we are unable to prove that any of the genus 5 covers of C_{Edge} have infinitely many primes of supersingular reduction. It remains to leave a breadcrumb trail that some traveler might stumble down again.

Conjecture 4.8.1. *For each of the classes $\alpha \in H^1_{\text{ét}}(C_{\mathrm{Edge},\overline{\mathbb{Q}}}, \mathbb{F}_2)$ in orbits #6, 7, 9, 10, the genus 5 curve D_α (which is defined over $\mathbb{Q}$) has infinitely many primes of supersingular reduction.*

The most promising candidate is orbit #6 where $\mathrm{Jac}(D_\alpha)_{\overline{\mathbb{Q}}} \sim E^2_{\mathbb{Q}(\sqrt{-3})} \times E^3_{15a1}$. This curve is supersingular at the primes p of E_{15a1} at which $p \equiv -1 \mod 3$. Unfortunately, I do not see any way to rule out that all but finitely many of the supersingular primes of E_{15a1} happen to be 1 mod 3. The other three orbits #7, 9, 10 are more questionable. Here the Jacobian splits as $\mathrm{Jac}(D_\alpha)_{\overline{\mathbb{Q}}} \sim E^2 \times E^3_{15a1}$ where E is non-CM. However, in all three cases E has real j-invariant so we know it has infinitely many primes of supersingular reduction. The naive heuristic then shows that $\mathrm{Jac}(D_\alpha)$ should also have infinitely many supersingular primes although they are very sparse growing like $\sim \log\log x$ for $p \leq x$. So give it up, Sub-Subs! For by how much the more pains ye take to please the world, by so much the more shall ye for ever go thankless!

Chapter 5

Foliated Descent in General

> "Here is the Labyrinth," Albert said, gesturing towards a tall lacquered writing cabinet. "An ivory labyrinth!" I exclaimed. "A very small sort of labyrinth ..." "A labyrinth of symbols," he corrected me.
>
> Jorge Luis Borges

In the previous chapter, we developed an argument to show the non-uniruledness of product-quotient surfaces carrying a particular kind of symmetric form. The main technique is the principle of infinite descent. Here we will explore further applications of this technique to the unirationality of surfaces in positive characteristic. In particular, we will develop something of a general criterion, at least a sort of infinite algorithm, which in certain special cases will yield the desired answer to the question: is X covered by rational curves?

The descent argument has three main phases that must be understood in order to run such a procedure. These are:

1. a mechanism to force rational curves to lie tangent to a foliation, which is provided by a certain symmetric differential form

2. a degree bound which, assuming the foliation is p-closed so that the family of rational curves descends to the quotient, provides a discrete invariant of the family that decreases at each step thereby setting up a contradiction by infinite descent

3. a certification that the mechanism may be repeated to each surface appearing as a quotient in order that the degree bound may be invoked *ad infinitum*.

In this chapter, we will focus on the first and second in some generality. This will leave the third and most difficult stage to subsequent chapters in the case of Hilbert modular varieties and to further study in the future.

5.1 Descent using an arbitrary symmetric form

Let X be a smooth projective surface over a field k of positive characteristic. Given that X carries a symmetric differential form $\omega \in H^0(X, \operatorname{Sym}^m \Omega_X)$ for some $m > 0$ we would like to argue that any rational curve on X must lie tangent to some foliation as in the previous chapter. However, this is not true in general; first we must replace X by a suitable cover "extracting a linear differential equation" from the nonlinear differential equation induced by ω. This is done analogously to the arguments in [Bog77] and is reproduced as follows.

Consider the projective bundle $\pi : \mathbb{P}_X(\Omega_X) \to X$ and note that $\omega \in H^0(X, \operatorname{Sym}^m \Omega_X)$ defines a section of the tautological line bundle $s_\omega \in H^0(\mathbb{P}_X(\Omega_X), \mathscr{O}_{\mathbb{P}}(m))$ on the 3-fold $\mathbb{P} := \mathbb{P}_X(\Omega_X)$. Therefore, the vanishing locus $Z := Z(s_\omega)$ is a divisor in $\mathbb{P}$ whose support is the locus of pairs $(x, v) \in \mathbb{P}$ where $\omega_x(v) = 0$. We replace Z by its reduction and decompose Z into two collections of irreducible components: those that dominate X and those that are contracted onto a curve inside X. Denote the union of these by Z_{hor} and Z_{cont} respectively.

For any generically immersive map $f : C \to X$ from a smooth projective curve to X, there is a unique "holonomic" lift $t_f : C \to \mathbb{P}_X(\Omega_X)$ defined by the map of bundles

$$f^* : f^*\Omega_X \twoheadrightarrow \operatorname{im}(f^*) \subset \omega_C$$

where $\operatorname{im} f^*$ is torsion-free of rank 1 and hence a line bundle. This defines t_f in such a way that $t_f^*\mathscr{O}_{\mathbb{P}}(1) = \operatorname{im} f^*$. At the points $t \in C$ where f is an immersion, the map t_f is defined by $s \mapsto (f(s), \mathrm{d}f_s(\partial_s))$, viewing $\mathbb{P} := \mathbb{P}(\Omega_X)$ as the projectivization of the physical tangent bundle. Since C is a smooth projective curve and $\mathbb{P}_X(\Omega_X)$ is proper, the extension over the points where f is not an immersion exists uniquely.

Now if $f : C \to X$ is any curve such that $f^*\omega = 0$, we see that the canonical lift $\widetilde{f} := t_f : C \to \mathbb{P}$ lands inside Z giving a diagram:

$$\begin{tikzcd} & Z \arrow[r, hook] & \mathbb{P} \arrow[d, "\pi"] \\ C \arrow[ur, dashed, "\widetilde{f}"] \arrow[rr, "f"] & & X \end{tikzcd}$$

Now we will show that any covering family of X by genus ≤ 1 curves induces a covering family of some component of Z_{hor} by genus ≤ 1 curves that are furthermore tangent to an induced foliation.

Indeed, if C has genus 0 then because $\mathbb{P}^1$ carries no nonzero forms we must have that $f^*\omega = 0$. By definition, this means $t_f : \mathbb{P}^1 \to \mathbb{P}$ factors through Z. Similarly, for a genus 1 curve C and a map $f : C \to X$, either $f^*\omega$ is zero identically or is zero nowhere. Hence, if $V(\omega) = \{x \in X \mid \omega_x = 0\} \subset X$ contains a big divisor then at most finitely many curves may not meet $V(\omega)$. Except for a finite number of exceptions, for each genus 1 curve, $f^*\omega$ is therefore zero. For future convenience, we will use the following shorthand for the assumptions necessary to run the argument for genus 0 or genus 1 curves:

$(*)_0$ X carries a nonzero symmetric differential form $\omega \in H^0(X, \operatorname{Sym}^m \Omega_X)$

$(*)_1$ moreover we may choose ω so that $V(\omega) \subset X$ contains a big divisor

Since $\dim X = 2$, the divisor Z is a union of surfaces. Let $S \to Z$ be a resolution of singularities of a component of Z dominating X. Except for finitely many rational or elliptic curves – namely those which equal the image in X of a component of Z or which do not meet $V(\omega)$ – all rational or elliptic curves on X lift to a map $\widetilde{f} : C \to S$. Let $g : S \to X$ be the induced morphism. The quotient map $\pi^*\Omega_X \twoheadrightarrow \mathscr{O}_{\mathbb{P}}(1)$ defines a foliation on each component of Z.

Definition 5.1.1. The *induced holonomic foliation* on S is defined as follows. Consider the pushout diagram,

$$\begin{tikzcd} g^*\Omega_X \arrow[r, two heads] \arrow[d, hook] & \iota^*\mathscr{O}_{\mathbb{P}}(1) \arrow[d, hook] \\ \Omega_S \arrow[r, two heads] & \mathscr{Q} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $\iota : S \to \mathbb{P}$ is the obvious map. Since $\Omega_S \twoheadrightarrow \mathscr{Q}$ is surjective, its dual $\mathscr{F} \hookrightarrow \mathscr{T}_S$, with $\mathscr{F} = \mathscr{Q}^\vee$, is a saturated reflexive subsheaf of generic rank 1. Because $\mathscr{F}$ has rank 1 it is automatic that $[\mathscr{F}, \mathscr{F}] \subset \mathscr{F}$ so it defines a foliation.

Lemma 5.1.2. *Let $f : C \to X$ be a map from a smooth proper curve C such that $f^*\omega = 0$. Then the lift $\widetilde{f} : C \to S$ is a leaf of $\mathscr{F}$.*

Proof. The lifted map $t_f : C \to \mathbb{P}$ is induced by the factorization

$$f^*\Omega_X \to t_f^*\mathscr{O}_{\mathbb{P}}(1) \to \omega_C$$

using that t_f factors through S the following diagram shows that $\widetilde{f}^*\Omega_S \to \omega_C$ factors through $\widetilde{f}^*\Omega_S \twoheadrightarrow \widetilde{f}^*\mathscr{Q}$,

$$\begin{tikzcd}
\widetilde{f}^* g^* \Omega_X \arrow[r] \arrow[d] & \widetilde{f}^* \iota^* \mathscr{O}_{\mathbb{P}}(1) \arrow[d] \arrow[ddr, bend left] & \\
\widetilde{f}^* \Omega_S \arrow[r] \arrow[drr, bend right] & \widetilde{f}^* \mathscr{Q} \arrow[dr, dashed] & \\
& & \omega_C
\end{tikzcd}$$

□

5.1.1 The formulation in families

It will also be useful to understand how the above argument on a single immersed curve behaves in families. It may seem that, by studying the entire covering family, one loses the advantage conferred by setting aside a unirational parametrization $\mathbb{P}^2 \dashrightarrow X$ – which may be inseparable – in favor of a single generically chosen rational curve $f_t : \mathbb{P}^1 \to X$ which can always be replaced by an immersion but this is not so. Indeed, suppose that we are given a covering family (meaning just that f is dominant and generically finite),

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "f"] \arrow[d, "g"] & X \\
T &
\end{tikzcd}$$

by smooth curves parametrized by a smooth irreducible base T. Suppose that f is inseparable. This cannot be rectified by simply factoring out some iterate of Frobenius as could be done for each curve individually. However, we can always replace f by one whose general fiber $f_t : \mathscr{C}_t \to X$ is a generic immersion. Indeed, consider the cotangent maps

$$f^* \Omega_X \to \Omega_{\mathscr{C}} \to \Omega_{\mathscr{C}/T}$$

If this map is identically zero then $\mathrm{d}f : \mathscr{T}_{\mathscr{C}} \to f^* \mathscr{T}_X$ kills the foliation $\mathscr{T}_{\mathscr{C}/T} \subset \mathscr{T}_{\mathscr{C}}$ induced by g and hence must factor through the relative Frobenius of the map g:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F_{\mathscr{C}/T}"'] \arrow[rr, bend left, "f"] \arrow[d, "g"] & \mathscr{C}^{(p/T)} \arrow[r, dashed, "f'"'] \arrow[d, "g^{(p)}"] & X \\
T \arrow[r, equal] & T &
\end{tikzcd}$$

Let $\mathscr{C}' := \mathscr{C}^{(p/T)}$. Now we may repeat this argument with f' until

$$f'^* \Omega_X \to \Omega_{\mathscr{C}'} \to \Omega_{\mathscr{C}'/T}$$

is nonzero. This must happen after finitely many steps since $\deg f' = \frac{1}{p}\deg f$ and $\deg f$ is finite. After the factorization terminates, the pullback map is nonzero. Therefore, restricting to the general fiber, $f_t'^*\Omega_X \to \Omega_{\mathscr{C}'_t}$ is nonzero hence surjective at the generic point so $f'_t : \mathscr{C}'_t \to X$ is generically immersive.

Furthermore, the map

$$f'^*\Omega_X \to \Omega_{\mathscr{C}'} \to \Omega_{\mathscr{C}'/T}$$

induces a rational map

$$\widetilde{f'} : \mathscr{C}' \dashrightarrow \mathbb{P}$$

whose image lies inside $Z \subset \mathbb{P}$ by the argument above. Since f' dominates X, the image of this lift cannot lie inside Z_{cont} and must lie in some component of Z_{hor} since T (and hence $\mathscr{C}$) is irreducible. Therefore, choosing a resolution of singularities $S \to Z_i \subset Z_{\text{hor}}$ of this component we obtain a rational map $\widetilde{f'} : \mathscr{C}' \to S$. This map is defined away from codimension 2 on $\mathscr{C}'$ since $\mathscr{C}'$ is nonsingular and S is proper. This bad locus cannot dominate T so by shrinking T we may assume that $\widetilde{f'} : \mathscr{C}' \to S$ is a morphism. Likewise, as before, this is a family of leaves of the induced foliation on S.

5.2 Descent I: the algorithm

Now we describe the strategy to prove that X is not uniruled assuming $(*)_0$ and not covered by genus 1 curves assuming $(*)_1$. We denote the running assumption by $(*)_g$. Suppose that X had a covering family,

$$\begin{tikzcd} \mathscr{C} \arrow[r, "f"] \arrow[d, "g"] & X \\ T & \end{tikzcd}$$

by curves of genus 0 or 1. As before we may assume that f is generically immersive on fibers. Let $S_1, \ldots, S_r \to \mathbb{P}$ denote resolutions of the components of Z_{hor}. Since f dominates X it lifts to one of $S_1, \ldots, S_r$ and the lift consists of curves tangent to the foliation $\mathscr{F}_i$ on S_i. If its p-curvature $\psi_{p,\mathscr{F}_i}$ is nonzero, then the image of $\widetilde{f}$ lies inside $V(\psi_{p,\mathscr{F}_i}) \subset S_i$ which does not dominate X. Therefore, f must lift to a component S_i on which $\psi_{p,\mathscr{F}_i} = 0$ hence we may form the quotient. By Lemma 2.3.3 the family $\widetilde{f} : \mathscr{C} \to S_i$ is generically pulled back from a family of curves on $S_i/\mathscr{F}_i$ in the sense that there exists a diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F_{\mathscr{C}/T}"] \arrow[d, "\widetilde{f}"'] & \mathscr{C}^{(p/T)} \arrow[d, "\widetilde{f}'"] \\ S_i \arrow[r] & S_i/\mathscr{F}_i \end{tikzcd}$$

Eventually, the existence of such a "descent to $S_i/\mathscr{F}$" will force the "degree" of $\widetilde{f}$ to decrease and therefore we will reach a contradiction.

For each quotient $S_i/\mathscr{F}_i$ we choose a resolution of singularities $X_i' \to S_i/\mathscr{F}_i$. The process may continue to run as long as X_i' also satisfies $(*)_g$. Assuming this is true, we can form a growing tree:

$$\begin{tikzcd}[column sep=small] & \{S_i\}_{i|\psi_{p,\mathscr{F}_i}=0} \arrow[dl] \arrow[dr] & & & \{S_{i,j}\}_{j|\psi_{p,\mathscr{F}_{i,j}}=0} \arrow[dl] \arrow[dr] & \\ X & & \{S_i/\mathscr{F}_i\} & \{X_i\} \arrow[l] & & S_{i,j}/\mathscr{F}_{i,j} & \cdots \arrow[l] \end{tikzcd}$$

We say *the algorithm terminates successfully* at step k if the roof collection $\{S_{i_1,\ldots,i_k}\}$ is empty meaning that the foliations on $S_{i_1,\ldots,i_k}$ each have a nontrivial p-curvature. Since in this case, every genus $\leq g$ curve on X must lie in the finite collection of curves $V(\psi_{p,\mathscr{F}_{i_1,\ldots,i_k}})$ (or rather their images under the various roofs) we have proven the following.

Theorem 5.2.1. *If the algorithm terminates successfully then X is not uniruled.*

The author has not written down an example where the algorithm terminates successfully, but since most foliations are not p-closed, it is likely such examples exist.

We say *the algorithm terminates inconclusively* if assumption $(*)_g$ fails to hold for some $X_{i_1,\ldots,i_k}$ meaning that one of the quotients by a foliation has no smooth model carrying a symmetric form (or in the case $g = 1$ each symmetric form does not contain a big divisor in its vanishing locus). As we will see, the successful cases are when the algorithm either terminates successfully at step k or can be proven to run indefinitely. We will see in the next section how, in this latter case, descent can be performed.

5.3 Descent II: the degree bound

Let us assume the algorithm of the previous section does not terminate. Explicitly, this means the tree formed by taking successive roofs of the form

$$\begin{tikzcd} & \{S_i\}_{i|\psi_{p,\mathscr{F}_i}=0} \arrow[dl] \arrow[dr] & & \\ X & & \{S_i/\mathscr{F}_i\} & \{X_i\} \arrow[l] \end{tikzcd}$$

is infinite in size and therefore has at least one infinite branch by König's lemma. We specialize to this infinite branch to produce a sequence of roofs

$$
\begin{tikzcd}
 & S^{(0)} \arrow[dl] \arrow[dr] & & & S^{(1)} \arrow[dl] \arrow[dr] & & \\
X^{(0)} & & S^{(0)}/\mathscr{F}^{(0)} & X^{(1)} \arrow[l] & & S^{(1)}/\mathscr{F}^{(1)} & \cdots \arrow[l]
\end{tikzcd}
\tag{5.1}
$$

where the sequence of smooth projective surfaces $S^{(i)}$ carry p-closed foliations $\mathscr{F}^{(i)}$. Starting with a covering family of $X = X^{(0)}$, we must show it does not descend to a covering family arbitrarily far down this infinite branch. Indeed, if we can show this for every infinite branch we will obtain a contradiction because any covering family of X lifts to a covering family of some S_i and then descends down some roof. This process selects an infinite branch of the tree down which the covering family descends.

To reach a contradiction, we wish to show that some measure of "degree" of the covering family strictly decreases upon descending past each roof. If this measure is valued in a well-ordered set (e.g. in the positive integers) then the covering family may only descend a finite number of steps down the branch.

A natural candidate for this measure is the *canonical degree*:

$$\deg_{\mathscr{C}_t} f_t^* \omega_X = C \cdot f^* K_X$$

where C is the class of a fiber of $g : \mathscr{C} \to T$. There are a number of issues that need to be overcome in order to prove this is a good measure in the sense that it strictly decreases after descent down a roof. We turn our attention now to these issues.

5.3.1 Replacing X to modify the covering family

The most important obstacle is that $S/\mathscr{F}$ may be highly singular along the locus where $\mathscr{F}$ is not a subbundle of $\mathscr{T}_S$. If the general member $f_t : \mathscr{C}_t \to S$ passes through this locus then we face a series of escalating issues: (1) we will not be able to control $\deg f_t^* K_S$ in terms of $\deg f_t'^* K_{S/\mathscr{F}}$ due to contributions from the term $f_t^* c_1(\mathscr{F})$, (2) the canonical degree of a lift of f_t' to a resolution $X' \to S/\mathscr{F}$ may increase in terms of the singularities, and (3) worst of all, if the singularities are sufficiently bad $C \cdot f_t^* K_{S/\mathscr{F}}$ may not be well-defined or may be non-integral (e.g. this may occur if $S/\mathscr{F}$ is $\mathbb{Q}$-Gorenstein but not Gorenstein).

The remedy is to make sure that the covering family $\widetilde{f} : \mathscr{C} \to S$ can be chosen so that a generic curve does not intersect the singular locus of $\mathscr{F}$ and hence $\widetilde{f'} : \mathscr{C}' \to S/\mathscr{F}$ lands inside the nonsingular locus of $S/\mathscr{F}$ after possibly shrinking T. This can be accomplished by blowing up X to replace the initial family $f : \mathscr{C} \to X$ by a quasi-finite covering family. Now replacing X by its blowup may increase the initial canonical degree of the family so the astute reader may worry that such a replacement, if needed at each roof, will overwhelm the reduction in degree given by descent along the foliation. However, we will see that given a covering of X, we can perform *once and for all* a sequence of blowups of X so that the problems are solved for the sufficiently general curve at every stage of the descent. Then the replacement is harmless since it only increases the canonical degree a finite amount initially and hence the argument by infinite descent still proceeds.

To see this argument in action, we need two lemmas: the first showing that after a finite sequence of blowups (depending on the given covering family $f : \mathscr{C} \to X$) we obtain a quasi-finite covering family, the second showing that once we start with a quasi-finite covering family, the descent along a roof remains quasi-finite, so this replacement procedure need not be repeated.

Lemma 5.3.1. *Let X be a nonsingular variety and suppose $g : \mathscr{C} \to T$ is a smooth proper family of curves over a nonsingular irreducible base variety T. Let $f : \mathscr{C} \to X$ be a dominant map. Then there exists a (not necessarily proper or quasi-finite) birational morphism $j : \widetilde{X} \to X$ from a nonsingular variety $\widetilde{X}$ so that the image of j has complement in codimension ≥ 3 along with a dense open set $T' \subset T$ fitting into a diagram*

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[d, "j"] \\
\mathscr{C}_{T'} \arrow[r, hook] \arrow[d, "g"] \arrow[urr, "\widetilde{f}"] & \mathscr{C} \arrow[r, "f"] \arrow[d, "g"] & X \\
T' \arrow[r, hook] & T &
\end{tikzcd}$$

so that the induced rational map $f' : \mathscr{C}_{T'} \to \widetilde{X}$ is a quasi-finite morphism.

Proof. Let $U \subset X$ be the locus where f is flat. Since X is irreducible and f is dominant, by generic flatness U is dense. We apply the flatification theorem of Raynaud–Gruson [RG71, Theorem 5.2.2] to obtain a U-admissible blowup $b : \widetilde{X}_0 \to X$ such that the strict transform $\widetilde{\mathscr{C}} \to \widetilde{X}_0$ is flat hence quasi-finite for reasons of dimension. Recall that the strict transform (the scheme theoretic closure of $\mathscr{C}_U$ inside $\mathscr{C}_{\widetilde{X}_0}$) is a $\mathscr{C}_U$-modification $\widetilde{\mathscr{C}} \to \mathscr{C}$. Since $\mathscr{C}$ is normal, by Zariski's main theorem $\widetilde{\mathscr{C}} \to \mathscr{C}$ has connected fibers and thus, by irreducibility of $\widetilde{\mathscr{C}}$ and dimension counting, $\widetilde{\mathscr{C}} \to \mathscr{C}$ is an isomorphism over an open set $V \subset \mathscr{C}$ of complementary codimension ≥ 2. Thus, $\mathscr{C} \setminus V \to T$ lands in a proper closed subset so we may shrink to an open set $T' \subset T$ so that $\widetilde{\mathscr{C}}_{T'} \to \mathscr{C}_{T'}$ is an isomorphism. Thus we obtain the diagram,

$$\begin{tikzcd} & & \widetilde{X}_0 \arrow[d, "b"] \\ \mathscr{C}_{T'} \arrow[r, hook] \arrow[urr, "\widetilde{f}_0"] \arrow[d, "g"] & \mathscr{C} \arrow[r, "f"] \arrow[d, "g"] & X \\ T' \arrow[r, hook] & T & \end{tikzcd}$$

However, $\widetilde{X}_0$ is likely not smooth since it involves arbitrary U-admissible blowups. In characteristic zero, we may simply resolve the singularities of $\widetilde{X}_0$. In fact, applying Hironaka's resolution algorithm via blowups at smooth centers gives a refinement of Raynaud–Gruson producing flatification via a sequence of U-admissible blowups at smooth centers [Ryd25, Theorem A]. Luckily for our positive characteristic applications, we will not need this. It is enough to resolve singularities in codimension ≤ 2 which is accomplished in all characteristics by the Zariski–Lipman normalized blowup algorithm [Lip78]. A sequence of normalizations and blowups at the singular locus of $\widetilde{X}_0$ will produce, in a finite number of steps, a variety $\widetilde{X}_N$ whose singular locus has codimension ≥ 3. Since $\mathscr{C}_{T'}$ is normal, each normalization step $\widetilde{X}_{i+1} \to \widetilde{X}_i$ admits a lifting

$$\begin{tikzcd} & \widetilde{X}_{i+1} \\ \mathscr{C}_{T'} \arrow[r, "\widetilde{f}_i"] \arrow[ur, dashed, "\widetilde{f}_{i+1}"] & \widetilde{X}_i \end{tikzcd}$$

automatically. For a blowup step $\widetilde{X}_{i+1} \to \widetilde{X}_i$, since the singular locus of the normal variety $\widetilde{X}_i$ lives in codimension ≥ 2, the blowup is an isomorphism over an open set of $\widetilde{X}_i$ of complementary codimension ≥ 2. Since $\widetilde{f}_i : \mathscr{C}_{T'} \to \widetilde{X}_i$ is quasi-finite, the preimage of the center of the blowup is also codimension ≥ 2 in $\mathscr{C}$ hence we may shrink T' further to obtain a lift $f_{i+1} : \mathscr{C}_{T''} \to \widetilde{X}_{i+1}$. Iterating this process and replacing T' by the successive smaller open sets results in a diagram,

$$\begin{tikzcd} & & \widetilde{X}_N \arrow[d] \\ & & \widetilde{X}_0 \arrow[d, "b"] \\ \mathscr{C}_{T'} \arrow[r, hook] \arrow[uurr, "\widetilde{f}_N"] \arrow[d, "g"] & \mathscr{C} \arrow[r, "f"] \arrow[d, "g"] & X \\ T' \arrow[r, hook] & T & \end{tikzcd}$$

Finally, the singular locus of $\widetilde{X}_N$ lives in codimension ≥ 3 so, after possibly shrinking T' further, $\widetilde{f}_N : \mathscr{C}_{T'} \to \widetilde{X}_N$ has image inside the smooth locus of $\widetilde{X}_N$. Hence taking $\widetilde{X} := (\widetilde{X}_N)^{\mathrm{sm}}$ completes the asserted construction. □

Remark 5.3.2. The full power of flatification is really overkill for our purposes. In fact, the arguments will not really use quasi-finiteness but rather the weaker property that $\widetilde{f} : \mathscr{C} \to \widetilde{X}$ does not contract any horizontal divisor of $\mathscr{C} \to T$ to a lower-dimensional subvariety of $\widetilde{X}$. The existence of a modification $j : \widetilde{X} \to X$ forcing this property can be established simply by blowing up successively the images of each divisor of $\mathscr{C}$ contracted by $\widetilde{f}$. After sufficiently many blowups, the codimension of the image of a divisor $E \subset \mathscr{C}$ strictly decreases.

Returning to the situation that X is a smooth projective surface satisfying $(*)_g$ we prove that quasi-finiteness is preserved after descent along a roof.

Lemma 5.3.3. *Suppose that $f : \mathscr{C} \to X$ is a quasi-finite covering family of smooth curves. As in §5.1.1, consider the lift and descent of these covering families,*

$$\begin{tikzcd}
 & \mathscr{C}_{T'} \arrow[rrrr, "\widetilde{f}"] \arrow[dddl, hook] \arrow[ddddd] \arrow[ddr] & & & & S \arrow[dddl] \arrow[dd] & \\
 & & & \mathscr{C}_{T''}^{(p/T)} \arrow[rrr, "\widetilde{f}''"] \arrow[dl, hook] \arrow[dddl] & & & X' \arrow[dl] \\
 & & \mathscr{C}_{T'}^{(p/T)} \arrow[rrr, "\widetilde{f}'"'] \arrow[dddl] & & & S/\mathscr{F} & \\
\mathscr{C} \arrow[rrrr, "f"] \arrow[ddd, "g"'] & & & & X & & \\
 & & T'' \arrow[dl, hook] & & & & \\
 & T' \arrow[dl, hook] & & & & & \\
T & & & & & &
\end{tikzcd}$$

then $\widetilde{f}'' : \mathscr{C}_{T''}^{(p/T)} \to X'$ is quasi-finite.

Proof. We repeatedly use the following fact: let f and g be finite type maps such that $f \circ g$ is quasi-finite then g is quasi-finite. Moreover, if g is surjective then f is also quasi-finite. We start with the assumption that $f : \mathscr{C} \to X$ is quasi-finite. The square

$$\begin{tikzcd} \mathscr{C}_{T'} \arrow[r, "\widetilde{f}"] \arrow[d, hook] & S \arrow[d] \\ \mathscr{C} \arrow[r, "f"] & X \end{tikzcd}$$

implies that $\widetilde{f} : \mathscr{C}_{T'} \to S$ is quasi-finite. Next, the square

$$\begin{tikzcd} \mathscr{C}_{T'} \arrow[r, "\widetilde{f}"] \arrow[d, two heads] & S \arrow[d] \\ \mathscr{C}_{T'}^{(p/T)} \arrow[r, "\widetilde{f}'"] & S/\mathscr{F} \end{tikzcd}$$

proves that $\widetilde{f}' : \mathscr{C}_{T'}^{(p/T)} \to S/\mathscr{F}$ is quasi-finite using that $\mathscr{C}_{T'} \to \mathscr{C}_{T'}^{(p/T)}$ is surjective and $S \to S/\mathscr{F}$ is finite hence $\mathscr{C}_{T'} \to S \to S/\mathscr{F}$ is quasi-finite. Finally, the square

$$\begin{tikzcd} \mathscr{C}_{T''}^{(p/T)} \arrow[r, "\widetilde{f}''"] \arrow[d, two heads] & X' \arrow[d] \\ \mathscr{C}_{T'}^{(p/T)} \arrow[r, "\widetilde{f}'"] & S/\mathscr{F} \end{tikzcd}$$

implies that $\widetilde{f}'' : \mathscr{C}_{T''}^{(p/T)} \to X'$ is quasi-finite. □

5.3.2 Canonical degree change in the quotient

The goal of this section is to understand how the canonical degree behaves when a curve is pulled back along the quotient $S \to S/\mathscr{F}$ by a p-closed foliation. Since we have shown the covering family lifts to a family of leaves of $\mathscr{F}$, these curves arise as pullbacks along the quotient and we must understand the canonical degree of the curves on $S/\mathscr{F}$. However, if the curves pass through the singular points of the foliation $\mathscr{F}$ then they descend to curves passing through the singularities of $S/\mathscr{F}$ which may be quite wild. We have already seen the remedy is to start with a quasi-finite covering family so that the generic curve does not pass through $\mathrm{Sing}(\mathscr{F}) \subset S$ since it lies in codimension 2.

Lemma 5.3.4. *Let $f : C \to X$ be a generic immersion from a smooth projective curve to a smooth variety X carrying a rank 1 foliation $\mathscr{F}$. Suppose that f is a leaf of $\mathscr{F}$ and that* $\mathrm{im}\, f$ *does not intersect* $\mathrm{Sing}(\mathscr{F})$. *Then* $\deg f^*\mathscr{F} \geq \chi(C) := 2 - 2g(C)$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{Q} \longrightarrow 0$$

By definition, over $U := X \setminus \mathrm{Sing}(\mathscr{F})$ this is an exact sequence of vector bundles. Since $f : C \to X$ factors through $U \hookrightarrow X$, applying f^* preserves exactness of the above sequence. Therefore, we obtain a diagram,

$$\begin{tikzcd} 0 \arrow[r] & f^*\mathscr{F} \arrow[r] & f^*\mathscr{T}_X \arrow[r] & f^*\mathscr{Q} \arrow[r] & 0 \\ & & \mathscr{T}_C \arrow[u] \arrow[ul, dashed] & & \end{tikzcd}$$

and the map $\mathscr{T}_C \to f^*\mathscr{T}_X \to f^*\mathscr{Q}$ is zero because $f : C \to X$ is a leaf of $\mathscr{F}$. Hence, $\mathscr{T}_C \to f^*\mathscr{T}_X$ factors through the kernel $f^*\mathscr{F}$. But $f^*\mathscr{F}$ is a line bundle so the nonzero map $\mathscr{T}_C \to f^*\mathscr{F}$ implies that

$$\deg f^*\mathscr{F} \geq \deg \mathscr{T}_C = 2 - 2g(C).$$

□

Remark 5.3.5. In the above argument, it is important that $f : C \to X$ does not hit $\mathrm{Sing}(\mathscr{F})$. Otherwise, the pullback exact sequence may not be left exact. Recall the definition of a leaf requires only that the tangent spaces at general points agree with $\mathscr{F}$. Since the vanishing of the map $\mathscr{T}_C \to f^*\mathscr{Q}$ may be checked at the generic point assuming $f^*\mathscr{Q}$ is torsion-free, we can state the property of being a leaf as the global condition that $\mathscr{T}_C \to f^*\mathscr{Q}$ vanishes.

Proposition 5.3.6. *Let $f : \mathscr{C} \to S$ be a quasi-finite covering family of smooth curves of genus $g \leq 1$ tangent to $\mathscr{F}$ and let $f' : \mathscr{C}_{T'}^{(p/T)} \to X'$ be the induced family on the resolution $X' \to S/\mathscr{F}$ from the descent of f to $S/\mathscr{F}$. Then,*

$$\deg f_t'^* \omega_{X'} \leq \frac{1}{p} \deg f_t^* \omega_S.$$

Moreover, if the family has genus 0 *then this inequality is strict.*

Proof. Let $\pi : S \to S/\mathscr{F}$ denote the quotient map. Recall from Theorem 2.3.2(2), the relationship between the canonical divisors for the quotient by a foliation:

$$K_S = \pi^* K_{S/\mathscr{F}} + (p-1)c_1(\mathscr{F})$$

which holds over the smooth locus $U \subset S$ of the foliation. Although the divisor $K_{S/\mathscr{F}}$ is well-defined because $S/\mathscr{F}$ is normal, if a curve passed through the singularities of $S/\mathscr{F}$ its intersection with $K_{S/\mathscr{F}}$ may not be defined. Moreover, the pullback $\pi^* K_{S/\mathscr{F}}$ may not make sense if $K_{S/\mathscr{F}}$ is not $\mathbb{Q}$-Cartier. However, since $f : \mathscr{C} \to S$ is quasi-finite and dominant, the general curve misses $\mathrm{Sing}(\mathscr{F}) \subset X$ since it lies in codimension 2. Hence, the general curve in the family $\mathscr{C}^{(p/T)} \to S/\mathscr{F}$ misses the singular points so $f_t^* K_{S/\mathscr{F}} = f_t^* K_{X'}$ is well-defined and equals $c_1(f_t^* \omega_{S/\mathscr{F}})$. From the diagram,

$$\begin{tikzcd} \mathscr{C}_{T'} \arrow[r, "F_{\mathscr{C}/T}"] \arrow[d, "f"] & \mathscr{C}_{T'}^{(p/T)} \arrow[d, "f'"] \\ S \arrow[r, "\pi"] & S/\mathscr{F} \end{tikzcd}$$

and since, for $t \in T'$ possibly in an open subset of T', f_t has image inside U,

$$\deg f_t^* \omega_S = \deg \mathrm{Frob}^* f_t'^* \omega_{X'} + (p-1) \deg f_t^* \mathscr{F}.$$

However, $\deg \mathrm{Frob}^* f_t'^* \omega_{X'} = p \cdot \deg f_t'^* \omega_{X'}$ and $\deg f_t^* \mathscr{F} \ge 2 - 2g$ by Lemma 5.3.4. Since $g \le 1$, the Lemma gives $\deg f_t^* \mathscr{F} \ge 0$. Rearranging gives,

$$\deg f_t'^* \omega_{X'} = \frac{1}{p} \deg f_t^* \omega_S - \frac{p-1}{p} \deg f_t^* \mathscr{F} \le \frac{1}{p} \deg f_t^* \omega_S.$$

Moreover, if $g = 0$ then we obtain a stronger bound $\deg f_t^* \mathscr{F} \ge 2$ so in fact,

$$\deg f_t'^* \omega_{X'} = \frac{1}{p} \deg f_t^* \omega_S - 2 \cdot \frac{p-1}{p} < \frac{1}{p} \deg f_t^* \omega_S.$$

□

5.3.3 Canonical degree change in the covering construction

Now we must understand how the canonical degree of a quasi-finite family $f : \mathscr{C} \to X$ changes when passing to the lift $\widetilde{f} : \mathscr{C}_{T'} \to S$. The following lemma will be useful in the comparison of $\deg \widetilde{f}_t^* \omega_S$ to $\deg f_t^* \omega_X$.

Lemma 5.3.7. *Let $\pi : X \to Y$ be a finite birational map of varieties with Y Gorenstein and X Cohen–Macaulay. Then there is a map $\omega_X \to \pi^* \omega_Y$ between their dualizing sheaves so that the induced map*

$$\pi_* \omega_X \to \pi_* \pi^* \omega_Y \to \omega_Y$$

is the canonical trace. In particular, $\omega_X \to \pi^* \omega_Y$ is the canonical isomorphism over the locus that π is an isomorphism and hence is injective.

Proof. For any proper morphism π, there is always a trace map $t_\pi : \pi_* \omega_X \to \omega_Y$ witnessing Grothendieck duality. In particular, in the proper case, t_π is determined by requiring it to induce an isomorphism $H^n(X, \omega_X) \to H^n(Y, \omega_Y)$ compatible with the trace maps $H^n(X, \omega_X) \to k$ and hence t_π is nontrivial. Consider the diagram,

$$\begin{tikzcd} & \pi^*\pi_*\omega_X \arrow[dl] \arrow[dr] & \\ \omega_X \arrow[rr, dashed] & & \pi^*\omega_Y \end{tikzcd}$$

and we must show the existence of the dashed arrow. First, $\pi^*\pi_*\omega_X \to \omega_X$ is surjective. Indeed, this holds for any affine map π since the statement is local on the base and it amounts to the fact that for a ring map $A \to B$ and a B-module M, the multiplication map $B \otimes_A M \to M$ is surjective. Furthermore, letting $U \subset Y$ be the locus over which π is an isomorphism, $\pi^*\pi_*\omega_X \to \omega_X$ is an isomorphism over $\pi^{-1}(U)$. Hence $\ker(\pi^*\pi_*\omega_X \to \omega_X)$ is torsion. However, Y is Gorenstein so ω_Y is a line bundle. Therefore, $\pi^*\omega_Y$ is a line bundle so

$$\ker(\pi^*\pi_*\omega_X \to \omega_X) \to \pi^*\omega_Y$$

must be zero proving the factorization in the diagram. Since ω_X is torsion-free of rank 1, we also see that $\omega_X \to \pi^*\omega_Y$ must be injective. □

Recall that the lift diagram

$$\begin{tikzcd} & Z \arrow[r, hook] & \mathbb{P} \arrow[d, "\pi"] \\ C \arrow[ur, dashed, "\widetilde{f}"] \arrow[rr, "f"] & & X \end{tikzcd}$$

is defined by the map of bundles on C

$$f^* : f^*\Omega_X \twoheadrightarrow \mathrm{im}\,(f^*) \subset \omega_C$$

so by definition $\widetilde{f}^*\mathcal{O}_{\mathbb{P}}(1) = \mathrm{im}\, f^* \hookrightarrow \omega_C$. We first calculate the canonical degree of $\widetilde{f}$ on $\mathbb{P}$. To do this, recall that the Euler sequence for $\mathbb{P} := \mathbb{P}_X(\Omega_X) \to X$,

$$0 \to \Omega_{\mathbb{P}/X} \to \pi^*\Omega_X \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}} \to 0$$

implies that

$$\omega_{\mathbb{P}} \cong \pi^*\omega_X^{\otimes 2}(-2).$$

Therefore,

$$\deg \widetilde{f}_t^*\omega_{\mathbb{P}} = 2\deg f_t^*\omega_X - 2\deg \widetilde{f}_t^*\mathcal{O}_{\mathbb{P}}(1) \tag{5.2}$$

Now, let $Z_i \subset Z_{\mathrm{hor}}$ be the component containing the image of $\widetilde{f}$. We split the resolution $S \to Z_i$ into $S \to \widetilde{Z}_i \to Z_i$ where $\widetilde{Z}_i \to Z_i$ is the normalization and $S \to \widetilde{Z}_i$ is an isomorphism away from codimension 2 on the base.

Note that the total divisor $Z \subset \mathbb{P}$ is cut out by the symmetric form viewed as section $s_\omega \in H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(m))$ so its divisor class equals

$$Z = \sum_{j=1}^{r} m_j Z_j = m\xi$$

where $\xi := c_1(\mathscr{O}_{\mathbb{P}}(1))$ and $Z_1, \ldots, Z_r$ are the irreducible components of Z with multiplicities $m_1, \ldots, m_r$. Since Z_i is a divisor, adjunction gives

$$\omega_{Z_i} = \omega_{\mathbb{P}}(Z_i)|_{Z_i}.$$

Therefore,

$$\begin{aligned} \deg \widetilde{f_t}^* \omega_{Z_i} &= \deg \widetilde{f_t}^* \omega_{\mathbb{P}} + \deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(Z_i) \\ &= \deg \widetilde{f_t}^* \omega_{\mathbb{P}} + \frac{1}{m_i}\left[m \deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(1) - \sum_{j\neq i} m_j \deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(Z_j) \right]. \end{aligned}$$

Furthermore, for a generic element $t \in T$ the curve $\widetilde{f_t} : \mathscr{C}_t \to Z_i$ is not contained in Z_j for $j \neq i$. This is because $f : \mathscr{C} \to X$ is dominant and otherwise its image would be contained in the image of $Z_i \cap Z_j$ which is 1-dimensional. Therefore $\deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(Z_j) \geq 0$ for $j \neq i$. Although this inequality is correct for all components Z_j of Z, it will be useful to separate the contributions from Z_{cont} and Z_{hor}. Indeed, $Z_{\mathrm{cont}} = \pi^*[V(\omega)]^{\mathrm{div}}$ where $[V(\omega)]^{\mathrm{div}}$ is the divisorial part of the vanishing locus $V(\omega) \subset X$ of the symmetric form defining Z.

Together with equation (5.2) this implies,

$$\begin{aligned} \deg \widetilde{f_t}^* \omega_{Z_i} &\leq \deg \widetilde{f_t}^* \omega_{\mathbb{P}} + \frac{m}{m_i} \deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(1) - \frac{1}{m_i} \deg f_t^* [V(\omega)]^{\mathrm{div}} && (5.3) \\ &= 2 \deg f_t^* \omega_X + \left(\frac{m}{m_i} - 2\right) \deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(1) - \frac{1}{m_i} \deg f_t^* [V(\omega)]^{\mathrm{div}}. && (5.4) \end{aligned}$$

Since $Z_i \subset \mathbb{P}$ is a divisor, it has Gorenstein (in fact lci) singularities. Therefore, we may apply Lemma 5.3.7 to the finite morphism $\widetilde{Z_i} \to Z_i$ to obtain a map

$$\widetilde{f'}^* \omega_{\widetilde{Z_i}} \to \widetilde{f}^* \omega_{Z_i}.$$

Since $\widetilde{f'}$ covers $\widetilde{Z_i}$ the above map is nontrivial hence

$$\deg \widetilde{f'}^* \omega_{\widetilde{Z_i}} \leq \deg \widetilde{f}^* \omega_{Z_i} \tag{5.5}$$

Furthermore, since f is quasi-finite, $\widetilde{f'}$ must also be quasi-finite and therefore, the generic curve misses the singular locus of $\widetilde{Z_i}$ meaning it lives in the locus where $S \to \widetilde{Z_i}$ is an isomorphism. This implies that $\widetilde{f'}^* \omega_S = \widetilde{f'}^* \omega_{\widetilde{Z_i}}$. Combining this with inequalities (5.5) and (5.3) gives

$$\deg \widetilde{f'}^* \omega_S \leq 2 \deg f_t^* \omega_X + \left(\frac{m}{m_i} - 2\right) \deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(1) - \frac{1}{m_i} \deg f_t^* [V(\omega)]^{\text{div}}. \tag{5.6}$$

Now we must analyze the second term on the right for a family of curves of genus ≤ 1. Note that $\widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(1) \hookrightarrow \omega_{\mathscr{C}_t}$ by the construction of the lift to $\mathbb{P}(\Omega_X)$. Therefore,

$$\deg \widetilde{f_t}^* \mathscr{O}_{\mathbb{P}}(1) \leq \deg \omega_{\mathscr{C}_t} \leq 0$$

since the curves are smooth and have genus ≤ 1. However, this term might still be problematic if $\left(\frac{m}{m_i} - 2\right) < 0$ since if $\deg \widetilde{f_t}^{(k)*} \mathscr{O}_{\mathbb{P}}(1)$ is large and negative, together they may give an unbounded positive contribution. However, recall that these $m_1, \dots, m_r$ are the multiplicities appearing in the effective decomposition

$$Z = \sum_{j=1}^{r} m_j Z_j = m\xi$$

and m is the degree of the map $Z \to X$. Setting $d_j := \deg (Z_j \to X)$ then we can write the relation $m = m_1 d_1 + \cdots + m_r d_r$. Hence, $\frac{m}{m_i} < 2$ implies that $d_i = 1$ since $m_j, d_j \geq 0$, for $j \neq i$, and $d_i > 0$. Therefore the problematic case occurs only when $Z_i \to X$ is birational. But in this case, the induced foliation on S, the resolution of Z_i, induces a foliation on X for which the curves $f : \mathscr{C} \to X$ are already tangent. Hence, in this case we may actually choose $S = X$ with the obvious corollary that,

$$\deg \widetilde{f'}^* \omega_S = \deg f_t^* \omega_X.$$

Putting these two cases together we obtain,

$$\deg \widetilde{f'}^* \omega_S \leq \max \left\{ \deg f_t^* \omega_X, 2 \deg f_t^* \omega_X - \frac{1}{m_i} \deg f_t^* [V(\omega)]^{\text{div}} \right\}. \tag{5.7}$$

5.3.4 The main inequality

Theorem 5.3.8. *In the notation of diagram 5.1, if each $X^{(k)}$ is of general type then any covering family can descend through only finitely many roofs in this infinite branch.*

Proof. Let $f : \mathscr{C} \to X$ be a covering family. Using Lemma 5.3.1 we modify X by a sequence of blowups $j : \widetilde{X} \to X$ to obtain a new covering family $f' : \mathscr{C}_{T'} \to \widetilde{X}$ that is quasi-finite. Since we may pull back foliations and symmetric forms, and since quotients by foliations are functorial, we obtain a morphism between branches,

$$\begin{tikzcd}
 & \widetilde{S}^{(0)} \arrow[dl] \arrow[dr] \arrow[dd, dashed] & & & \widetilde{S}^{(1)} \arrow[dl] \arrow[dr] \arrow[dd, dashed] & & \\
\widetilde{X}^{(0)} \arrow[dd, dashed] & & \widetilde{S}^{(0)}/\widetilde{\mathscr{F}}^{(0)} \arrow[dd, dashed] & \widetilde{X}^{(1)} \arrow[l] \arrow[dd, dashed] & & \widetilde{S}^{(1)}/\widetilde{\mathscr{F}}^{(1)} \arrow[dd, dashed] & \cdots \arrow[l] \\
 & S^{(0)} \arrow[dl] \arrow[dr] & & & S^{(1)} \arrow[dl] \arrow[dr] & & \\
X^{(0)} & & S^{(0)}/\mathscr{F}^{(0)} & X^{(1)} \arrow[l] & & S^{(1)}/\mathscr{F}^{(1)} & \cdots \arrow[l]
\end{tikzcd}$$

After shrinking the base, any covering family of $X^{(k)}$ lifts to a covering family of $\widetilde{X}^{(k)}$ so it suffices to show that the quasi-finite family $f' : \mathscr{C}_{T'} \to \widetilde{X}$ does not descend arbitrarily far down this branch. Assume that f' has descended to a quasi-finite covering family $f^{(k)} : \mathscr{C}^{(k)} \to \widetilde{X}^{(k)}$. Suppose $f^{(k)}$ descends to $f^{(k+1)} : \mathscr{C}^{(k+1)} \to \widetilde{X}^{(k+1)}$ meaning first it lifts to $\widetilde{S}^{(k)}$ as a covering family by leaves of $\widetilde{\mathscr{F}}^{(k)}$ and then we form the descent to $\widetilde{S}^{(k)}/\widetilde{\mathscr{F}}^{(k)}$ and, after possibly shrinking $T^{(k)}$, we lift to $\widetilde{X}^{(k+1)}$. By Lemma 5.3.3 $f^{(k+1)}$ is also quasi-finite. Furthermore, combining Proposition 5.3.6 and Equation (5.7) produces the inequality,

$$\deg f_t^{(k+1)*}\omega_{\widetilde{X}^{(k+1)}} \le \frac{1}{p}\max\left\{\deg f_t^{(k)*}\omega_{\widetilde{X}^{(k)}}, 2\deg f_t^{(k)*}\omega_{\widetilde{X}^{(k)}} - \frac{1}{m_i}\deg f_t^{(k)*}[V(\omega)]^{\mathrm{div}}\right\}$$

which is strict in the case of genus 0. For genus 0, we obtain

$$\deg f_t^{(k+1)*}\omega_{\widetilde{X}^{(k+1)}} < \frac{2}{p}\deg f_t^{(k)*}\omega_{\widetilde{X}^{(k)}} \le \deg f_t^{(k)*}\omega_{\widetilde{X}^{(k)}}$$

proving that the canonical degree strictly decreases at each roof. Likewise, for genus 1, we are under assumption $(*)_1$, meaning that $V(\omega)$ contains a big divisor. Since $f^{(k)}$ is a covering family, the term $\deg f_t^{(k)*}[V(\omega)]^{\mathrm{div}}$ is strictly positive. Hence, we also conclude that

$$\deg f_t^{(k+1)*}\omega_{\widetilde{X}^{(k+1)}} < \frac{2}{p}\deg f_t^{(k)*}\omega_{\widetilde{X}^{(k)}} \le \deg f_t^{(k)*}\omega_{\widetilde{X}^{(k)}}.$$

Assuming that $\widetilde{X}^{(k)}$ is of general type, the canonical degree of a covering family must be positive (since $K_{\widetilde{X}^{(k)}}$ is $\mathbb{Q}$-effective). Since $\deg f_t^{(0)*}\omega_{\widetilde{X}^{(0)}}$ is finite, it can only decrease finitely many times while remaining positive proving the theorem. $\square$

Therefore, together with the discussion of this and the previous sections, we have proven the following criterion.

Theorem 5.3.9. *If the main algorithm does not terminate and each $X_{i_1,\dots,i_k}$ appearing in the tree is of general type, then X is not uniruled. Furthermore, assuming $(*)_1$ for all $X_{i_1,\dots,i_k}$ in the tree, then X is not covered by genus 1 curves.*

In the next section, we will see that Theorem 5.3.9 applies to prove certain Hilbert modular surfaces are not covered by rational or elliptic curves and we will generalize this argument to higher-dimensional Hilbert modular varieties.

Chapter 6

Non-uniruled Hilbert Modular Varieties

> Tis true without lying, certain and most true.
> That which is below is like that which is above
> and that which is above is like that which is below.
>
> Pseudo-Aristotle (Trans. Isaac Newton)

In this chapter we will review the theory of Hilbert modular varieties and their canonical models, investigate their special fiber mod p, and use their tautological foliation structure to prove non-uniruledness along the lines of the previous chapter. The advantage of this method over one based on étale cohomology will be that it applies to those primes living in a constructible subset of $\operatorname{Spec} \mathbb{Z}$ (i.e. to all but finitely many primes) since the constructions only rely on geometric gadgets as opposed to arithmetic properties of the Hecke action on cohomology.

6.1 Hilbert Modular Varieties

Roughly speaking, Hilbert modular varieties are Shimura varieties parametrizing abelian varieties with the action of an order in a totally real number field $F/\mathbb{Q}$ of maximal degree. The story begins with the Albert classification of endomorphisms of simple abelian varieties.

Theorem 6.1.1 (Albert, [Mum08, Ch. IV, §21]). *Let $A/\mathbb{C}$ be a simple abelian variety. Then $D := \mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division algebra over $\mathbb{Q}$. Let F be its center so $F/\mathbb{Q}$ is a number field. Then there are four possible cases:*

1. *$D = F$ and F is totally real*

2. *D is a quaternion algebra over a totally real field F which is totally split at infinity*

3. *D is a quaternion algebra over a totally real field F which is totally non-split at infinity*

4. *D is a central division algebra over a CM-field F*

We call the first case "real multiplication" (RM), the second and third cases "quaternionic multiplication" (QM), and the fourth "complex multiplication" (CM). Hilbert modular varieties parametrize abelian varieties with maximal RM meaning $\dim A = [F : \mathbb{Q}] = d$. Let F be a totally real field and Σ be the set of embeddings $F \hookrightarrow \mathbb{R}$.

Recall that, over $\mathbb{C}$, an abelian variety is determined by a lattice $\Lambda \subset \mathbb{C}^d$ of rank $2d$ with a Riemann form giving the polarization. For $\tau \in \mathfrak{H}^\Sigma$ we get a lattice $L_\tau \subset \mathbb{C}^d$ defined by

$$\mathscr{O}_F^{\oplus 2} \hookrightarrow \mathbb{C}^d \quad \text{via} \quad (\alpha, \beta) \mapsto (\alpha_\sigma \tau_\sigma + \beta_\sigma)_{\sigma \in \Sigma}$$

where we write $\alpha_\sigma := \sigma(\alpha)$. The action of $\delta \in \mathscr{O}_F$ via $\delta \cdot (z_\sigma)_{\sigma\in\Sigma} = (\delta_\sigma z_\sigma)_{\sigma\in\Sigma}$ preserves the lattice since σ is multiplicative. Therefore, $\mathscr{O}_F$ acts on the abelian variety $\mathbb{C}^d/L_\tau$ preserving a particular polarization form.

Definition 6.1.2. Let F be a totally real field. Let $\Gamma \subset \mathrm{GL}_2^+(\mathscr{O}_F)$ be a finite index subgroup. Then consider the action of Γ on $\mathfrak{H}^\Sigma$ via the embeddings

$$\Gamma \hookrightarrow \mathrm{GL}_2^+(\mathscr{O}_F) \hookrightarrow \mathrm{GL}_2^+(\mathbb{R})^\Sigma.$$

A *Hilbert modular form* at level Γ of weight $[k_\sigma]_{\sigma\in\Sigma}$ is a holomorphic function $f : \mathfrak{H}^\Sigma \to \mathbb{C}$ bounded at the cusps such that

$$f(\gamma \cdot \underline{\tau}) = f(\underline{\tau}) \prod_{\sigma\in\Sigma} j(\gamma_\sigma, \tau_\sigma)^{k_\sigma}$$

for all $\gamma \in \Gamma$, where for $\tau \in \mathfrak{H}$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ the *factor of automorphy* j is

$$j(\gamma, \tau) = (c\tau + d) \cdot (\det \gamma)^{-1/2} \qquad \text{for} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Remark 6.1.3. The form of the factor of automorphy $j(\gamma, \tau) = (c\tau + d) \cdot (\det \gamma_\sigma)^{-1/2}$ can be remembered from the case of scalar matrices $\gamma = \lambda \cdot \mathrm{id}$ for $\lambda \in (\mathscr{O}_F^\times)^+$. In this case, γ acts trivially on $\mathfrak{H}^\Sigma$ so we must have that $f(\gamma \cdot \underline{\tau}) = f(\underline{\tau})$ and indeed $j(\gamma, \tau) = \lambda \cdot (\lambda^2)^{-1/2} = 1$. Furthermore, $\mathrm{d}\tau_\sigma$ transforms as

$$\mathrm{d}(\gamma_\sigma \cdot \tau_\sigma) = \frac{a(c\tau + d) - c(a\tau + d)}{(c\tau + d)^2} \cdot \mathrm{d}\tau = \frac{\det \gamma}{(c\tau + d)^2} \cdot \mathrm{d}\tau = j(\gamma, \tau)^{-2}\mathrm{d}\tau.$$

Therefore, if f is a Hilbert modular form of weight $(2, \dots, 2)$ for Γ then $f \mathrm{d}\tau_1 \wedge \cdots \wedge \mathrm{d}\tau_d$ is invariant under Γ.

Based on this discussion, the analytic space $\mathfrak{H}^\Sigma/\Gamma$ is our first candidate for the Hilbert modular variety.

6.1.1 Shimura Varieties

As in our previous discussion of Shimura curves, we follow the conventions of [Mil05b] for Shimura data and axioms. First we recall the definitions of a Shimura datum and a Shimura variety. Let $\mathbb{S} := \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_{m,\mathbb{C}}$ be the Deligne torus and let $G^{\mathrm{ad}} := G/Z(G)$ denote the adjoint group associated to a linear algebraic group.

Definition 6.1.4. A *Shimura datum* is a pair (G, X) of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of maps $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

(SV1) for all $h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\}$$

(SV2) for all $h \in X$, the element $\mathrm{ad}(h(i))$ is a Cartan involution[1] of $G_{\mathbb{R}}^{\mathrm{ad}}$

(SV3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Let $\mathbb{A}^\infty$ denote the finite adeles of $\mathbb{Q}$. Associated to a Shimura datum (G, X) and a compact open subgroup $K \subset G(\mathbb{A}^\infty)$ defining a *level structure* there is a Shimura variety defined on the following analytic space.

Definition 6.1.5. The *Shimura variety* defined by the Shimura datum (G, X) at level K is the double coset space

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty)/K.$$

However, Shimura varieties are often disconnected objects. It will be useful to have a description of their connected components. These are described by *connected Shimura data* as follows. Let $G(\mathbb{R})^+$ denote the connected component of the identity in the *real topology* and $G(\mathbb{Q})^+ := G(\mathbb{Q}) \cap G(\mathbb{R})^+$.

[1] An involution θ of G is *Cartan* if $G^{(\theta)}(\mathbb{R}) := \{g \in G(\mathbb{C}) \mid g = \theta(\bar{g})\}$ is compact.

Definition 6.1.6. A *connected Shimura datum* is a pair (G, X^+) consisting of a semisimple algebraic group G over $\mathbb{Q}$ and a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X^+ of homomorphisms $h : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the following conditions:

(SV1) for all $h \in X^+$, only the characters $z/\bar{z}, 1, \bar{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ h$

(SV2) for all $h \in X^+$, the element $\mathrm{ad}(h(i))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$

(SV3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

We refer to any quotient of the form $\Gamma \backslash X^+$ where Γ is an arithmetic subgroup[2] of $G(\mathbb{Q})^+$ containing the image of a congruence subgroup of $G(\mathbb{Q})$ as a *connected Shimura variety* relative to (G, X^+). One shows that X^+ is a Hermitian symmetric domain [Mil05b, Proposition 4.8] on which the arithmetic subgroup $\Gamma := G(\mathbb{Q})^+ \cap K$ acts so the quotient $\Gamma \backslash X^+$ has a canonical structure as a quasi-projective variety induced by realizing it as a Zariski open of the Baily–Borel compactification $\overline{\Gamma \backslash X^+}^{BB}$, see [Mil05b, Theorem 3.12].

Proposition 6.1.7. *Let (G, X) be a Shimura datum and X^+ be a connected component of X regarded as a $G(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$. Then (G^{der}, X^+) is a connected Shimura datum and there is a homeomorphism*

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty)/K \xrightarrow{\sim} \bigsqcup_{g \in G(\mathbb{Q})^+ \backslash G(\mathbb{A}^\infty)/K} \Gamma_g \backslash X^+$$

where $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})^+$.

Proof. [Mil05b, Corollary 5.8] and [Mil05b, Lemma 5.13]. □

In particular, this shows that Shimura varieties are quasi-projective.

6.1.2 Hilbert Modular Varieties as Shimura Varieties

The most fundamental way Hilbert modular varieties appear as Shimura varieties is through the Weil restriction for the group GL_2 from a totally real field $F/\mathbb{Q}$.

Definition 6.1.8. Let $G^{\mathrm{Hilb}} := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$ and consider the homomorphism

$$h_0 : \mathbb{S} \to G^{\mathrm{Hilb}}_{\mathbb{R}} = \prod_{\sigma \in \Sigma} \mathrm{GL}_{2,\mathbb{R}}$$

[2] A subgroup $\Gamma \subset G(\mathbb{Q})$ is *arithmetic* if it is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ for some (equivalently, any) faithful representation $G \hookrightarrow \mathrm{GL}_n$. Equivalently, we can ask that Γ is commensurable with $\mathscr{G}(\mathbb{Z})$ for some integral model $\mathscr{G}$ of G.

given by

$$x + iy \mapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \dots, \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right) \in \prod_{\sigma \in \Sigma} \mathrm{GL}_{2,\mathbb{R}}$$

and let X^{Hilb} be the $G_{\mathbb{R}}^{\mathrm{Hilb}}$-conjugacy class generated by h_0. Then $(G^{\mathrm{Hilb}}, X^{\mathrm{Hilb}})$ is a Shimura datum and the associated Shimura varieties $\mathrm{Sh}_K(G^{\mathrm{Hilb}}, X^{\mathrm{Hilb}})$ are called the *Hilbert modular varieties* associated to F.

Note that we may identify $X^{\mathrm{Hilb}} \cong (\mathfrak{H}^{\pm})^{\Sigma}$ with the action by fractional linear transformations of $G_{\mathbb{R}}^{\mathrm{Hilb}} = \prod_{\sigma \in \Sigma} \mathrm{GL}_{2,\mathbb{R}}$; see [Mil05b, Example 1.23].

Example 6.1.9. Suppose we take the level subgroup $K = G^{\mathrm{Hilb}}(\mathbb{A}^{\infty}) = \mathrm{GL}_2(\mathbb{A}_F^{\infty})$ consisting of matrices whose entries have no denominators at any place of F. We would like to compute the associated Hilbert modular variety,

$$\mathrm{Sh}_K(G^{\mathrm{Hilb}}, X^{\mathrm{Hilb}}) := \mathrm{GL}_2(F) \backslash X \times \mathrm{GL}_2(\mathbb{A}_F^{\infty}) / \mathrm{GL}_2(\widehat{\mathscr{O}}_F) \xrightarrow{\sim} \bigsqcup_{g \in \mathrm{GL}_2(F)^+ \backslash \mathrm{GL}_2(\mathbb{A}_F^{\infty}) / \mathrm{GL}_2(\widehat{\mathscr{O}}_F)} \Gamma_g \backslash X^+$$

Here, $X^+ = \mathfrak{H}^{\Sigma}$ is a choice of connected component of X^{Hilb} in the real topology, which is itself a $\mathrm{GL}_2(F \otimes \mathbb{R})^+$-conjugacy class. First we compute the index set,

$$\mathrm{GL}_2(F)^+ \backslash \mathrm{GL}_2(\mathbb{A}_F^{\infty}) / \mathrm{GL}_2(\widehat{\mathscr{O}}_F).$$

The valuations of the determinant at each finite place of F defines a map

$$\mathrm{GL}_2(F)^+ \backslash \mathrm{GL}_2(\mathbb{A}_F^{\infty}) / \mathrm{GL}_2(\widehat{\mathscr{O}}_F) \xrightarrow{\det} \mathscr{C}\ell^+(F)$$

which is clearly surjective via building diagonal matrices whose entries are $1, \varpi_v^{a_v}$ where a_v are the valuations so that $\prod \mathfrak{p}_v^{a_v}$ is a specified generator of $\mathscr{C}\ell^+(F)$. The map is well-defined because we only quotient by elements $\mathrm{GL}_2(F)^+ \subset \mathrm{GL}_2(F)$ which, in particular, have totally positive determinants. Furthermore, this map is injective by the strong approximation theorem (cf. [Mil05b, Theorem 4.16]).

We may choose double coset representatives:

$$g_{\mathfrak{a}} = \begin{pmatrix} \pi_v^{\mathrm{val}_v(\mathfrak{a})} & 0 \\ 0 & 1 \end{pmatrix}$$

where $\mathfrak{a} \subset \mathscr{O}_F$ runs over a set of representatives for $\mathscr{C}\ell^+(F)$. Then

$$\Gamma_g := \mathrm{GL}_2(F)^+ \cap g\mathrm{GL}_2(\widehat{\mathscr{O}}_F)g^{-1}$$

so we first compute,

$$g\mathrm{GL}_2(\widehat{\mathcal{O}}_F)g^{-1} = \left\{ \begin{pmatrix} a_v & \pi_v^{-\mathrm{val}_v(\mathfrak{a})} b_v \\ \pi_v^{\mathrm{val}_v(\mathfrak{a})} c_v & d_v \end{pmatrix} \middle| a_v, b_v, c_v, d_v \in \mathcal{O}_{F_v} \right\}$$

and therefore

$$\Gamma_{g_{\mathfrak{a}}} = \Gamma(\mathcal{O}_F \oplus \mathfrak{a}) := \mathrm{GL}(\mathcal{O}_F \oplus \mathfrak{a})^+.$$

The Hilbert modular varieties of the above generality are coarse moduli spaces for abelian varieties with real multiplication, but they are often not fine moduli spaces, even in the absence of elliptic points. At its source, this defect is witnessed by the failure of axiom (SV5) for the pair $(G^{\mathrm{Hilb}}, X^{\mathrm{Hilb}})$.

Definition 6.1.10. We say that a Shimura datum (G, X) satisfies the additional axiom

(SV5) if $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}^\infty)$ where $Z := Z(G)$ is the scheme theoretic center.

Example 6.1.11. In the case $(G^{\mathrm{Hilb}}, X^{\mathrm{Hilb}})$, we have $Z = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ and must consider $Z(\mathbb{Q}) = F^\times$ lying inside $Z(\mathbb{A}^\infty) = \mathbb{A}_{F,f}^\times$ via the natural embedding. However, the image is not discrete whenever $F \neq \mathbb{Q}$ due to the presence of infinitely many units. Indeed, the Dirichlet unit theorem implies that the units of $\mathcal{O}_F$ have rank $[F : \mathbb{Q}] - 1$. Furthermore, any open neighborhood of $1 \in \mathbb{A}_F^{\times,\infty}$ (in the idele topology) contains all sufficiently divisible elements of the unit group $\mathcal{O}_F^\times$ meaning it cannot be discrete.

Failure of (SV5) precludes the existence of a universal family of polarized abelian varieties over the Shimura variety. We will witness a manifestation of this defect when we form an arithmetic model Y_U for the Hilbert modular variety $\mathrm{Sh}_U(G^{\mathrm{Hilb}}, X^{\mathrm{Hilb}})$ (see §6.2.2) constructed from finite quotients of PEL Shimura varieties. The PEL variant, discussed in the subsequent section, does carry a universal family of polarized abelian varieties. However, the totally positive elements of the center, $(\mathcal{O}_F^\times)^+$, will act nontrivially on the universal family. In order to construct a quotient abelian scheme over Y_U, from the universal family with full level N structure, the subgroup

$$\{\gamma^2 \mid \gamma \in \mathcal{O}_F^\times : \gamma \equiv 1 \mod N\} \subset (\mathcal{O}_F^\times)^+$$

must be trivial for some $N > 1$. This is exactly the condition (SV5) that $F^\times \subset (\mathbb{A}_F^\infty)^\times$ is discrete. Unfortunately, this condition only holds when $F = \mathbb{Q}$.

6.1.3 PEL Hilbert modular varieties

However, we will find that a variant of the Hilbert modular varieties defined above is a fine (at sufficiently large level) moduli space for certain abelian varieties with Polarization, Endomorphism (real multiplication), and Level (PEL) structures. To build this alternative Hilbert modular Shimura datum we need to review the theory of PEL Shimura data.

PEL Shimura varieties

Let $(B, *)$ be a semisimple k-algebra with an involution.

Definition 6.1.12. A *symplectic* $(B, *)$*-module* is a B-module V equipped with a skew-symmetric k-bilinear form $\psi : V \times V \to k$ such that

$$\psi(b^* u, v) = \psi(u, bv) \text{ for all } b \in B, \text{ and } u, v \in V.$$

In what follows, let $k = \mathbb{Q}$. Given $(B, *)$ a semisimple $\mathbb{Q}$-algebra with involution and (V, ψ) a faithful $(B, *)$-symplectic module, define the $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}_B(V)$

$$G_{B,V}(\mathbb{Q}) = \{g \in \mathrm{GL}_B(V) \mid \psi(gx, gy) = \mu(g) \cdot \psi(x, y) \text{ for some } \mu(g) \in \mathbb{Q}^\times\}.$$

We think of this group as a variant of GSp with the base ring B. However, notice that we require $\mu(g) \in \mathbb{Q}^\times$ and ψ is not B-linear. The identity component of $G_{B,V}$ is reductive but $G_{B,V}$ is not necessarily connected.

Write G for $G_{B,V}$. If there exists a homomorphism $h : \mathbb{S} \to G_\mathbb{R}$ such that (V, h) has type $\{(-1, 0), (0, -1)\}$ as a rational Hodge structure and the form $\psi(u, h(i)v)$ is symmetric and positive-definite then let X denote the $G(\mathbb{R})$-conjugacy class.

Definition 6.1.13. Given that it exists and G is connected, the pair (G, X) above is called a *PEL Shimura datum*.

The $\mathbb{C}$-points of $\mathrm{Sh}_K(G, X)$ for (G, X) a PEL Shimura datum and $K \subset G(\mathbb{A}^\infty)$ a compact open subgroup parametrize certain tuples (A, s, ι, η) where A is a complex abelian variety, s is a polarization compatible with ψ in some way, ι is an action of B rationally on A, and η is a level structure associated to the subgroup K. For details, see [Mil05b, Chapter 8].

Viewing ψ as a $\mathbb{Q}$-bilinear symplectic form gives a map of Shimura data $(G, X) \to (\mathrm{GSp}(\psi), X(\psi))$ with $X(\psi)$ the space of complex structures on $V(\mathbb{R})$ [Mil05b, p.69]. We call the pair $(\mathrm{GSp}(\psi), X(\psi))$ the associated *Siegel Shimura datum* and the map gives an associated map of Shimura varieties realizing the universal family of abelian varieties over the PEL Shimura varieties attached to (G, X).

PEL Shimura datum for a Hilbert modular variety

We will use the above strategy to form a variant Shimura datum for certain Hilbert modular varieties. As seen previously, the main obstacle to be overcome is the failure of (SV5) for $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$. The new data we build fixes this in a minimal way by modifying the center of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$. As before, let F be a totally real field of degree $d := [F : \mathbb{Q}]$. We consider $B = F$ as a $\mathbb{Q}$-algebra with the trivial involution $*$. We then build a symplectic $(B, *)$-module (V, ψ) of rank 2 in the standard way: let $V = F^{\oplus 2}$ and consider the F-linear alternating form $\langle -, - \rangle_F$ defined by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and define the $\mathbb{Q}$-bilinear form $\psi : V \times V \to \mathbb{Q}$ via

$$\psi(u, v) := \mathrm{tr}_{F/\mathbb{Q}} \langle u, v \rangle_F \,.$$

Now we form the PEL Shimura datum from this symplectic module. Note that for $g \in \mathrm{GL}_2(F)$ and $u, v \in V$ we compute

$$\psi(gu, gv) = \mathrm{tr}_{F/\mathbb{Q}}(\det g \cdot \langle u, v \rangle_F).$$

The condition $\psi(g-, g-) = \mu(g) \cdot \psi(-, -)$ for $\mu(g) \in \mathbb{Q}^\times$ then forces $\mu(g) = \det g \in \mathbb{Q}^\times$. Hence $G^{\mathrm{PEL}} := G_{B,V}$ is the subgroup of $G^{\mathrm{Hilb}} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ consisting of matrices with rational determinant meaning precisely it is formed via the fiber product,

$$\begin{tikzcd} G^{\mathrm{PEL}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{G}_{m,\mathbb{Q}} \arrow[d, hook] \\ \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F} \arrow[r, "\det"] & \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F} \end{tikzcd}$$

The standard homomorphism $h_0 : \mathbb{S} \to G^{\mathrm{Hilb}}_{\mathbb{R}}$ given by

$$x + iy \mapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \ldots, \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right) \in \prod_{\sigma \in \Sigma} \mathrm{GL}_{2,\mathbb{R}}$$

actually lands in $G^{\mathrm{PEL}}_{\mathbb{R}} \subset G^{\mathrm{Hilb}}_{\mathbb{R}}$. Indeed, because the natural map $\mathrm{GL}_{2,\mathbb{Q}} \hookrightarrow \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$ becomes the diagonal embedding

$$\mathrm{GL}_{2,\mathbb{R}} \to \prod_{\sigma \in \Sigma} \mathrm{GL}_{2,\mathbb{R}}$$

after base change from $\mathbb{Q}$ to $\mathbb{R}$ we see that h_0 lands inside $(\mathrm{GL}_{2,\mathbb{Q}} \hookrightarrow G^{\mathrm{Hilb}})_{\mathbb{R}}$ and, in particular, is contained in $G^{\mathrm{PEL}}_{\mathbb{R}}$. Let X^{PEL} be the $G^{\mathrm{PEL}}(\mathbb{R})$-orbit of h_0 under conjugacy. Since the center acts trivially under the conjugation action, as previously, $X^{\mathrm{PEL}} = (\mathfrak{H}^{\pm})^{\Sigma}$ acted on by fractional linear transformations.

Example 6.1.14. Suppose we take the level subgroup $K = G^{\mathrm{PEL}}(\mathbb{A}^{\infty}) \cap \mathrm{GL}_2(\widehat{\mathscr{O}_F})$ consisting of matrices in $\mathrm{GL}_2(\mathbb{A}_F^{\infty})$ whose entries have no denominators at any place of F and whose determinant lies in $\mathbb{A}^{\infty} \subset \mathbb{A}_F^{\infty}$. We would like to compute the associated Hilbert modular variety,

$$\mathrm{Sh}_K(G^{\mathrm{PEL}}, X^{\mathrm{PEL}}) := G^{\mathrm{PEL}}(\mathbb{Q})^+ \backslash X^{\mathrm{PEL}} \times G^{\mathrm{PEL}}(\mathbb{A}^{\infty})/K \xrightarrow{\sim} \coprod_{g \in G^{\mathrm{PEL}}(\mathbb{Q})^+ \backslash G^{\mathrm{PEL}}(\mathbb{A}^{\infty})/K} \Gamma_g \backslash X^+$$

Here, $X^+ = \mathfrak{H}^{\Sigma}$ is a choice of connected component of X^{PEL} in the real topology. First we compute the index set,

$$g \in G^{\mathrm{PEL}}(\mathbb{Q})^+ \backslash G^{\mathrm{PEL}}(\mathbb{A}^{\infty})/K.$$

As before, we may consider the valuations of the determinant of an element of $G^{\mathrm{PEL}}(\mathbb{A}^{\infty}) \subset \mathrm{GL}_2(\mathbb{A}_F^{\infty})$. However, by definition, the determinant lives in $\mathbb{A}^{\times,\infty}$ and $\mathscr{C}\ell^+(\mathbb{Q}) = 0$ so the natural class-group valued invariants are trivial. In fact, by the strong approximation theorem (see [Mil05b, Theorem 4.16]) applied to $G^{\mathrm{PEL,der}} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$, this double coset space is a singleton.

Finally we compute

$$\Gamma := G^{\mathrm{PEL}}(\mathbb{Q})^+ \cap \mathrm{GL}_2(\widehat{\mathscr{O}}_F) = \mathrm{SL}_2(\mathscr{O}_F)$$

because the intersection consists of matrices with rational integral positive determinant but 1 is the only positive unit in $\mathbb{Z}$. Therefore

$$\mathrm{Sh}_K(G^{\mathrm{PEL}}, X^{\mathrm{PEL}}) = \mathrm{SL}_2(\mathscr{O}_F) \backslash \mathfrak{H}^{\Sigma}.$$

6.2 Moduli Problems

In the following sections, we largely follow the notation and exposition of [GdS24, DS23, Dia23].

As before, fix a totally real field F of degree $d := [F : \mathbb{Q}]$. Let $D = \mathrm{disc}_{F/\mathbb{Q}}$ be the absolute discriminant of F.

Definition 6.2.1. Let $\mathfrak{c}$ be a fractional ideal of F and $N \geq 3$ an integer. A *family of $\mathfrak{c}$-polarized Hilbert–Blumenthal abelian varieties with full level structure N* over a base scheme S is a four-tuple

$$\underline{A} = (A, \iota, \lambda, \eta)$$

where

1. $A \to S$ is an abelian scheme of relative dimension $d = [F : \mathbb{Q}]$
2. $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}(A/S)$ is an injective ring homomorphism such that[3] $\mathrm{Lie}(A/S)$ is, locally on S, a free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$-module of rank 1
3. λ is a $\mathfrak{c}$-polarization in the sense of [Kat78] (1.0.7). Precisely, $\lambda : \mathfrak{c} \otimes_{\mathcal{O}_F} A \xrightarrow{\sim} A^\vee$ is an isomorphism[4] compatible with the $\mathcal{O}_F$-structures (where $a \in \mathcal{O}_F$ acts on $\mathfrak{c} \otimes_{\mathcal{O}_F} A$ via $a \otimes 1$ and on $A^\vee$ via $\iota(a)^\vee$) and symmetric in the sense that the induced map of étale sheaves
$$\mathfrak{c} \subset \mathrm{Hom}_{S,\mathcal{O}_F}(A, \mathfrak{c} \otimes_{\mathcal{O}_F} A) \xrightarrow{\sim} \mathrm{Hom}_S(A, A^\vee)$$
lands inside the subsheaf of symmetric elements. Furthermore, we require that the induced map of étale sheaves
$$\lambda_* : \mathfrak{c} \to \mathrm{Hom}_S^{\mathrm{sym}}(A, A^\vee)$$
is an isomorphism identifying $\mathfrak{c}^+$ with the sheaf of polarizations (symmetric maps $A \to A^\vee$ which are locally induced by an ample line bundle)
4. an $\mathcal{O}_F$-linear isomorphism
$$\eta : (\mathcal{O}_F/N)^{\oplus 2} \xrightarrow{\sim} A[N]$$

[3]This "Rapoport condition" is lacking from the moduli problem considered in [DP94]. However, when D is invertible on S this condition is automatically satisfied (see [DP94, Corollary 2.9]) hence over $\mathrm{Spec}\, \mathbb{Z}[(ND)^{-1}]$ there is no ambiguity about the moduli problems considered.

[4]The scheme $\mathfrak{c} \otimes_{\mathcal{O}_F} A$ is defined by the functor $T \mapsto A(T) \otimes_{\mathcal{O}_F} \mathfrak{c}$ using the structure ι to make $A(T)$ an $\mathcal{O}_F$-module. This is easily seen to be represented by an abelian scheme.

of étale sheaves on S.

Let $\mathscr{M}^{\mathfrak{c}}(N)$ be the moduli functor parametrizing such $\mathfrak{c}$-polarized Hilbert–Blumenthal abelian varieties with full level structure N.

The moduli functor $\mathscr{M}^{\mathfrak{c}}(N)$ is represented by a smooth (see [Rap78, Theorem 1.20 and Theorem 1.22] and [DP94, Theorem 2.2]) quasi-projective (see [Cha85, Theorem 1.4]) $\operatorname{Spec}(\mathbb{Z}[(ND)^{-1}])$-scheme of relative dimension d which we shall denote by $Y^{\mathfrak{c}}(N)$ and call a *Hilbert modular scheme*.

Remark 6.2.2. Since the structure of the $\mathscr{O}_F$-action ι forces A to have many automorphisms which are, by definition, *compatible* with the polarization and level structures (and commute with ι since $\mathscr{O}_F$ is commutative), we should ask how it can be possible that this moduli problem is represented by a scheme since it appears to have nontrivial stabilizers. The resolution is located in precisely understanding the sense in which the polarization is equivariant for the $\mathscr{O}_F$-module structure and how this differs from saying that certain automorphisms of A are automorphisms of the polarized pair (A, λ). Recall that, as in the Siegel moduli problem, a morphism $f : (A, \lambda_A) \to (B, \lambda_B)$ of polarized abelian varieties is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda_A"'] & B \arrow[d, "\lambda_B"] \\ A^\vee & B^\vee \arrow[l, "f^\vee"] \end{tikzcd}$$

where the direction of the map $f^\vee : B^\vee \to A^\vee$ is forced upon us when we do not identify A with B. However, in the case that $A = B$ and we are investigating the automorphisms of the pair (A, λ_A) one might ask about the commutativity of the same diagram with the direction of the dual arrow flipped. Precisely, this is the sense in which we say that an endomorphism $f : A \to A$ is *compatible* with the polarization or that the polarization is *equivariant* or *compatible* with a set of automorphisms. Explicitly, f is compatible with λ_A if the diagram,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda_A"'] & A \arrow[d, "\lambda_A"] \\ A^\vee \arrow[r, "f^\vee"] & A^\vee \end{tikzcd}$$

commutes. In the particular case of $\mathfrak{c}$-polarizations of HBAVs, we ask that the polarization is equivariant for the $\mathscr{O}_F$-structures in this second sense expressed by commutativity of the following diagrams,

$$\begin{tikzcd}
\mathfrak{c} \otimes_{\mathcal{O}_F} A \arrow[r, "\mathrm{id}\otimes\iota(\nu)"] \arrow[d, "\lambda"'] & \mathfrak{c} \otimes_{\mathcal{O}_F} A \arrow[d, "\lambda"] \\
A^\vee \arrow[r, "\iota(\nu)^\vee"'] & A^\vee
\end{tikzcd}$$

for $\nu \in \mathcal{O}_F$. However, we can also ask if $\nu \in \mathcal{O}_F$ defines an endomorphism of the polarized pair (A, λ) in the primary Siegel moduli sense. This requires that the reversed dual arrow diagram

$$\begin{tikzcd}
\mathfrak{c} \otimes_{\mathcal{O}_F} A \arrow[r, "\mathrm{id}\otimes\iota(\nu)"] \arrow[d, "\lambda"'] & \mathfrak{c} \otimes_{\mathcal{O}_F} A \arrow[d, "\lambda"] \\
A^\vee & A^\vee \arrow[l, "\iota(\nu)^\vee"']
\end{tikzcd}$$

commutes. Combining these diagrams and noting that λ is an isomorphism gives $\iota(\nu)^\vee \circ \iota(\nu)^\vee = \mathrm{id}_{A^\vee}$ and hence $\nu = \pm 1$ in $\mathcal{O}_F$ since ι is injective. Therefore, if $N \geq 3$, the combination of the polarization and level structures rigidify away all automorphisms.

6.2.1 Unit action on polarizations

The group $(\mathcal{O}_F^\times)^+$ of totally positive units acts on $Y^{\mathfrak{c}}(N)$ by modifying the polarization as follows:

$$\nu \cdot (A, \iota, \lambda, \eta) = (A, \iota, \nu \cdot \lambda, \eta)$$

where $\nu \cdot \lambda : \mathfrak{c} \otimes_{\mathcal{O}_F} A \to A^\vee$ is the isomorphism given by $a \otimes x \mapsto \iota(\nu)^\vee \cdot \lambda(a \otimes x)$. Likewise, there is an obvious $\mathrm{GL}_2(\mathcal{O}_F/N)$-action on the level structure:

$$u \cdot (A, \iota, \lambda, \eta) = (A, \iota, \lambda, \eta \circ u^{-1})$$

for $u \in \mathrm{GL}_2(\mathcal{O}_F/N)$ viewed as an automorphism of $(\mathcal{O}_F/N)^{\oplus 2}$. These two actions are compatible in the following sense: for all $\mu \in \mathcal{O}_F^\times$ the action of $\mu^2 \in (\mathcal{O}_F^\times)^+$ agrees with the action of $\mu^{-1} \cdot \mathrm{id} \in \mathrm{GL}_2(\mathcal{O}_F/N)$ via the natural isomorphism:

$$(A, \iota, \lambda, \eta \circ \mu \cdot \mathrm{id}) \xrightarrow{\sim} (A, \iota, \mu^2 \cdot \lambda, \eta) \tag{6.1}$$

induced by $\iota(\mu) : A \to A$. Since $\mathcal{O}_F$ is commutative, this defines an isomorphism of HBAVs which are $\mathfrak{c}$-polarized via polarizations determined by the commutativity of the following diagram

$$
\begin{tikzcd}
\mathfrak{c} \otimes_{\mathcal{O}_F} A \arrow[r, "\mathrm{id}\otimes\iota(\mu)"] \arrow[d, "\mu^2\cdot\lambda"'] & \mathfrak{c} \otimes_{\mathcal{O}_F} A \arrow[d, "\lambda"] \\
A^\vee & A^\vee \arrow[l, "\iota(\mu)^\vee"]
\end{tikzcd}
$$

since compatibility says $\iota(\mu)^\vee \circ \lambda \circ (\mathrm{id} \otimes \iota(\mu)) = \iota(\mu^2)^\vee \circ \lambda$ which is $\mu^2 \cdot \lambda$ by definition.

6.2.2 Hilbert modular varieties of level U

The action of $\mathrm{GL}_2(\mathcal{O}_F/N)$ on $Y^{\mathfrak{c}}(N)$ induces an action of $(\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathrm{GL}_{2,\mathbb{Z}})(\widehat{\mathbb{Z}}) = \mathrm{GL}_2(\widehat{\mathcal{O}_F})$ via the quotient $\mathrm{GL}_2(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_2(\mathcal{O}_F/N)$. Since the unit action on polarizations commutes with reparametrization of the level structure, we get an action

$$(\mathcal{O}_F^\times)^+ \times \mathrm{GL}_2(\widehat{\mathcal{O}_F}) \circlearrowright Y^{\mathfrak{c}}(N)$$

which is trivial on the subgroup

$$\{(\mu^2, u) \in (\mathcal{O}_F^\times)^+ \times \mathrm{GL}_2(\mathcal{O}_F) \mid \mu \in \mathcal{O}_F^\times : u \equiv \mu \cdot \mathrm{id} \mod N\}.$$

Now choose a compact open subgroup $U \subset \mathrm{GL}_2(\widehat{\mathcal{O}_F})$ such that $U(N) \subset U$ where

$$U(N) := \ker(\mathrm{GL}_2(\widehat{\mathcal{O}_F}) \to \mathrm{GL}_2(\mathcal{O}_F/N)).$$

Then we form the subquotient,

$$G_{U,N} := ((\mathcal{O}_F^\times)^+ \times U)/\{(\mu^2, u) \mid \mu \in \mathcal{O}_F^\times, u \in U : u \equiv \mu \cdot \mathrm{id} \mod N\}.$$

Since $Y^{\mathfrak{c}}(N) \to \mathrm{Spec}\,(\mathbb{Z}[(ND)^{-1}])$ is quasi-projective, the quotient $Y^{\mathfrak{c}}(N)/G_{U,N}$ by the finite group $G_{U,N}$ is representable by a quasi-projective scheme over $\mathrm{Spec}\,(\mathbb{Z}[(ND)^{-1}])$ (by [Gro71, Prop.V.1.8]) which is smooth and $Y^{\mathfrak{c}}(N) \to Y^{\mathfrak{c}}(N)/G_{U,N}$ is étale when $G_{U,N}$ acts without fixed points. A sufficient condition for $G_{U,N}$ to act freely is given in [DS23, Lemma 2.4.1] which requires U to be sufficiently small (in a way that depends only on F).

Definition 6.2.3. For a compact open subgroup $U \subset \mathrm{GL}_2(\widehat{\mathcal{O}_F})$ such that $U(N) \subset U$, the *Hilbert modular variety* Y_U is defined as the disjoint union of quotients

$$Y_U := \coprod_{[\mathfrak{c}] \in \mathscr{Cl}^+(F)} Y^{\mathfrak{c}}(N)/G_{U,N}.$$

First, we note that the isomorphism class of $Y^{\mathfrak{c}}(N)$ is well-defined for an ideal class

$[\mathfrak{c}] \in \mathscr{C}\ell^+(F)$. Indeed, if $\mathfrak{c} = \gamma\mathfrak{c}'$ where $\gamma \gg 0$, i.e. $\gamma \in F$ is totally positive, then $\lambda \circ (\gamma \otimes 1) : \mathfrak{c}' \otimes_{\mathscr{O}_F} A \xrightarrow{\sim} A^\vee$ is a $\mathfrak{c}'$-polarization of A. Thus there is an isomorphism $\gamma : Y^{\mathfrak{c}}(N) \xrightarrow{\sim} Y^{\mathfrak{c}'}(N)$ of moduli schemes depending on the choice of γ sending,

$$(A, \iota, \lambda, \eta) \mapsto (A, \iota, \lambda \circ (\gamma \otimes 1), \eta).$$

The definition of Y_U is independent of N for all N large enough that $U(N) \subset U$ (see [DS23, § 2.5.]) and produces an integral model for the Hilbert modular variety defined as a complex Shimura variety in the following sense.

Theorem 6.2.4. $Y_U(\mathbb{C}) \cong \mathrm{Sh}_U(G^{\mathrm{Hilb}}, X^{\mathrm{Hilb}})$ *canonically.*

Proof. See §2.7 of [DS23] or §2.2 of [Dia23]. □

Remark 6.2.5. While the geometric connected components of Y_U indexed by $\overline{\mathbb{Q}}$-points of $Z^{\mathfrak{c}}(N)$ are permuted by the Galois action, the disjoint union over classes $[\mathfrak{c}] \in \mathscr{C}\ell^+(F)$ is done over $\mathbb{Q}$ so the Galois action preserves this decomposition. This aligns with the Galois action on the components of a Shimura variety is discussed in [Mil05b, p.119].

6.2.3 Connected components

Denote by $\mathfrak{d} = \mathfrak{d}_{F/\mathbb{Q}}$ the absolute different ideal of F. This ideal is relevant to the duality of locally free $\mathscr{O}_F$-modules because of the isomorphism of $\mathscr{O}_F$-modules,

$$\mathfrak{d}^{-1} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathscr{O}_F, \mathbb{Z})$$

via the trace pairing. Furthermore, for any $\mathscr{O}_F$-module M (or sheaf of $\mathscr{O}_F \otimes \mathscr{O}_S$-modules) we have a natural isomorphism of $\mathscr{O}_F$-modules,

$$M^{\otimes -1} := \mathrm{Hom}_{\mathscr{O}_F}(M, \mathscr{O}_F) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes_{\mathscr{O}_F} \mathfrak{d}^{-1}$$

defined by the following map induced from the trace pairing,

$$\varphi \mapsto (m \otimes a \mapsto \mathrm{tr}_{F/\mathbb{Q}}(a\varphi(m))).$$

Definition 6.2.6. Let $Z^{\mathfrak{c}}(N)$ denote the $\mathbb{Z}[(ND)^{-1}]$-scheme representing $\mathscr{O}_F$-linear isomorphisms

$$\mathfrak{c} \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mu_N.$$

If $\underline{A} = (A, \iota, \lambda, \eta)$ is a $\mathfrak{c}$-polarized HBAV with full level structure N over S, then $\lambda \otimes \wedge^2 \eta$ defines an isomorphism

$$\mathfrak{c} \otimes (\mathbb{Z}/N\mathbb{Z}) = \mathfrak{c} \otimes_{\mathcal{O}_F} \wedge^2_{\mathcal{O}_F} (\mathcal{O}_F/N)^2 \xrightarrow{\sim} \mathrm{Hom}_S^{\mathrm{sym}}(A, A^\vee) \otimes_{\mathcal{O}_F} \wedge^2_{\mathcal{O}_F} A[N]$$

But the Weil pairing

$$\langle -, - \rangle_\lambda : A[N] \otimes A[N] \to \mu_N$$

for $\lambda \in \mathrm{Hom}_S^{\mathrm{sym}}(A, A^\vee)$ defines an isomorphism

$$\mathrm{Hom}_S^{\mathrm{sym}}(A, A^\vee) \otimes_{\mathcal{O}_F} \wedge^2_{\mathcal{O}_F} A[N] \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_F, \mu_N) = \mathfrak{d}^{-1} \otimes \mu_N$$

and hence an element of $Z^{\mathfrak{c}}(N)(S)$. This produces a map

$$Y^{\mathfrak{c}}(N) \to Z^{\mathfrak{c}}(N)$$

Proposition 6.2.7. *The fibers of $Y^{\mathfrak{c}}(N) \to Z^{\mathfrak{c}}(N)$ are geometrically integral.*

Proof. This is discussed in [DP94, §2.4] (cf. [DS23, §2.2]) and [DP94, §2.13]. □

It will be convenient to introduce a second standard level structure studied by Katz in [Kat78] which has the advantage of defining a single connected component integrally. This is also the level structure used in [GdS24] from which we will take many of the subsequent results.

Definition 6.2.8. Let $\mathfrak{c}$ be a fractional ideal of F and $N \geq 4$ an integer[5]. A *family of $\mathfrak{c}$-polarized Hilbert–Blumenthal abelian varieties with $\Gamma_{00}(N)$-level structure* over a base scheme S is a four-tuple

$$\underline{A} = (A, \iota, \lambda, \eta)$$

where (A, ι, λ) is a $\mathfrak{c}$-polarized HBAV and η is a $\Gamma_{00}(N)$-level structure on A in the sense of [Kat78] (1.0.8), i.e. a closed embedding $\eta : \mathfrak{d}^{-1} \otimes \mu_N \hookrightarrow A[N]$ of S-group schemes compatible with ι. Let $\mathscr{M}_{00}^{\mathfrak{c}}(N)$ be the moduli functor parametrizing such $\mathfrak{c}$-polarized Hilbert–Blumenthal abelian varieties with $\Gamma_{00}(N)$-level structure.

As before, the moduli functor $\mathscr{M}_{00}^{\mathfrak{c}}(N)$ is represented by a smooth quasi-projective $\mathrm{Spec}\,(\mathbb{Z}[(ND)^{-1}])$-scheme of relative dimension d which we shall denote by $Y_{00}^{\mathfrak{c}}(N)$ (see

[5]Here we impose the further condition that $N \geq 4$ to ensure that the Hilbert–Blumenthal data has no automorphisms. There can be automorphisms even with $\Gamma_{00}(N)$-level structure for $N = 3$. For example, the elliptic curve with CM by $\mathbb{Z}[\zeta_3]$ has an automorphism of order 3 preserving a 3-torsion point.

[Rap78, Kat78] for more discussion). The fibers of $Y_{00}^{\mathfrak{c}}(N) \to \operatorname{Spec}(\mathbb{Z}[(ND)^{-1}])$ are geometrically integral by [DR80, Corollary 4.6] (cf. [DP94, §2.4 and §2.13]).

Because of the map $Y^{\mathfrak{c}}(N) \to Z^{\mathfrak{c}}(N)$, given an inclusion $\ell : \mathfrak{c} \otimes (\mathbb{Z}/N\mathbb{Z}) \hookrightarrow (\mathscr{O}_F/N)^{\oplus 2}$ of $\mathscr{O}_F$-modules, we obtain a forgetful map

$$Y^{\mathfrak{c}}(N) \xrightarrow{j} Y_{00}^{\mathfrak{c}}(N)$$

sending $\underline{A} = (A, \iota, \lambda, \eta) \mapsto (A, \iota, \lambda, \eta \circ \ell)$ where $\eta \circ \ell$ is the $\Gamma_{00}(N)$-level structure:

$$\mathfrak{d}^{-1} \otimes \mu_N \xrightarrow{\lambda \otimes \wedge^2 \eta} \mathfrak{c} \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\ell} (\mathscr{O}_F/N)^{\oplus 2} \xrightarrow{\eta} A[N].$$

These maps are finite étale since $A[N]$ is finite étale over $\operatorname{Spec} \mathbb{Z}[(ND)^{-1}]$ and the various choices of a map ℓ are permuted by the action of $\mathrm{GL}_2(\mathscr{O}_F/N)$ on $Y^{\mathfrak{c}}(N)$ by acting on the level structure (since $\mathrm{GL}_2(\mathscr{O}_F/N)$ acts transitively on the set of such maps ℓ).

6.2.4 Tautological bundles and linearizations

Let $\mathscr{M}$ denote a Hilbert modular scheme developed in the previous sections carrying a universal family $\pi : \mathscr{A} \to \mathscr{M}$ (meaning either $Y^{\mathfrak{c}}(N)$ or $Y_{00}^{\mathfrak{c}}(N)$ but not Y_U in general) of $\mathfrak{c}$-polarized HBAVs. Write

$$\underline{\omega} = \pi_* \Omega^1_{\mathscr{A}/\mathscr{M}}$$

for the *Hodge bundle* of $\mathscr{M}$. Let $\underline{\mathrm{Lie}} := \mathrm{Lie}(\mathscr{A}/\mathscr{M})$ denote the Lie algebra sheaf of the universal abelian scheme over $\mathscr{M}$ which is a locally free $\mathscr{O}_F \otimes \mathscr{O}_{\mathscr{M}}$-module of rank 1 identified with $\underline{\omega}^\vee := \mathrm{Hom}_{\mathscr{O}_{\mathscr{M}}}(\underline{\omega}, \mathscr{O}_{\mathscr{M}})$ as a sheaf of $\mathscr{O}_F \otimes \mathscr{O}_{\mathscr{M}}$-modules. Hence, they are related by a $\mathscr{O}_F$-pairing

$$\underline{\mathrm{Lie}} \otimes_{\mathscr{O}_F \otimes \mathscr{O}_{\mathscr{M}}} \underline{\omega} \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mathscr{O}_{\mathscr{M}}$$

whose composition with the trace map

$$\mathfrak{d}^{-1} \otimes \mathscr{O}_{\mathscr{M}} \xrightarrow{\mathrm{tr}_{F/\mathbb{Q}}} \mathbb{Z} \otimes \mathscr{O}_{\mathscr{M}} = \mathscr{O}_{\mathscr{M}}$$

yields the standard duality $\mathscr{O}_{\mathscr{M}}$-pairing

$$\underline{\mathrm{Lie}} \otimes_{\mathscr{O}_{\mathscr{M}}} \underline{\omega} \twoheadrightarrow \underline{\mathrm{Lie}} \otimes_{\mathscr{O}_F \otimes \mathscr{O}_{\mathscr{M}}} \underline{\omega} \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mathscr{O}_{\mathscr{M}} \xrightarrow{\mathrm{tr}_{F/\mathbb{Q}}} \mathscr{O}_{\mathscr{M}}.$$

Now focus on the case $\mathscr{M} = Y^{\mathfrak{c}}(N)$. This modular scheme carries an action $(\mathscr{O}_F^\times)^+ \times \mathrm{GL}_2(\widehat{\mathscr{O}_F}) \circlearrowright Y^{\mathfrak{c}}(N)$ extending to an equivariant action on the universal abelian scheme

$\pi : \mathscr{A} \to \mathscr{M}$. Indeed, $g = (\nu, u) \in G := (\mathscr{O}_F^\times)^+ \times \mathrm{GL}_2(\widehat{\mathscr{O}_F})$ acts via the map of moduli functors

$$g : (A, \iota, \lambda, \eta) \mapsto (A, \iota, \nu\lambda, \eta \circ u^{-1})$$

meaning that $g : \mathscr{M} \to \mathscr{M}$ is equipped with a unique isomorphism $\alpha_g : \mathscr{A} \to g^*\mathscr{A}$ sending $g^*\lambda$ to $\nu\lambda$ and $g^*\eta$ to $\eta \circ u^{-1}$. These isomorphisms α_g give $\pi : \mathscr{A} \to \mathscr{M}$ the structure of an equivariant family inducing G-equivariant structures on the sheaves

$$\underline{\omega},\ \underline{\mathrm{Lie}},\ R^1\pi_*\mathscr{O}_{\mathscr{A}}$$

commuting with the $\mathscr{O}_F$-module structures. In particular, the pairing

$$\underline{\mathrm{Lie}} \otimes_{\mathscr{O}_F \otimes \mathscr{O}_{\mathscr{M}}} \underline{\omega} \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mathscr{O}_{\mathscr{M}}$$

is equivariant. However, more care must be taken with another isomorphism between these sheaves. Since $\mathscr{A}$ is $\mathfrak{c}$-polarized, there is an isomorphism

$$R^1\pi_*\mathscr{O}_{\mathscr{A}} = \mathrm{Lie}(\mathscr{A}^\vee/\mathscr{M}) \xrightarrow{\lambda^{-1}} \mathrm{Lie}(\mathfrak{c} \otimes_{\mathscr{O}_F} \mathscr{A}/\mathscr{M}) = \mathfrak{c} \otimes_{\mathscr{O}_F} \underline{\mathrm{Lie}} \tag{6.2}$$

However, this isomorphism does not respect the equivariant structures on $\underline{\mathrm{Lie}}$ and $R^1\pi_*\mathscr{O}_{\mathscr{A}}$ because λ is not preserved by α_g. Rather, the following compatibility diagram

$$\begin{tikzcd}
g^*\underline{\mathrm{Lie}} \otimes_{\mathscr{O}_F} \mathfrak{c} \arrow[r, "g^*\lambda"] & g^*R^1\pi_*\mathscr{O}_{\mathscr{A}} \arrow[d, "\alpha_g^*"] \\
\underline{\mathrm{Lie}} \otimes_{\mathscr{O}_F} \mathfrak{c} \arrow[u, "(\alpha_g)_*"] \arrow[r, "\nu\lambda"] & R^1\pi_*\mathscr{O}_{\mathscr{A}}
\end{tikzcd}$$

commutes. Since $R^1\pi_*\mathscr{O}_{\mathscr{A}}$ is given an equivariant structure via the isomorphisms α_g^*, in order to make a consistent choice of the directions of the maps in the conventions for an equivariant sheaf $\underline{\mathrm{Lie}}$ should be given an equivariant structure via the maps $(\alpha_g)_*^{-1} : g^*\underline{\mathrm{Lie}} \to \underline{\mathrm{Lie}}$. Therefore, absorbing the scaling factor ν into the equivariant structure, we obtain a commutative diagram

$$\begin{tikzcd}
g^*\underline{\mathrm{Lie}} \arrow[r, "g^*\lambda"] \arrow[d, "(\alpha_g)_*^{-1}\otimes\nu"'] & g^*R^1\pi_*\mathscr{O}_{\mathscr{A}} \arrow[d, "\alpha_g^*"] \\
\underline{\mathrm{Lie}} \otimes_{\mathscr{O}_F} \mathfrak{c} \arrow[r, "\lambda"] & R^1\pi_*\mathscr{O}_{\mathscr{A}}
\end{tikzcd}$$

which has the pleasing consequence that, if we equip $\mathfrak{c} \otimes \mathscr{O}_{\mathscr{M}}$ with the equivariant structure given by multiplication by ν to give an equivariant sheaf of $\mathscr{O}_{\mathscr{M}}$-modules denoted by $\underline{\mathfrak{c}}$, then we obtain an equivariant isomorphism of $\mathcal{O}_F$-modules with G-actions

$$\underline{\mathrm{Lie}} \otimes_{\mathcal{O}_F \otimes \mathcal{O}_S} \underline{\mathfrak{c}} \xrightarrow{\sim} R^1\pi_*\mathcal{O}_{\mathcal{A}}. \tag{6.3}$$

6.2.5 The Kodaira–Spencer isomorphism

The Hodge-theoretic Kodaira–Spencer map for the universal family,

$$\delta : \pi_*\Omega_{\mathcal{A}/\mathcal{M}} \to \Omega_{\mathcal{M}/\mathbb{Z}} \otimes_{\mathcal{O}_{\mathcal{M}}} R^1\pi_*\mathcal{O}_{\mathcal{A}}$$

arises as the connecting map for $R^1\pi_*$ applied to the pullback exact sequence

$$0 \to \pi^*\Omega_{\mathcal{M}/\mathbb{Z}} \to \Omega_{\mathcal{A}/\mathbb{Z}} \to \Omega_{\mathcal{A}/\mathcal{M}} \to 0$$

and is therefore induced as a graded part of the Gauss–Manin connection on relative de Rham cohomology of the universal family. This Kodaira–Spencer map δ is actually $\mathcal{O}_F$-equivariant with respect to the $\mathcal{O}_F$-module structures on $\pi_*\Omega_{\mathcal{A}/\mathcal{M}} = \underline{\omega}$ and $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ and is equivariant for the $(\mathcal{O}_F^\times)^+ \times \mathrm{GL}_2(\widehat{\mathcal{O}_F})$-action in the case $\mathcal{M} = Y^{\mathfrak{c}}(N)$ since the pullback exact sequence is equivariant. Since $\underline{\omega}$ and $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ are invertible $\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{M}}$-modules, by dualizing, we obtain a morphism

$$\mathrm{KS} : \mathcal{T}_{\mathcal{M}/\mathbb{Z}} \otimes \mathcal{O}_F \to \underline{\omega}^{\otimes -1} \otimes R^1\pi_*\mathcal{O}_{\mathcal{A}}$$

called the Kodaira–Spencer map. Note that the tensor operations on the right are as $\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{M}}$-modules. Using the isomorphism (6.3) produces

$$\mathrm{KS} : \mathcal{T}_{\mathcal{M}/\mathbb{Z}} \to \underline{\omega}^{\otimes -1} \otimes \underline{\mathrm{Lie}} \otimes \underline{\mathfrak{c}} = \underline{\mathrm{Lie}}^{\otimes 2} \otimes \underline{\mathfrak{dc}} \tag{6.4}$$

where $\underline{\mathrm{Lie}}^{\otimes 2}$ is the tensor square as an invertible $\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{M}}$-module as are the other unmarked tensor operations. This map KS is an equivariant isomorphism; see [Kat78] formulas (1.0.19)-(1.0.21). Likewise, its $\mathcal{O}_{\mathcal{M}}$-dual is an isomorphism

$$\mathrm{KS}^\vee : \underline{\omega}^{\otimes 2} \otimes (\underline{\mathfrak{c}})^{-1} \xrightarrow{\sim} \Omega_{\mathcal{M}/\mathbb{Z}}$$

6.2.6 Descent to Y_U

Recall that the action

$$(\mathcal{O}_F^\times)^+ \times \mathrm{GL}_2(\widehat{\mathcal{O}_F}) \circlearrowright Y^{\mathfrak{c}}(N)$$

factors through a finite quotient. In particular, $G_{U,N} \circlearrowright Y^{\mathfrak{c}}(N)$ is well-defined. However, not all subgroups acting trivially on $Y^{\mathfrak{c}}(N)$ act trivially on the universal abelian scheme and hence the tautological bundles are not actually $G_{U,N}$-equivariant. Indeed, the subgroup

$$T_{U,N} := \{(\mu^2, u) \mid \mu \in \mathcal{O}_F^\times, u \in U : u \equiv \mu \cdot \mathrm{id} \mod N\}$$

acts trivially on $Y^{\mathfrak{c}}(N)$ but for $g = (\mu^2, u) \in T_{U,N}$ the map $\alpha_g : \mathscr{A} \to g^*\mathscr{A}$ may be identified with $\iota(\mu) : \mathscr{A} \to \mathscr{A}$, the isomorphism (6.1). Hence g acts on $\underline{\omega}$ and $R^1\pi_*\mathcal{O}_{\mathscr{A}}$ as multiplication by μ through the $\mathcal{O}_F$-structure and on $\underline{\mathrm{Lie}}$ as multiplication by μ^{-1}. Note this respects the isomorphism (6.3) $R^1\pi_*\mathcal{O}_{\mathscr{A}} \xrightarrow{\sim} \underline{\mathrm{Lie}} \otimes \underline{\mathfrak{c}}$ since $\underline{\mathfrak{c}}$ is acted as multiplication by $\nu = \mu^2$. In particular, if we consider the invertible $\mathcal{O}_F \otimes \mathcal{O}_{\mathscr{M}}$-module

$$\underline{\omega}^{\otimes k} \otimes \underline{\mathfrak{a}}^{\otimes l}$$

for $\mathfrak{a}$ a fractional ideal of $\mathcal{O}_F$ then $T_{U,N}$ acts trivially if and only if the condition:

$$(*) \qquad \text{for all } \mu \in \mathcal{O}_F^\times \cap U \text{ we have } \mu^{k+2l} = 1$$

is satisfied. When this is the case, $\underline{\omega}^{\otimes k} \otimes \underline{\mathfrak{a}}^{\otimes l}$ is a $G_{U,N}$-equivariant bundle giving a descent datum and thus a canonical choice of bundle on Y_U pulling back to $\underline{\omega}^{\otimes k} \otimes \underline{\mathfrak{a}}^{\otimes l}$.

In particular, the target $\underline{\mathrm{Lie}}^{\otimes 2} \otimes \underline{\mathfrak{d}\mathfrak{c}}$ of the Kodaira–Spencer isomorphism (6.4) has $k + 2l = 0$ and hence descends to Y_U along with its isomorphism to the tangent bundle of Y_U. This was, of course, guaranteed by KS being G-equivariant for the usual equivariant structure on $\mathcal{T}_{\mathscr{M}/\mathbb{Z}}$ since $\mathcal{T}_{\mathscr{M}/\mathbb{Z}}$ is always equivariant for any group action and thus $T_{U,N}$ must act trivially.

6.3 Tautological Foliations

Hilbert modular varieties are equipped with d tautological rank 1 foliations. Thinking analytically about the quotient $\mathfrak{H}^\Sigma/\Gamma$, these foliations are given by the directions ∂_{z_i} which are preserved by the Γ-quotient since $\Gamma \circlearrowright \mathfrak{H}^\Sigma$ through $\mathrm{GL}_2^+(F) \to \mathrm{GL}_2^+(\mathbb{R})^\Sigma$ which is "diagonal" in the sense that it does not mix the factors indexed by Σ. These analytic foliations define algebraic foliations on the underlying Shimura variety. To access these foliations in mixed characteristic, we employ a moduli interpretation. Let $\pi : A^{\mathrm{univ}} \to Y^{\mathfrak{c}}(N)$ be the universal abelian scheme. The moduli interpretation of these foliations arises from decomposing the Hodge bundle $\underline{\omega} := \pi_*\Omega_{A^{\mathrm{univ}}/Y^{\mathfrak{c}}(N)}$ according to the action of $\mathcal{O}_F$ by various characters. In order for this decomposition to exist, we require an assumption about the base scheme S, namely that $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$ decomposes as the direct sum of d copies of $\mathcal{O}_S$. This occurs, for example, when $\mathcal{O}_S$ contains the images of all embeddings $\sigma : F \hookrightarrow \mathbb{R}$. Since we are moving towards understanding the foliations in positive or mixed characteristic, we now fix a prime and work over a fixed extension of $\mathbb{Z}_p$ needed to ensure this splitting result holds.

Fix a prime number p which is unramified in F and coprime to N so $p \in \mathrm{Spec}\,(\mathbb{Z}[(ND)^{-1}])$. By changing the representative of $\mathfrak{c}$ we may also assume p is relatively prime to $\mathfrak{c}$. Write,

$$p\mathcal{O}_F = \mathfrak{p}_1 \dots \mathfrak{p}_r \qquad f_i := f(\mathfrak{p}_i/p)$$

where $f(\mathfrak{p}_i/p)$ is the inertia degree of $\mathfrak{p}_i$. Let κ be the finite field $\mathbb{F}_{p^f}$ where $f = \mathrm{lcm}\{f_1, \dots, f_r\}$ so that $\kappa(\mathfrak{p}_i) \hookrightarrow \kappa$ for each i. Let $W(\kappa)$ be the Witt ring of κ which we may identify with the ring of integers of the compositum of the completions of all embeddings of F into $\overline{\mathbb{Q}}_p$. There is a decomposition,

$$\mathbb{B} = \mathrm{Hom}(F, W(\kappa)[1/p]) = \bigsqcup_{\mathfrak{p}|p} \mathbb{B}_{\mathfrak{p}}$$

indexed by the r primes of $\mathcal{O}_F$ over p, where $\sigma : F \hookrightarrow W(\kappa)[1/p]$ lies in $\mathbb{B}_{\mathfrak{p}}$ if $\sigma^{-1}(pW(\kappa)) \cap \mathcal{O}_F = \mathfrak{p}$. The Frobenius ϕ of $W(\kappa)$ acts on $\mathbb{B}$ via $\sigma \mapsto \phi \circ \sigma$ permuting each $\mathbb{B}_{\mathfrak{p}}$ cyclically.

This set $\mathbb{B}$ plays the p-adic role of Σ in indexing the foliations but, unlike Σ, it carries a canonical action of Frob_p.

Let $\mathscr{M}$ denote the base change of a Hilbert modular scheme of the previous section (one carrying a universal family: either $Y^{\mathfrak{c}}(N)$ or $Y^{\mathfrak{c}}_{00}(N)$ but not Y_U in general) from $\mathrm{Spec}\,(\mathbb{Z}[(ND)^{-1}])$ to $W(\kappa)$. We denote by G the group canonically acting on $\mathscr{M}$. As previously, we let $\pi : \mathscr{A} \to \mathscr{M}$ denote the universal abelian scheme over $\mathscr{M}$ and

$$\underline{\omega} = \pi_* \Omega^1_{\mathscr{A}/\mathscr{M}}$$

the *Hodge bundle*. Because $p \nmid \mathrm{disc}_{F/\mathbb{Q}}$ and $W(\kappa)$ contains all embeddings of F, there is a decomposition

$$\mathcal{O}_F \otimes \mathcal{O}_{\mathscr{M}} = \bigoplus_{\sigma \in \mathbb{B}} \mathcal{O}_{\mathscr{M}}$$

as a coherent sheaf of $\mathcal{O}_{\mathscr{M}}$-algebras. Recall the "Rapoport condition" that $\mathrm{Lie}(A/S)$ is, locally on S, a free $\mathcal{O}_F \otimes \mathcal{O}_S$-module of rank 1 hence its dual $\pi_* \Omega^1_{\mathscr{A}/\mathscr{M}} = e^* \Omega^1_{\mathscr{A}/\mathscr{M}}$ is also locally free of rank 1. Hence, it decomposes under the $\mathcal{O}_F$-action into a sum of d line bundles

$$\underline{\omega} = \bigoplus_{\sigma \in \mathbb{B}} \omega_\sigma$$

where each ω_σ is a line bundle on $\mathscr{M}$ identified with the subsheaf

$$\omega_\sigma = \{\alpha \in \underline{\omega} \mid \iota(\nu)^*(\alpha) = \sigma(\nu)\alpha : \forall \nu \in \mathscr{O}_F\}.$$

Recall that formula (6.4) gives the Kodaira–Spencer isomorphism

$$\mathrm{KS} : \mathscr{T}_{\mathscr{M}/\mathbb{Z}} \xrightarrow{\sim} \underline{\mathrm{Lie}}^{\otimes 2} \otimes \underline{\mathfrak{d}\mathfrak{c}} \xrightarrow{\sim} \underline{\omega}^{\otimes -2} \otimes \underline{\mathfrak{c}\mathfrak{d}}^{-1}.$$

We also write

$$\underline{\mathfrak{c}\mathfrak{d}}^{-1} = \bigoplus_{\sigma \in \mathbb{B}} \delta_\sigma.$$

Since $(p, \mathfrak{d}\mathfrak{c}) = 1$, these line bundles δ_σ are trivial but do carry a nontrivial G-action. Then we have an isomorphism

$$\mathscr{T}_{\mathscr{M}/\mathbb{Z}} \cong \bigoplus_{\sigma \in \mathbb{B}} \omega_\sigma^{-2} \otimes \delta_\sigma$$

of G-equivariant $\mathscr{O}_{\mathscr{M}}$-modules. Denote by $\mathscr{F}_\sigma$ the canonical sub-bundles of $\mathscr{T}_{\mathscr{M}/\mathbb{Z}}$ corresponding to the summand $\omega_\sigma^{-2} \otimes \delta_\sigma$.

6.3.1 p-curvature and Hasse invariants

Let $\Theta \subset \mathbb{B}$ and consider the sub-bundle

$$\mathscr{F}_\Theta := \bigoplus_{\sigma \in \Theta} \mathscr{F}_\sigma \subset \mathscr{T}_{\mathscr{M}/W(\kappa)}$$

which we now show is a foliation. The following is Lemma 3.2.1 of [GdS24] which we reproduce for the reader's convenience.

Lemma 6.3.1. *The sub-bundle $\mathscr{F}_\Theta$ is involutive meaning that if ξ, η are sections of $\mathscr{F}_\Theta$ so is $[\xi, \eta]$.*

Proof. It suffices to prove that each $\mathscr{F}_\sigma$ is involutive. Recall that the Hodge-theoretic Kodaira–Spencer map is the associated graded of the Gauss–Manin connection

$$\nabla : H^1_{\mathrm{dR}}(\mathscr{A}/\mathscr{M}) \to H^1_{\mathrm{dR}}(\mathscr{A}/\mathscr{M}) \otimes_{\mathscr{O}_{\mathscr{M}}} \Omega^1_{\mathscr{M}/W(\kappa)}$$

which is a flat connection meaning that for sections ξ, η of $\mathscr{T}_{\mathscr{M}/W(\kappa)}$

$$\nabla_{[\xi,\eta]} = \nabla_\xi \circ \nabla_\eta - \nabla_\eta \circ \nabla_\xi.$$

Furthermore, the Gauss–Manin connection is functorial for actions on $\pi : \mathscr{A} \to \mathscr{M}$ relative to $\mathscr{M}$. In particular, it is $\mathscr{O}_F$-linear meaning that ∇_ξ commutes with $\iota(a)^*$ for $a \in \mathscr{O}_F$ and hence preserves the σ-isotypic component $H^1_{\mathrm{dR}}(\mathscr{A}/\mathscr{M})[\sigma]$ for each $\sigma \in \mathbb{B}$. By definition, $\xi \in \mathscr{F}_\sigma$ if for each $\tau \neq \sigma$ the operator ∇_ξ maps the subspace

$$\omega_\tau = \underline{\omega}[\tau] \subset H^1_{\mathrm{dR}}(\mathscr{A}/\mathscr{M})$$

to itself since this means the associated graded is trivial on ω_τ for $\tau \neq \sigma$. Using the flatness relation, if this holds for ∇_ξ and ∇_η it also holds for $\nabla_{[\xi,\eta]}$. □

Now we reduce mod p and consider the foliations $\mathscr{F}_\Theta$ on the special fiber $M := \mathscr{M} \times_{W(\kappa)} \kappa$. Since these are now foliations in pure characteristic p, there is a p-curvature map

$$\psi_{\mathscr{F}_\Theta,p} : \mathrm{Frob}_M^* \mathscr{F}_\Theta \to \mathscr{T}_M/\mathscr{F}_\Theta \xrightarrow{\sim} \mathscr{F}_{\Theta^c}.$$

It turns out these obstructions to p-closedness are related to the *partial Hasse invariants* of a HBAV viewed as special functions on M which are the mod p variants of Hilbert modular forms. These are sections of the *automorphic line bundles*

$$\mathscr{L}^{\underline{k},\underline{\ell}} := \bigotimes_{\sigma \in \mathbb{B}} \omega_\sigma^{k_\sigma} \otimes \delta_\sigma^{\ell_\sigma}$$

for *weight vectors* $\underline{k} := [k_\sigma]_{\sigma\in\mathbb{B}}$ and $\underline{\ell} := [\ell_\sigma]_{\sigma\in\mathbb{B}}$ of integers. We saw in § 6.2.6 that if $(\underline{k}, \underline{\ell})$ is *paritious*, meaning $k_\sigma + 2\ell_\sigma$ is independent of σ, then $\mathscr{L}^{\underline{k},\underline{\ell}}$ descends to a line bundle $\mathscr{L}_U$ on Y_U. A *Hilbert modular form* of weight $(\underline{k}, \underline{\ell})$ and level U over a $W(\kappa)$-scheme S is a section $s \in H^0((Y_U)_S, \mathscr{L}_U^{\underline{k},\underline{\ell}})$ of the automorphic line bundle $\mathscr{L}_U^{\underline{k},\underline{\ell}}$ base changed to S. For more on the definitions and important properties of Hilbert modular forms in the arithmetic setting, see [Kat78] and [Gor02, §2]. We will sometimes write the weight vector $\underline{k}$ as a formal sum $\sum_{\sigma\in\mathbb{B}} k_\sigma[\sigma] \in \mathbb{Z}[\mathbb{B}]$ and drop $\underline{\ell}$ for U a standard congruence subgroup.

The *ordinary locus* $M^{\mathrm{ord}} \subset M$ is the open subscheme where the fibers of the universal abelian variety are ordinary. There are *partial Hasse invariants* h_σ for $\sigma \in \mathbb{B}$ which are Hilbert modular forms of weight $p\left[\phi^{-1} \circ \sigma\right] - [\sigma]$ over κ whose simultaneous nonvanishing defines M^{ord}; see [Gor02, §2.3].

Lemma 6.3.2. *[GdS24, Lemma 3.4.1] For $M = Y_{00}^{\mathfrak{c}}(N)_\kappa$, let $\sigma \neq \tau$. We have,*

$$\Gamma(M, \omega_\sigma^{2p} \otimes \omega_\tau^{-2}) = \begin{cases} 0 & \tau \neq \phi \circ \sigma \\ \kappa \cdot h_\tau^2 & \tau = \phi \circ \sigma \end{cases}$$

For $M = Y^{\mathfrak{c}}(N)_\kappa$ it is likely also true but we would need to repeat the proof since pulling back along a forgetful map $j : Y^{\mathfrak{c}}(N) \to Y_{00}^{\mathfrak{c}}(N)$ might introduce additional sections. However, the analogous result for $Y^{\mathfrak{c}}(N)_\kappa$ will not be necessary.

Consider the p-curvature map on $M = Y_{00}^{\mathfrak{c}}(N)$.

$$\begin{array}{ccc} \mathrm{Frob}_M^* \mathscr{F}_\sigma & \xrightarrow{\psi_{\mathscr{F}_\sigma, p}} & \mathscr{T}_M / \mathscr{F}_\sigma \\ \| & & \| \\ \omega_\sigma^{-2p} \otimes \delta_\sigma^p & \longrightarrow & \bigoplus_{\tau \neq \sigma} \omega_\tau^{-2} \otimes \delta_\tau \end{array}$$

By the lemma, only the projection to the $\tau = \phi \circ \sigma$ factor is nonzero and must be a multiple of the square partial Hasse invariant h_τ^2. Abusing notation, we write $\psi_{\mathscr{F}_\sigma, p} = u_\sigma \cdot h_{\phi\circ\sigma}^2$.

Theorem 6.3.3 (cf. [GdS24, Theorem 3.2.2])**.** *The foliation $\mathscr{F}_\Theta$ is p-closed if and only if Θ is ϕ-stable. In particular $\mathscr{F}_\sigma$ is p-closed if and only if $\mathbb{B}_\mathfrak{p} = \{\sigma\}$ or equivalently $f(\mathfrak{p}_\sigma/p) = 1$. Otherwise, u_σ is a unit so $\psi_{\mathscr{F}_\sigma, p}$ vanishes exactly on the locus where $h_{\phi\circ\sigma}$ does.*

Proof. For $M = Y_{00}^{\mathfrak{c}}(N)$ this follows immediately from [GdS24, Theorem 3.2.2]. For $M = Y^{\mathfrak{c}}(N)$ we use one of the forgetful maps $j : Y^{\mathfrak{c}}(N) \to Y_{00}^{\mathfrak{c}}(N)$ which is finite étale and the compatibility of p-curvature with pullback by Proposition 2.2.12. This argument works because the tautological bundles are compatible with pullback since they are built from the Hodge bundle and Kodaira–Spencer isomorphism which are both compatible. □

6.3.2 Quotients and Iwahori level structure

Let $I \subset \mathrm{Spec}\,(\mathscr{O}_F/p)$ be a subset of the primes of $\mathscr{O}_F$ over p. We introduce the notation $\mathfrak{p}_I$ for the product of the primes in I i.e.

$$\mathfrak{p}_I := \prod_{\mathfrak{p}\in I} \mathfrak{p}.$$

We also write $\mathbb{B}_I$ for the union of $\mathbb{B}_\mathfrak{p}$ as $\mathfrak{p}$ ranges over the primes in I,

$$\mathbb{B}_I := \bigcup_{\mathfrak{p}\in I} \mathbb{B}_\mathfrak{p}.$$

Notice that any ϕ-stable subset $\Theta \subset \mathbb{B}$ is of the form $\mathbb{B}_I$ for some I since the collection of $\mathbb{B}_{\mathfrak{p}} \subset \mathbb{B}$ is exactly the ϕ-orbit decomposition of $\mathbb{B}$.

Hilbert modular schemes with Iwahori level structure at p

Definition 6.3.4. Let $\mathscr{M}^{\mathfrak{c}}$ be a Hilbert modular scheme carrying a universal family of $\mathfrak{c}$-polarized HBAVs with level structure coprime to p. We define the *Iwahori $\Gamma(\mathfrak{p}_I)$-level moduli space $\mathscr{M}_0^{\mathfrak{c}}(\mathfrak{p}_I)$* as the scheme representing the moduli problem parametrizing tuples $(\underline{A}, H)$ over S where $\underline{A} \in \mathscr{M}^{\mathfrak{c}}$ and $H \subset A[\mathfrak{p}_I]$ is a finite S-flat $\mathscr{O}_F$-invariant *isotropic* subgroup scheme of rank $N(\mathfrak{p}_I) = p^{\sum_{\mathfrak{p} \in I} f(\mathfrak{p}/p)}$.

We will now explain the meaning of *isotropic* in this context. Since we can take $\mathfrak{c}$ to be prime-to-p, there is a canonical isomorphism $\mathfrak{c} \otimes_{\mathscr{O}_F} A[p] \xrightarrow{\sim} A[p]$ sending $\alpha \otimes u \mapsto \iota(\alpha)u$. Thus, the canonical pairing $A[p] \otimes A^\vee[p] \to \mu_p$ induces, via λ, a perfect Weil pairing

$$\langle -, - \rangle_\lambda : A[p] \otimes A[p] \to \mu_p$$

restricting to a perfect pairing on $A[\mathfrak{p}_I]$ because the Rosati involution corresponds to the trivial involution on $\mathscr{O}_F$. The subgroups $H \subset A[\mathfrak{p}_I]$ considered above are maximal isotropic subgroups for this pairing.

The scheme $\mathscr{M}_0^{\mathfrak{c}}(\mathfrak{p}_I)$ is not smooth over $\mathbb{Z}[(ND)^{-1}]$ but it is flat and lci hence with normal Cohen–Macaulay total space [Pap95, Theorem 2.2.2 and Corollary 2.2.3]. We will see that its fiber over p is the union of at least two components. The forgetful map $\pi : \mathscr{M}_0(\mathfrak{p}_I) \to \mathscr{M}$ is proper.

The Atkin–Lehner map

Let $M^{\mathfrak{c}}$ (resp. $M_0^{\mathfrak{c}}(\mathfrak{p}_I)$) denote the special fiber of $\mathscr{M}^{\mathfrak{c}}$ (resp. $\mathscr{M}_0^{\mathfrak{c}}(\mathfrak{p}_I)$) over the field κ of characteristic p. Let $M^{\mathfrak{c},\mathrm{ord}} \subset M^{\mathfrak{c}}$ be the ordinary locus, the open dense subset where the partial Hasse invariants h_σ are all nonvanishing. A $\bar{\kappa}$-point of $M^{\mathfrak{c}}$ lies in $M^{\mathfrak{c},\mathrm{ord}}$ if and only if the fiber of the universal abelian variety over that point is ordinary. Let $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{ord}}$ be the preimage of $M^{\mathfrak{c},\mathrm{ord}}$. Note that while $M^{\mathfrak{c},\mathrm{ord}} \subset M^{\mathfrak{c}}$ is dense, $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{ord}} \subset M_0^{\mathfrak{c}}(\mathfrak{p}_I)$ need not be dense in general due to the existence of components lying over the non-ordinary locus.

Then $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{ord}}$ contains two distinguished disjoint smooth irreducible components: the component $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{ord,m}}$ called the *multiplicative component* classifies pairs $(\underline{A}, H)$ where H is multiplicative (equivalent in the ordinary case to being a connected infinitesimal group scheme) meaning its Cartier dual is étale. The component $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{ord,ét}}$ classifies pairs $(\underline{A}, H)$ where H is étale. In total, $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{ord}}$ has $2^{\#I}$ irreducible components parametrizing pairs $(\underline{A}, H)$ where we specify the choice of multiplicative or étale for each order $p^{f(\mathfrak{p}/p)}$ subgroup $H[\mathfrak{p}]$ for $\mathfrak{p} \in I$.

For any finite flat group scheme there is a canonical exact sequence

$$0 \to H^\circ \to H \to H^{\text{ét}} \to 0$$

picking out connected and étale pieces respectively. Applying this to $A[\mathfrak{p}_I]$, the forgetful map $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ord,m}} \to M_0$ is an isomorphism with inverse given by the section $\sigma : \underline{A} \mapsto (\underline{A}, A[\mathfrak{p}_I]^\circ)$. We can also write $A[\mathfrak{p}_I]^\circ = A[\mathrm{Fr}] \cap A[\mathfrak{p}_I]$ where $A[\mathrm{Fr}]$ is the kernel of the relative Frobenius map $\mathrm{Fr} = \mathrm{Fr}_{A/S} : A \to A^{(p/S)}$ since $A[\mathrm{Fr}] = A[p]^\circ$ if A is ordinary. On the other hand, the forgetful map $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ord,ét}} \to M_0^{\mathfrak{c}}$ is finite flat and purely inseparable of degree $N(\mathfrak{p}_I)$.

The results of this section are due to Pappas [Pap95], Stamm [Sta97], and Goren–Kassaei [GK12]; see [GdS24, p. 13] for more detailed attribution. By analogy with the case of modular curves, we define the Atkin–Lehner map

$$w : \mathscr{M}_0^{\mathfrak{c}}(\mathfrak{p}_I) \to \mathscr{M}_0^{\mathfrak{c}\mathfrak{p}_I}(\mathfrak{p}_I) \quad \text{via} \quad w : (\underline{A}, H) \mapsto (\underline{A}/H, A[\mathfrak{p}_I]/H)$$

where $\underline{A}/H$ is the tuple $(A/H, \iota', \lambda', \eta')$ where ι' and η' are induced by ι and η in the obvious way since H has order coprime to N. Parallel to the discussion in [GdS24, p. 14], we will spell out the construction of the polarization λ' since in that reference it is only done in the case that I is a singleton. Strictly speaking, the notation $\mathscr{M}_0^{\mathfrak{c}\mathfrak{p}_I}(\mathfrak{p}_I)$ makes no sense because the polarization module $\mathfrak{c}\mathfrak{p}_I$ is not coprime to p so our definition of isotropic does not apply. There are two potential remedies: (1) we can define $\mathscr{M}_0^{\mathfrak{c}}(\mathfrak{p}_I)$ for general $\mathfrak{c}$ as a moduli space of isogenies between HBAVs with a particular type of compatibility between their polarizations as done in [Pap95] or (2) choose an element $\gamma \gg 0$ so that $\mathfrak{c}\mathfrak{p}_I = \gamma\mathfrak{c}'$ for $\mathfrak{c}'$ coprime to p and define $\mathscr{M}_0^{\mathfrak{c}\mathfrak{p}_I}(\mathfrak{p}_I) := \mathscr{M}_0^{\mathfrak{c}'}(\mathfrak{p}_I)$ remembering then that strictly speaking the map w depends on the choice of γ. For maximum canonicity, we make the former choice.

Definition 6.3.5. An S-object of $\mathscr{M}_0^{\mathfrak{c}}(\mathfrak{p}_I)$ is a pair of S-objects $\underline{A}_1 \in \mathscr{M}^{\mathfrak{c}}$ and $\underline{A}_2 \in \mathscr{M}^{\mathfrak{c}\mathfrak{p}_I}$ and an $\mathscr{O}_F$-linear isogeny $\phi : A_1 \to A_2$ of degree $N(\mathfrak{p}_I)$ with kernel annihilated by $\mathfrak{p}_I$. Furthermore, we require that the level structures are compatible in the sense that $\eta_2 = \phi \circ \eta_1$ and the polarizations are compatible in the sense that for all $a \in \mathfrak{c}\mathfrak{p}_I$,

$$\phi^\vee \circ \lambda_{2*}(a) \circ \phi = \lambda_{1*}(a)$$

as maps $A_1 \to A_1^\vee$.

In this language, the Atkin–Lehner map sends

$$(\underline{A}_1, \underline{A}_2, \phi : A_1 \to A_2) \mapsto (\underline{A}_2, \underline{A}_1/A_1[\mathfrak{p}_I], \phi' : A_2 \to A_1/A_1[\mathfrak{p}_I]).$$

The polarization λ' we needed to construct above, is exactly the polarization λ_2 on A_2 in the new definition. For any isotropic H, such a polarization λ_2 exists (see [GdS24, p.14]) proving that the definitions are equivalent for $\mathfrak{c}$ coprime to p. If the class of $\mathfrak{p}_I$ in $\mathscr{C}\ell^+(F)$ is nontrivial then w is not an involution since

$$w^2 : (\underline{A}_1, \underline{A}_2, \phi : A_1 \to A_2) \mapsto (\underline{A}_1/A_1[\mathfrak{p}_I], \underline{A}_2/A_2[\mathfrak{p}_I], \bar{\phi} : A_1/A_1[\mathfrak{p}_I] \to A_2/A_2[\mathfrak{p}_I]) \\ \in \mathscr{M}_0^{\mathfrak{c}\mathfrak{p}_I^2}(\mathfrak{p}_I).$$

This Atkin–Lehner map swaps the fully multiplicative and fully étale components in the central fiber. Define $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{m}}$ and $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}}$ to be the Zariski closures in $M_0^{\mathfrak{c}}(\mathfrak{p}_I)$ of $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{ord,m}}$ and $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ord,ét}}$ respectively. The following is an obvious generalization of [GdS24, Proposition 3.3.3].

Proposition 6.3.6. *Both $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^m$ and $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}}$ are finite flat over $M^{\mathfrak{c}}$. The forgetful morphism $\pi : M_0^{\mathfrak{c}}(\mathfrak{p}_I)^m \to M^{\mathfrak{c}}$ is an isomorphism and $\pi : M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}} \to M^{\mathfrak{c}}$ is purely inseparable of height 1 and degree $N(\mathfrak{p}_I)$. The map w induces isomorphisms*

$$w : M_0^{\mathfrak{c}}(\mathfrak{p}_I)^m \xrightarrow{\sim} M_0^{\mathfrak{c}\mathfrak{p}_I}(\mathfrak{p}_I)^{\text{ét}} \quad \textit{and} \quad w : M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}} \xrightarrow{\sim} M_0^{\mathfrak{c}\mathfrak{p}_I}(\mathfrak{p}_I)^m.$$

Hence, both $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^m$ and $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}}$ are smooth over κ.

In the proof, it is shown that the relative Frobenius $\mathrm{Fr} : M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}} \to M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}}$ factors as $\theta_I \circ \pi$ proving that both are finite flat purely inseparable maps of height 1. This map θ_I has a nice moduli interpretation:

$$\theta_I : \underline{A} \mapsto (\underline{A}/A[\mathrm{Fr}], A[\mathfrak{p}_I]/A[\mathrm{Fr}]).$$

In [GdS24, 3.4.2] it is proved that θ_I is the quotient of $M^{\mathfrak{c}}$ by the p-closed foliation $\mathscr{F}_{\mathbb{B}_{I^c}}$ (or rather their argument in the case that I is a singleton generalizes directly). Therefore, we have the following result.

Theorem 6.3.7. *The quotient map $M^{\mathfrak{c}} \to M^{\mathfrak{c}}/\mathscr{F}_{\mathbb{B}_{I^c}}$ by the p-closed foliation $\mathscr{F}_{\mathbb{B}_{I^c}} = \bigoplus_{\sigma\notin\mathbb{B}_I} \mathscr{F}_\sigma$ is isomorphic to the map $\theta_I : M^{\mathfrak{c}} \to M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}}$. Hence the composition of this isomorphism with $\pi \circ w : M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}} \xrightarrow{\sim} M^{\mathfrak{c}\mathfrak{p}_I}$ produces an isomorphism $M^{\mathfrak{c}}/\mathscr{F}_{\mathbb{B}_{I^c}} \xrightarrow{\sim} M^{\mathfrak{c}\mathfrak{p}_I}$.*

Proof. See [GdS24, Theorem 3.2.2 and Proposition 3.3.3]. □

Hence, quotienting by the tautological foliation gives back another mod p fiber of a Hilbert modular variety with a twisted polarization module.

6.3.3 Descent to non-PEL Hilbert modular varieties

As we saw in § 6.2.6, the foliations $\mathscr{F}_\Theta$ descend to Y_U. We will abuse notation by denoting the induced foliations on Y_U as $\mathscr{F}_\Theta$ as well.

Theorem 6.3.8. *For any subset $\Theta \subset \mathbb{B}$, the foliation $\mathscr{F}_\Theta$ on $M = Y_U \times_{W(\kappa)} \kappa$ satisfies the following:*

1. *$\mathscr{F}_\Theta$ is p-closed if and only if Θ is preserved by ϕ in which case Θ is of the form $\mathbb{B}_{I^c}$ for some index set I*

2. *the p-curvature map of $\mathscr{F}_\Theta$ under the identifications*

$$\begin{tikzcd} \mathrm{Frob}_{Y_U}^* \mathscr{F}_\Theta \arrow[r, "\psi_{\mathscr{F}_\Theta, p}"] \arrow[d, equal] & \mathscr{T}_M / \mathscr{F}_\Theta \arrow[d, equal] \\ \bigoplus_{\sigma \in \Theta} \mathscr{F}_\sigma^p \arrow[r] & \bigoplus_{\tau \in \Theta^c} \mathscr{F}_\tau \end{tikzcd}$$

is given by an off-diagonal matrix whose nonzero terms are maps $\mathscr{F}_\sigma^p \to \mathscr{F}_{\phi \circ \sigma}$ which are nonzero on the ordinary locus

3. *the quotient by $\mathscr{F}_{\mathbb{B}_{I^c}}$ may be identified with the map*

$$M \to M$$

arising from the equivariant maps $s : Y^{\mathfrak{c}}(N)_\kappa \to Y^{\mathfrak{c}\mathfrak{p}_I}(N)_\kappa$ where multiplication by $\mathfrak{p}_I$ induces a bijection $\mathscr{C}\ell^+(F) \to \mathscr{C}\ell^+(F)$.

Proof. This follows from the corresponding results on $Y^{\mathfrak{c}}(N)$ Theorems 6.3.3 and 6.3.7 (cf. [GdS24, Theorem 3.2.2 and Proposition 3.3.3]) by descent along the finite quotient $Y^{\mathfrak{c}}(N) \to Y_U$; see §6.2.6 for the descent of foliations. □

6.4 Compactifications

The basic result behind both the existence of good compactifications of Hilbert modular varieties as well as their algebraic structure is finite generation for a ring of modular forms and that the analytic Hilbert modular varieties are embedded via taking sufficiently many modular forms.

Theorem 6.4.1 ([BB66]). *The ring $M_\bullet(\Gamma)$ of modular forms at level Γ is finitely generated if Γ is congruence meaning it contains a congruence subgroup*

$$\Gamma(N) := \ker\left(\mathrm{GL}_2^+(\mathcal{O}_F) \to \mathrm{GL}_2(\mathcal{O}_F/N)\right)$$

Furthermore, these functions separate points and tangent vectors on $\mathfrak{H}^\Sigma/\Gamma$.

Therefore, we may define the Baily–Borel compactification:

$$\mathfrak{H}^\Sigma/\Gamma \to \overline{\mathfrak{H}^\Sigma/\Gamma}^{BB} := \operatorname{Proj} M_\bullet(\Gamma)$$

giving $\mathfrak{H}^\Sigma/\Gamma$ the structure of a Zariski open inside a *singular* algebraic variety. Furthermore, this Baily–Borel compactification is quite small: the boundary consists of finitely many points called *cusps*. These cusps may be resolved by a combinatorial blowup procedure to obtain a toroidal resolution that looks locally like a toric variety corresponding to a certain polyhedral cone complex that requires a choice of additional data [AMRT10]. In the arithmetic setting, we will follow the description of a combinatorially defined arithmetic toroidal compactification following the exposition of Chai [Cha90]. The cusps are parametrized by the following data, which, in the arithmetic setting, we take as a definition.

Definition 6.4.2. A *cusp* of $Y^{\mathfrak{c}}(N)$ consists of the following data:

1. projective rank-one $\mathcal{O}_F$-modules $\mathfrak{a}, \mathfrak{b}$

2. an exact sequence of projective $\mathcal{O}_F$-modules

$$0 \to \mathfrak{d}^{-1}\mathfrak{a}^{-1} \to H \to \mathfrak{b} \to 0$$

3. an isomorphism $\beta : \mathfrak{b}^{-1}\mathfrak{a} \xrightarrow{\sim} \mathfrak{c}$

4. an $\mathcal{O}_F$-linear isomorphism

$$\gamma : H \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \mathcal{O}_F \otimes (\mathbb{Z}/N\mathbb{Z}).$$

The set $C^{\mathfrak{c}}(N)$ of isomorphism classes of cusps of $Y^{\mathfrak{c}}(N)$ is finite. Note when $N = 1$, the set $C^{\mathfrak{c}}(1)$ has cardinality $h = \#\mathcal{C}\ell(\mathcal{O}_F)$ since the exact sequences are split.

Remark 6.4.3. For comparison with Chai's text [Cha90] recall that a projective rank-one module *with positivity* $\mathfrak{c}^\# = (\mathfrak{c}, \mathfrak{c}^+)$ is a choice of positive direction in $\mathfrak{c} \otimes_{\mathscr{O}_F,\tau} \mathbb{R}$ for each $\tau : F \hookrightarrow \mathbb{R}$ so that $\mathfrak{c}^+ \subset \mathfrak{c}$ is the subset of elements which are positive for each embedding. Note that every $\mathfrak{c}^\#$ is isomorphic to a fractional ideal $\mathfrak{c}' \subset F$ with the notion of positivity inherited from F. Indeed, any choice $\gamma \in \mathfrak{c}^+$ of a totally positive element defines an embedding $\mathfrak{c} \hookrightarrow F$ via $\alpha \mapsto \frac{\alpha}{\gamma}$ compatible with the notions of positivity.

For each cusp, we define $M := \mathfrak{a}\mathfrak{b}$ and $M^\vee := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = \operatorname{Hom}_{\mathscr{O}_F}(M, \mathfrak{d}^{-1}) = \mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{d}^{-1} \xrightarrow{\sim} \mathfrak{c}\mathfrak{a}^{-2}\mathfrak{d}^{-1}$. Notice that $M^\vee$ has a notion of positivity inherited from $\mathfrak{c}$ and $\mathfrak{d}$ as fractional ideals. So we may consider the cone $M_{\mathbb{R}}^{\vee+} \cup \{0\}$.

Definition 6.4.4. A $\Gamma(N)$-*admissible polyhedral cone decomposition* is a collection $\{\sigma_\alpha^c\}_{c \in C^{\mathfrak{c}}(N), \alpha \in I_c}$ of polyhedral cones running over all cusps and for each cusp $c \in C^{\mathfrak{c}}(N)$ an index set I_c satisfying the following conditions:

1. for $c \in C^{\mathfrak{c}}(N)$ the family $\{\sigma_\alpha^c\}_{\alpha \in I_c}$ is a *rational* polyhedral cone decomposition of $M_{\mathbb{R}}^{\vee+} \cup \{0\}$
2. the family $\{\sigma_\alpha^c\}_{\alpha \in I_c}$ is invariant under the natural action of U_N^2 on $M^\vee$ where $U_N = \{x \in \mathscr{O}_F^\times \mid u \equiv 1 \mod N\}$ and $\{\sigma_\alpha^c\}_{\alpha \in I_c}/U_N^2$ is a finite conical complex
3. the assignment $c \mapsto \{\sigma_\alpha^c\}_{\alpha \in I_c}$ is compatible with isomorphisms $c \xrightarrow{\sim} c'$ of cusps of $Y^{\mathfrak{c}}(N)$, where an isomorphism $c \xrightarrow{\sim} c'$ induces an isomorphism $M^\vee \xrightarrow{\sim} M'^\vee$ in the obvious way.

A $\Gamma(N)$-admissible polyhedral cone decomposition always exists. In fact, we can choose the cone decomposition Σ to be invariant under the action of $G_{U,N}$ by taking a common smooth refinement of the $G_{U,N}$-orbit of any $\Gamma(N)$-admissible polyhedral cone decomposition ([Rap78, Lemme 4.2(i)] for refinements, [Dim04, Corollaire 7.5] for smoothness) (cf. [Yan25, p.9]). Such a choice will lead to a smooth compactification so that the natural action on $Y^{\mathfrak{c}}(N)$ extends to $\overline{Y^{\mathfrak{c}}(N)}^{\Sigma}$.

A $\Gamma(N)$-admissible polyhedral cone decomposition will define, at each cusp, a toric variety $S_N(\Sigma_c)$ which is the union of affine toric varieties $\operatorname{Spec} \mathbb{Z}[\frac{1}{N}M \cap (\sigma_\alpha^c)^\vee]$. The toric variety $S_N(\Sigma^c)$ carries a U_N^2-action by the invariance of the cone complexes. The main result is that there is a relative toroidal compactification locally modeled on these toric varieties which extends the universal family as a semi-abelian scheme.

Theorem 6.4.5 ([Cha90, § 3.6])**.** *Let $\Sigma = \{\sigma_\alpha^c\}_{c,\alpha}$ be a $\Gamma(N)$-admissible polyhedral cone decomposition. There is an open immersion $j : Y^{\mathfrak{c}}(N) \hookrightarrow \overline{Y^{\mathfrak{c}}(N)}^{\Sigma}$ of schemes and a semi-abelian scheme $\mathscr{A}^{\mathrm{tor}} \to \overline{Y^{\mathfrak{c}}(N)}^{\Sigma}$ with a $\mathfrak{c}$-polarization, an action of $\mathscr{O}_F$, and $\Gamma(N)$-level* *structure extending the universal abelian scheme $\mathscr{A} \to Y^{\mathfrak{c}}(N)$ along with an isomorphism of formal schemes*

$$\varphi : \coprod_{c \in C^{\mathfrak{c}}(N)} S_N(\Sigma^c)^\wedge / U_N^2 \times Z^{\mathfrak{c}}(N) \to \left(\overline{Y^{\mathfrak{c}}(N)}^{\Sigma}\right)^\wedge$$

where the left-hand side is formally completed along the toric boundary and the right-hand side is completed along the complement of j. Furthermore, the restriction $\mathscr{A}^{\mathrm{tor}}$ to each formal cusp via φ agrees with the Mumford construction [Mum72] of a generalized Tate degeneration (see [Cha90, §3] for details).

Furthermore, the scheme $\overline{Y^{\mathfrak{c}}(N)}^{\Sigma} \to \operatorname{Spec} \mathbb{Z}[(ND)^{-1}]$ is smooth and proper with $j : Y^{\mathfrak{c}}(N) \hookrightarrow \overline{Y^{\mathfrak{c}}(N)}^{\Sigma}$ having snc boundary [Rap78, Corollaire 5.3]

The extension of the universal family over the boundary as a semi-abelian scheme gives a particularly nice way of understanding the behavior of the tautological foliations along the boundary of the toroidal compactification. Let $\overline{\mathscr{M}} := \overline{Y^{\mathfrak{c}}(N)}^{\Sigma}$ denote a particular choice of toroidal compactification and $\mathscr{A}^{\mathrm{tor}} \to \overline{\mathscr{M}}$ the universal semi-abelian scheme. We can define the extended Hodge bundle

$$\underline{\omega}_{\overline{\mathscr{M}}} := e^* \Omega_{\mathscr{A}^{\mathrm{tor}}/\overline{\mathscr{M}}}$$

and similarly the universal Lie algebra via the dual construction. These extensions are locally-free $\mathscr{O}_F \otimes \mathscr{O}_{\overline{\mathscr{M}}}$-modules of rank 1 (cf. [Rap78, Corollaire 4.4]). Hence, after extending scalars to $W(\kappa)$, there is an isotypic decomposition

$$\underline{\omega}_{\overline{\mathscr{M}}} = \bigoplus_{\sigma \in \mathbb{B}} \omega_{\overline{\mathscr{M}}}[\sigma]$$

into line bundles on $\overline{\mathscr{M}}$. The Kodaira–Spencer isomorphism of §6.3 will extend to an isomorphism on $\overline{\mathscr{M}}$ in the logarithmic sense giving a decomposition not of $\Omega_{\mathscr{M}/W(\kappa)}$ but of $\Omega_{\mathscr{M}/W(\kappa)}(\log E)$ where E is the toroidal boundary. Indeed, there is an (equivariant) isomorphism [Lan13, Theorem 6.4.1.1] (cf. [TX16, §2.11])

$$\mathrm{KS}^\vee : \underline{\omega}^{\otimes 2} \otimes (\underline{\mathfrak{c}})^{-1} \xrightarrow{\sim} \Omega_{\overline{\mathscr{M}}/\mathbb{Z}}(\log E) \tag{6.5}$$

induced by the logarithmic Gauss–Manin connection on relative de Rham cohomology of $\mathscr{A}^{\mathrm{tor}} \to \overline{\mathscr{M}}$. A choice of a $G_{U,N}$-invariant $\Gamma(N)$-admissible polyhedral cone decomposition

Σ then defines a compactification of Y_U via

$$j : Y_U = \coprod_{[\mathfrak{c}]\in\mathscr{C}\ell^+(F)} Y^{\mathfrak{c}}(N)/G_{U,N} \hookrightarrow \coprod_{[\mathfrak{c}]\in\mathscr{C}\ell^+(F)} \overline{Y^{\mathfrak{c}}(N)}^{\Sigma}/G_{U,N}$$

which we denote by $\overline{Y_U}^{\Sigma}$ or X_U^{tor} to suppress the dependence on Σ. This quotient is smooth and projective for sufficiently small U (e.g. for U neat) see [Yan25, p.9].

6.5 Existence of Symmetric Forms

In this section, we consider the existence of symmetric forms inducing the tautological foliations on a Hilbert modular variety. The argument and techniques here largely follow [RT18] with some additional attention paid to the arithmetic situation. We ought first to spell out in detail what we mean by a symmetric form inducing a collection of corank 1 foliations. Let X be an n-dimensional smooth variety and $\mathscr{F}_1, \ldots, \mathscr{F}_n \subset \mathscr{T}_X$ be corank one foliations. Recall that $\mathrm{Sing}\,(\mathscr{F}_i) \subset X$ is the subset where $\mathscr{F}_i \subset \mathscr{T}_X$ fails to be a subbundle. We can give $\mathrm{Sing}\,(\mathscr{F}_i)$ a subscheme structure by defining its ideal sheaf via an exact sequence

$$0 \to \mathscr{F}_i \to \mathscr{T}_X \to \mathscr{N}_{\mathscr{F}_i} \otimes \mathscr{I}_{\mathrm{Sing}\,(\mathscr{F}_i)} \to 0$$

where $\mathscr{N}_{\mathscr{F}_i}$, defined as the reflexive hull of the quotient, is a line bundle called the normal bundle of $\mathscr{F}_i$. We say the foliations are *in general position* if the wedge dual map,

$$\mathscr{N}_{\mathscr{F}_1}^{\vee} \otimes \cdots \otimes \mathscr{N}_{\mathscr{F}_n}^{\vee} \to \omega_X$$

is nonzero. When this holds, we obtain an isomorphism

$$\mathscr{N}_{\mathscr{F}_1}^{\vee} \otimes \cdots \otimes \mathscr{N}_{\mathscr{F}_n}^{\vee} \xrightarrow{\sim} \omega_X(-H)$$

where $H \subset X$ is an effective divisor called the *tangential divisor* $H := \tan(\mathscr{F}_1, \ldots, \mathscr{F}_n)$ of the foliations $\mathscr{F}_1, \ldots, \mathscr{F}_n$. The support of H is the locus where $\mathscr{F}_1, \ldots, \mathscr{F}_n$ fail to be in general position. If $\mathscr{F}_i$ is locally cut out as the kernel of a 1-form ω_i then H is locally cut out by the vanishing of

$$\omega_1 \wedge \cdots \wedge \omega_n$$

which defines the map $\mathscr{N}_{\mathscr{F}_1}^{\vee} \otimes \cdots \otimes \mathscr{N}_{\mathscr{F}_n}^{\vee} \to \omega_X$. Hence $\mathrm{Sing}\,(\mathscr{F}_i) \subset H$ since locally the singular locus of $\mathscr{F}_i$ is given by $V(\omega_i)$.

6.5.1 Birational invariance of effective foliation data

As in the previous section, we consider the data $(X, \{\mathscr{F}_i\}_{i\in I}, \omega)$ consisting of a smooth variety X of dimension n and a collection $\{\mathscr{F}_i\}_{i\in I}$ of n distinct corank 1 foliations $\mathscr{F}_i \subset \mathscr{T}_X$ that are in general position in the sense of the previous section along with a section

$$\omega \in H^0(X, \omega_X(-H)^{\otimes m})$$

where $H := \tan(\{\mathscr{F}_i\}_{i\in I})$ and $m \geq 1$ is an integer. The maps

$$\omega_X(-H) \xrightarrow{\sim} \bigotimes_{i\in I} \mathscr{N}^\vee_{\mathscr{F}_i} \to \operatorname{Sym}^n \Omega_X$$

allows us to view this top form ω vanishing along H as a symmetric nm-form. We call the tuple an *effective foliation datum* or we say that X is equipped with an effective foliation datum $(\{\mathscr{F}_i\}_{i\in I}, \omega)$. The vital property of these data is their "birational invariance" in the following sense.

Proposition 6.5.1. *Suppose that $f : X' \to X$ is a dominant generically finite separable morphism of varieties and X is equipped with an effective foliation datum $(\{\mathscr{F}_i\}_{i\in I}, \omega)$. Then f induces an effective foliation datum on X' functorially.*

Proof. The foliations are induced on X' in the usual way (Definition 2.1.3)

$$\mathscr{F}'_i := f^{[*]}\mathscr{F}_i := (f^*\mathscr{F}_i \times_{f^*\mathscr{T}_X} \mathscr{T}_{X'})^{\mathrm{sat}}.$$

Now there is a diagram of exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'_i \arrow[r] \arrow[d] & \mathscr{T}_{X'} \arrow[r] \arrow[d] & \mathscr{N}_{\mathscr{F}'_i} \otimes \mathscr{I}_{\mathrm{Sing}(\mathscr{F}'_i)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im}(f^*\mathscr{F}_i \to f^*\mathscr{T}_X) \arrow[r] & f^*\mathscr{T}_X \arrow[r] & f^*(\mathscr{N}_{\mathscr{F}_i} \otimes \mathscr{I}_{\mathrm{Sing}(\mathscr{F}_i)}) \arrow[r] & 0
\end{tikzcd}$$

Indeed, the map $\mathscr{F}'_i \to \mathscr{T}_{X'}$ factors through the image of $f^*\mathscr{F}_i \to f^*\mathscr{T}_X$ because the bottom row is exact and $\mathscr{F}'_i \to f^*\mathscr{T}_X \to f^*(\mathscr{N}_{\mathscr{F}_i} \otimes \mathscr{I}_{\mathrm{Sing}(\mathscr{F}_i)})$ is zero since $\mathscr{F}'_i$ is torsion free and the map is zero at the generic point of X using the separability assumption to know that $X' \to X$ is étale over the generic point. This observation also produces the map $\mathscr{N}_{\mathscr{F}'_i} \otimes \mathscr{I}_{\mathrm{Sing}(\mathscr{F}'_i)} \to f^*(\mathscr{N}_{\mathscr{F}_i} \otimes \mathscr{I}_{\mathrm{Sing}(\mathscr{F}_i)})$. Since $(\mathscr{N}_{\mathscr{F}'_i} \otimes \mathscr{I}_{\mathrm{Sing}(\mathscr{F}'_i)})^{\vee\vee} = \mathscr{N}_{\mathscr{F}'_i}$ and $\mathscr{N}_{\mathscr{F}_i}$ is a line bundle, this map induces a diagram of nonzero maps,

$$\begin{tikzcd}
\mathcal{N}_{\mathcal{F}_i'} \otimes \mathcal{I}_{\mathrm{Sing}(\mathcal{F}_i')} \arrow[r, hook] \arrow[d] & \mathcal{N}_{\mathcal{F}_i'} \arrow[d, dashed] \\
f^*(\mathcal{N}_{\mathcal{F}_i} \otimes \mathcal{I}_{\mathrm{Sing}(\mathcal{F}_i)}) \arrow[r] & f^*\mathcal{N}_{\mathcal{F}_i}
\end{tikzcd}$$

In particular, the dual gives a well-defined pullback map $f^*\mathcal{N}_{\mathcal{F}_i}^\vee \to \mathcal{N}_{\mathcal{F}_i'}^\vee$ induced by the pullback map $f^*\Omega_X \to \Omega_{X'}$. The tensor product of these maps induces a diagram,

$$\begin{tikzcd}
\omega_{X'}(-H') \arrow[r, "\sim"] & \bigotimes_{i\in I} \mathcal{N}_{\mathcal{F}_i'}^\vee \arrow[r] & \mathrm{Sym}^n \Omega_{X'} \\
f^*\omega_X(-H) \arrow[r, "\sim"] \arrow[u] & \bigotimes_{i\in I} f^*\mathcal{N}_{\mathcal{F}_i}^\vee \arrow[r] \arrow[u] & f^*\mathrm{Sym}^n \Omega_X \arrow[u]
\end{tikzcd}$$

compatible with the pullback maps on differential forms. Then the m-th symmetric power of this diagram defines $f^*\omega \in H^0(X', \omega_{X'}(-H')^{\otimes m}) \hookrightarrow H^0(X', \mathrm{Sym}^{nm}\Omega_{X'})$. □

Proposition 6.5.2. *If $U \subset X$ is an open set with complementary codimension ≥ 2 equipped with an effective foliation datum then there is a unique effective foliation datum on X extending the one on U.*

Proof. Let $j : U \hookrightarrow X$ be the open embedding. The subsheaves $j_*\mathcal{F}_i \subset j_*\mathcal{T}_U = \mathcal{T}_X$ are automatically saturated since $\mathcal{T}_X/j_*\mathcal{F}_i \hookrightarrow j_*(\mathcal{T}_U/\mathcal{F}_i)$ is torsion-free. We take $\mathcal{F}_i' := j_*\mathcal{F}_i$ to define the foliations on X. Then it is clear that $\omega_X(-H')|_U = \omega_U(-H)$ so the usual Hartogs property for line bundles implies that the section ω is in the image of the isomorphism

$$H^0(X, \omega_X(-H')^{\otimes m}) \to H^0(U, \omega_U(-H)^{\otimes m}).$$

□

Corollary 6.5.3. *If $f : X' \dashrightarrow X$ is a birational equivalence of smooth projective varieties then effective foliation data on X and X' are equivalent.*

Proof. The existence of a pullback follows immediately from combining Proposition 6.5.1 and Proposition 6.5.2 applied to the domain of definition $U := \mathrm{Dom}(f) \subset X'$ and the morphism $f : U \to X$. The equivalence follows from applying this argument to f^{-1} and functoriality. □

6.5.2 On toroidal compactifications of Hilbert modular varieties

Let X_U^{tor} be a toroidal compactification of the level U Hilbert modular variety Y_U and let $\mathcal{F}_\Theta$ denote the extension to X_U^{tor} of the tautological foliations on Y_U. We work relatively over the base $W(\kappa)$ for p coprime to the level so that the tautological foliations are defined. In particular, we consider the corank 1 foliations $\mathscr{F}^\sigma := \mathscr{F}_{\mathbb{B}\setminus\{\sigma\}}$ for $\sigma \in \mathbb{B}$. These corank 1 foliations are particularly important because we can compute the tangency divisor on X_U^{tor}. The following result is called the "tangency theorem" in [RT18]. Here we give a proof using the modular interpretation of the cover.

Theorem 6.5.4. *Let $E \subset X_U^{\mathrm{tor}}$ be the (reduced) toroidal boundary divisor. Then*

$$\tan\{\mathscr{F}^\sigma\}_{\sigma\in\mathbb{B}} = (d-1)E.$$

Equivalently, there is an isomorphism

$$\bigotimes_{\sigma\in\mathbb{B}} \mathscr{N}^\vee_{\mathscr{F}^\sigma} \cong \omega_{X_U^{\mathrm{tor}}/W(\kappa)}(-(d-1)E).$$

Proof. The extended Kodaira–Spencer isomorphism (6.5) gives an isomorphism

$$\mathrm{KS}^\vee : \bigoplus_{\sigma\in\mathbb{B}} \omega_\sigma^{\otimes 2} \otimes \delta_\sigma^{-1} = \underline{\omega}^2 \otimes (\underline{\mathfrak{c}})^{-1} \xrightarrow{\sim} \Omega_{X_U^{\mathrm{tor}}/W(\kappa)}(\log E).$$

Strictly speaking, on X_U the line bundle $\omega_\sigma^{\otimes 2} \otimes \delta_\sigma^{-1}$ is an irreducible object, not the tensor product of two different bundles since those do not descend from level $\Gamma(N)$ to level U. To simplify notation we write $\Omega_{X/W}$ in place of $\Omega_{X_U^{\mathrm{tor}}/W(\kappa)}$ and likewise for the tangent bundle. Now, the σ-isotypic parts $\omega_\sigma^{\otimes 2} \otimes \delta_\sigma^{-1}$ agree with the *logarithmic cotangent bundle* of the *logarithmic extension* of the tautological foliation $\mathscr{F}^\sigma$ since the decomposition gives an exact sequence,

$$0 \to \mathscr{F}^{\sigma,\log} \to \mathscr{T}_{X/W}(-\log E) \to \mathscr{N}_{\mathscr{F}^{\sigma,\log}} \to 0$$

where $\mathscr{N}_{\mathscr{F}^{\sigma,\log}} = \omega_\sigma^{\otimes -2} \otimes \delta_\sigma^1$ and $\mathscr{F}^{\sigma,\log}$ is the complement of the dual of the σ-isotypic part. Note that $\mathscr{F}^{\sigma,\log}$ is a *log-smooth* foliation since it is a sub-bundle of $\mathscr{T}_{X/W}(-\log E)$. Now consider the saturation sequence,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}^\sigma \arrow[r] & \mathscr{T}_{X/W} \arrow[r] & \mathscr{N}_{\mathscr{F}^\sigma} \otimes \mathscr{I}_{\mathrm{Sing}(\mathscr{F}^\sigma)} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F}^{\sigma,\log} \arrow[r] \arrow[u, hook] & \mathscr{T}_{X/W} \arrow[r] \arrow[u, hook] & \mathscr{N}_{\mathscr{F}^{\sigma,\log}} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Now considering the sequence

$$0 \to \mathscr{T}_{X/W}(-\log E) \to \mathscr{T}_{X/W} \to \bigoplus_{E_i\subset E} \mathscr{N}_{E_i} \to 0$$

we see that

$$[c_1(\mathscr{F}^\sigma) - c_1(\mathscr{F}^{\sigma,\log})] + [c_1(\mathscr{N}_{\mathscr{F}^\sigma}) - c_1(\mathscr{N}_{\mathscr{F}^{\sigma,\log}})] = E.$$

Since the divisors on the left-hand side are effective – using that the maps $\mathscr{F}^{\sigma,\log} \to \mathscr{F}^\sigma$ and $\mathscr{N}_{\mathscr{F}^{\sigma,\log}} \to \mathscr{N}_{\mathscr{F}^\sigma}$ have generically full rank – and supported on E, since E is reduced this means that each divisor is a complementary subset of the irreducible components E_i of the boundary. However, if E_i is contained in $c_1(\mathscr{F}^\sigma) - c_1(\mathscr{F}^{\sigma,\log})$ this implies that $\mathscr{F}^\sigma \subset \mathscr{T}_{X/W} \to \mathscr{N}_{E_i}$ is nonzero. However, each E_i is invariant under the foliations $\mathscr{F}^\sigma$ because they are contracted to the Baily–Borel compactification. This means the map $\mathscr{F}^\sigma \to \mathscr{N}_{E_i}$ is zero. Therefore, $\mathscr{N}_{\mathscr{F}^\sigma} = \mathscr{N}_{\mathscr{F}^{\sigma,\log}}(E)$. Thus, the Kodaira–Spencer isomorphism proves that,

$$\bigotimes_{\sigma\in\mathbb{B}} \mathscr{N}^\vee_{\mathscr{F}^\sigma} \cong \bigotimes_{\sigma\in\mathbb{B}} \mathscr{N}^\vee_{\mathscr{F}^{\sigma,\log}}(-E) \cong \det\Omega_{X/W}(\log E) \otimes \mathscr{O}_X(-dE) = \omega_{X/W}(-(d-1)E).$$

□

Remark 6.5.5. If we allow levels U so that Y_U is singular, then the same result holds when X_U^{tor} is the standard toroidal resolution of the finite quotient singularities on the toroidal compactification of Y_U. Then the above formula is still true when E is the union of the toroidal boundary and the (reduced) exceptional divisor of the resolution.

Therefore, the existence of symmetric differential forms on X_U^{tor} inducing the tautological foliations is reduced to proving that the divisor $K_{X_U^{\mathrm{tor}}} - (d-1)E$ is $\mathbb{Q}$-effective. This question exactly asks for the existence of certain cusp forms with large vanishing orders at the cusps compared to their level. This question has been substantially studied in the literature; we turn our attention in the next section to surveying the known results.

6.5.3 Existence of cusp forms with large vanishing order

Let $X := (X_U)^{\mathrm{tor}}$ and E be the toroidal boundary and the exceptional divisors of the resolution of any elliptic points. A fundamental question asks for which level subgroups U the Hilbert modular variety X is general type. In dimension $d > 2$, the problem of classifying all such U remains largely open. Hilbert modular forms of weight k and level U naturally define sections of $\mathscr{O}_X(k(K_X + E))$. Thus, significant effort has been devoted in the literature to studying the $\mathbb{Q}$-effectivity of the divisors $K_X + (1-\nu)E$ for $\nu \in \mathbb{Q}_{>0}$ with particular interest in reaching $\nu = 1$ where effectivity would show that X is general type. Heuristically, one can think of the linear series $|k(K_X + (1-\nu)E)|$ as consisting of Hilbert modular forms of weight k and level U "vanishing to order $k\nu$ at the cusps".

Our problem requires going beyond the general type considerations all the way to $\nu = d$. Luckily, in many cases this can be done due to the pioneering works of Tsuyumine and Bakker–Tsimerman.

Tsuyumine [Tsu85] studied the problem of computing the Kodaira dimension of Hilbert modular varieties over $\mathbb{C}$ by studying the asymptotics of the dimension of spaces of Hilbert modular forms with parallel weight $2k$, meaning the weight vector $\underline{k} = (2k, \ldots, 2k)$ is constant. A form f of parallel weight $2k$ of level $\Gamma \subset \mathrm{GL}_2^+(\mathcal{O}_F)$ defines a pluri-form

$$\omega_f = f(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d)^{\otimes k}$$

on $\mathfrak{H}^d$ invariant under Γ. It descends to $Y_\Gamma(\mathbb{C}) := \Gamma\backslash\mathfrak{H}^d$ and extends to a pluri-form on the toroidal compactification X_Γ^{tor} if and only if f vanishes to order at least $k-1$ at the cusps (see [Tsu85, Proposition 2] and the proof of [RT18, Proposition 6.3]). Tsuyumine calls the subspace of such forms $S_{2k}^{(0)}(\Gamma)$. He further defines a decreasing filtration of subspaces $S_{2k}^{(0)}(\Gamma) \supset S_{2k}^{(1)}(\Gamma) \supset \cdots$ where $S_{2k}^{(m)}(\Gamma)$ consists of forms vanishing to order at least $k-1+m$ at the cusps. In particular, a form $f \in S_{2k}^{(m)}(\Gamma)$ defines a pluri-form ω_f on X_Γ^{tor} vanishing to order at least m along the boundary divisor E i.e. a section of $kK_X - mE$. Hence, the existence of forms in $S_{2k}^{(\nu k)}(\Gamma)$ for some $k > 0$ implies that the divisor $K_{X_\Gamma^{\mathrm{tor}}} - \nu E$ is $\mathbb{Q}$-effective.

For $\Gamma = \Gamma_F = \mathrm{SL}_2(\mathcal{O}_F)$, Tsuyumine proved an asymptotic dimension bound for the spaces $S_{2k}^{(m)}(\Gamma_F)$ as $k \to \infty$. Note that Y_{Γ_F} is the PEL Hilbert modular variety $Y^{(1)}(1)_{\mathbb{C}}$.

Theorem 6.5.6 (Tsuyumine)**.** *(cf. [RT18, §6.4]) Let F be a totally real field of degree $d > 1$ and $\nu \geq 0$ an integer. Then*

$$\dim_{\mathbb{C}} S_{2k}^{(\nu k)}(\Gamma_F) \geq \left\{ \frac{\mathrm{disc}_{F/\mathbb{Q}}^{3/2} \zeta_F(2)}{2^{2d-1}\pi^{2d}} - \nu^d \cdot \frac{\mathrm{disc}_{F/\mathbb{Q}}^{1/2} h_F\, R\, 2^{d-1}}{d^d} \right\} k^d + O(k^{d-1})$$

where $\mathrm{disc}_{F/\mathbb{Q}}$ is the absolute discriminant of F, ζ_F is the Dedekind zeta function of F, h_F is the class number of F, and R is the positive regulator of F. In particular, if

$$\nu < \frac{d}{8\pi^2} \left(\frac{4\, \mathrm{disc}_{F/\mathbb{Q}}\, \zeta_F(2)}{h_F \cdot R} \right)^{1/d} \tag{6.6}$$

then $K_{X_{\Gamma_F}^{\mathrm{tor}}} - \nu E$ is big.

Proof. Combine [Tsu85, Lemma 5] with the dimension bound for $S_{2k}^{(m)}(\Gamma_F)$ given in [Tsu85, Lemma 4] and Shimizu's formula for the asymptotic dimension of $S_{2k}(\Gamma_F)$ proved in

[Shi63]. The case of $\nu = 1$ is recorded as equation (9) in [Tsu85] (with a minor typo in the exponent of 2 in the negative part of the leading term). □

Fixing d, a consequence of the Brauer–Siegel and Hermite–Minkowski theorems is that

$$\frac{d}{8\pi^2}\left(\frac{4\,\mathrm{disc}_{F/\mathbb{Q}}\,\zeta_F(2)}{h_F \cdot R}\right)^{1/d}$$

lands in any bounded interval for only finitely many number fields F. Hence, for any fixed d, for all but finitely many totally real fields F, the Hilbert modular variety $X_{\Gamma_F}^{\mathrm{tor}}$ has a symmetric form inducing the tautological foliations. Strictly speaking, our previous analysis does not apply for level $\Gamma_F = \mathrm{SL}_2(\mathcal{O}_F)$ due to the existence of elliptic points. Carrying out the same analysis on a resolution of the singularities appearing at elliptic points, we will obtain the same conclusion: there exists a global symmetric form on the resolution inducing the tautological foliations for all but finitely many totally real fields F (cf. [RT18, §6.5]). Alternatively, we can pass to the full congruence level $\Gamma(N)$ for $N \geq 3$ where there are no elliptic points. Of course, the same asymptotic formula still provides a lower bound proving that $X^{\mathfrak{c}}(N)^{\mathrm{tor}}$ has a symmetric form inducing the tautological foliations for all but finitely many totally real fields F and all $N \geq 3$.

For any F, we expect the same existence statement for forms of large vanishing order should hold at sufficiently large level. For a general congruence subgroup $\Gamma \subset \mathrm{GL}_2^+(\mathcal{O}_F)$ induced by a compact open subgroup $U \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$, the existence of forms with large vanishing order at the cusps is less well-understood. However, for the PEL case at level $\Gamma_{00}(N)$, Bakker–Tsimerman [BT18] proved the necessary existence result.

Theorem 6.5.7 (Bakker–Tsimerman)**.** *(cf. [BT18, Theorem 1.4]) Let F be a totally real field of degree $d > 1$ and $\mathfrak{n} \subset \mathcal{O}_F$ an ideal. Let $\Gamma_1(\mathfrak{n}) \subset \mathrm{SL}_2(\mathcal{O}_F)$ be the congruence subgroup*

$$\Gamma_1(\mathfrak{n}) = \left\{\gamma \in \mathrm{SL}_2(\mathcal{O}_F) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{\mathfrak{n}}\right\}.$$

Let $X_1(\mathfrak{n})^{\mathrm{tor}}$ be a toroidal compactification of the Hilbert modular variety $Y_1(\mathfrak{n}) := \Gamma_1(\mathfrak{n})\backslash\mathfrak{H}^d$. Then

$$K_{X_1(\mathfrak{n})^{\mathrm{tor}}} + \left(1 - \frac{d}{2\pi}|\mathrm{Nm}(\mathfrak{n})|^{\frac{1}{2d}}\right)E$$

is ample modulo E. In particular, if $\mathrm{Nm}(\mathfrak{n}) \geq (2\pi)^{2d}$ then $K_{X_1(\mathfrak{n})^{\mathrm{tor}}} - (d-1)E$ is ample modulo E and hence big.

Let $X \to \operatorname{Spec}\mathbb{Z}[(ND)^{-1}]$ be a toroidal compactification of a Hilbert modular variety of level U and E the toroidal boundary divisor. The divisors K_X and E are flat over

$\mathbb{Z}[(ND)^{-1}]$. We are interested in showing that $(K_X - (d-1)E)|_{X_{\mathbb{F}_p}}$ is $\mathbb{Q}$-effective (resp. big) for all primes $p \nmid ND$. By upper semicontinuity and flat base change, it suffices to prove that $K_X - (d-1)E$ is $\mathbb{Q}$-effective (resp. big) on the geometric generic fiber $X_{\mathbb{C}}$. Hence our consideration of Hilbert modular forms over $\mathbb{C}$ from the analytic perspective suffices to prove the necessary existence of sections mod p.

6.6 Descent and Non-Uniruledness

As previously, F is a totally real field of degree $d := [F : \mathbb{Q}]$. Here we will prove that if the discriminant of F or the level of U are large enough that the existence results on Hilbert modular forms surveyed in the previous sections apply, then the Hilbert modular variety has non-uniruled reduction at good primes. In particular, this automatically applies to all but finitely many primes in contrast to methods based on Galois representations (see the discussion in chapter 1).

Theorem 6.6.1. *Let $U \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$ be a compact open subgroup and X_U^{tor} a choice of toroidal compactification of Y_U. Suppose that $K_{X_U^{\mathrm{tor}}} - (d-1)E$ is $\mathbb{Q}$-effective (resp. big[6]). Then for any prime $p \in \operatorname{Spec}\mathbb{Z}$ unramified in F and coprime to the level of U, each geometric component of the reduction $(X_U^{\mathrm{tor}})_{\mathbb{F}_p}$ is not uniruled (resp. covered by genus ≤ 1 curves).*

Proof. First we give an outline of the argument which follows the structure of descent developed in the previous chapter. Consider a family of curves dominating a geometric component $X \subset (X_U^{\mathrm{tor}})_{\overline{\mathbb{F}}_p}$,

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle g} & & \\ T & & \end{array}$$

which as usual we assume is generically finite.

1. Replace f by a family of generically immersed curves.

2. Blow up X to replace f with a quasi-finite covering family.

3. The existence of a symmetric form inducing the tautological foliations implies that f is tangent to some $\mathscr{F}^\sigma$.

[6]Recall that X_U^{tor} may be disconnected. We call a divisor on a disjoint union of varieties $\mathbb{Q}$-*effective* (resp. *big*) if its restriction to each component is effective after taking some multiple (resp. is big). Equivalently, since there are finitely many components, we may ask for a uniform multiple so that there is a section which is nonvanishing at the generic point of each component.

4. Let Θ be a minimal subset so that f is tangent to $\mathscr{F}_\Theta$. By the p-curvature identities, Θ must be ϕ-stable. Hence $\Theta = \mathbb{B}_{I^c}$ for some subset $I \subset \operatorname{Spec}(\mathscr{O}_F/p)$.

5. Since $\mathscr{F}_\Theta = \mathscr{F}_{\mathbb{B}_{I^c}}$ is p-closed, f descends to a quasi-finite generically immersed family $f' : \mathscr{C}^{(p/T)} \to X/\mathscr{F}_\Theta$ along the quotient map $X \to X/\mathscr{F}_\Theta$.

6. We showed that $X' := X/\mathscr{F}_\Theta$ is birational to a (possibly different) geometric component of $(X_U^{\mathrm{tor}})_{\overline{\mathbb{F}}_p}$. Hence, we may repeat the above steps on the smooth locus of the normal variety X'.

7. At each step, there is a degree inequality $\deg f_t'^* \omega_{X'} < \deg f_t^* \omega_X$.

8. Iterating this construction contradicts the finiteness of $\deg f_t^* \omega_X$ by infinite descent.

In the subsequent sections, we spell out each step of the argument in more detail. □

6.6.1 Generically immersive replacement

The argument here is identical to that of § 5.1.1 where we only considered the case of surfaces. Suppose that f is inseparable and consider the cotangent pullback maps

$$f^*\Omega_X \to \Omega_{\mathscr{C}} \to \Omega_{\mathscr{C}/T}$$

If this map is identically zero then $\mathrm{d}f : \mathscr{T}_{\mathscr{C}} \to f^*\mathscr{T}_X$ kills the foliation $\mathscr{T}_{\mathscr{C}/T} \subset \mathscr{T}_{\mathscr{C}}$ induced by g and hence must factor through the relative Frobenius of the map g:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F_{\mathscr{C}/T}"'] \arrow[rr, bend left, "f"] \arrow[d, "g"] & \mathscr{C}^{(p/T)} \arrow[r, dashed, "f'"'] \arrow[d, "g^{(p)}"] & X \\
T \arrow[r, equal] & T &
\end{tikzcd}$$

Now we may repeat this argument to f' giving a diagram,

$$\begin{tikzcd}
\mathscr{C}^{(p/T)} \arrow[r, "f'"] \arrow[d, "F_{\mathscr{C}/T}"'] & X \arrow[d, equal] \\
\vdots \arrow[d] & \vdots \arrow[d, equal] \\
\mathscr{C}^{(p^\ell/T)} \arrow[r, dashed, "f^{(\ell)}"] & X
\end{tikzcd}$$

until

$$f^{(\ell)*}\Omega_X \to \Omega_{\mathscr{C}(p^\ell/T)} \to \Omega_{\mathscr{C}(p^\ell/T)/T}$$

is nonzero. This must happen after finitely many steps since $\deg f' = \frac{1}{p}\deg f$ and $\deg f$ is finite. Set $f' := f^{(\ell)}$. After the factorization terminates, the pullback map is nonzero. Therefore, restricting to the general fiber, $f_t'^*\Omega_X \to \Omega_{\mathscr{C}_t'}$ is nonzero hence surjective at the generic point so $f_t' : \mathscr{C}_t' \to X$ is generically immersive. Now we replace f by f' and shrink T so that we may assume that $f : \mathscr{C} \to X$ is generically immersive on each fiber.

6.6.2 Quasi-finite replacement

Using Lemma 5.3.1 we get a birational map $j : \widetilde{X} \to X$ so that after shrinking T further the map $f : \mathscr{C} \to X$ factors through a quasi-finite map $\widetilde{f} : \mathscr{C} \to \widetilde{X}$. Note, if we want to take $\widetilde{X}$ to be smooth, we may have to give up properness. However, this is no obstacle because the generic curve $\mathscr{C}_t$ is mapped to $\widetilde{X}$ so degrees and intersection numbers still behave well. Now we record the number $A := \deg \widetilde{f}_t^*\omega_{\widetilde{X}}$. This number may have increased from the analogous quantity computed for the original family mapping to X but we will not further increase it in the argument. Therefore, A is an upper bound (possibly depending on the family of curves as well as on the geometry of X) for the number of iterations possible in the subsequent descent steps.

6.6.3 Tangency to the tautological foliations

Let $\{\mathscr{F}^\sigma\}_{\sigma\in\mathbb{B}}$ be the collection of tautological corank 1 foliations on X. By hypothesis, $K_X - \tan(\{\mathscr{F}^\sigma\})$ is $\mathbb{Q}$-effective, which implies that the induced foliations $\widetilde{\mathscr{F}}^\sigma$ on $\widetilde{X}$ also induce a $\mathbb{Q}$-effective divisor $K_{\widetilde{X}} - \tan\{\widetilde{\mathscr{F}}^\sigma\}$. By the isomorphism

$$\mathscr{O}(K_{\widetilde{X}} - \tan\{\widetilde{\mathscr{F}}^\sigma\}) = \bigotimes_{\sigma\in\mathbb{B}} N^\vee_{\widetilde{\mathscr{F}}^\sigma} \hookrightarrow \operatorname{Sym}^d \Omega_{\widetilde{X}}$$

there exists a symmetric md-form $\omega \in H^0(\widetilde{X}, \operatorname{Sym}^{md}\Omega_{\widetilde{X}})$ inducing the foliations. This form vanishes along a divisor D in the linear series $|m(K_{\widetilde{X}} - \tan\{\widetilde{\mathscr{F}}^\sigma\})|$ by construction. If $\mathscr{C} \to T$ is a family of genus 0 curves then $\widetilde{f}^*\omega = 0$ in $\operatorname{Sym}^{md}\Omega_{\mathscr{C}/T}$ and if $K_{\widetilde{X}} - \tan\{\widetilde{\mathscr{F}}^\sigma\}$ is big and $\mathscr{C} \to T$ is a family of genus 1 curves then likewise $\widetilde{f}^*\omega = 0$ in $\operatorname{Sym}^{md}\Omega_{\mathscr{C}/T}$ because $\mathscr{C}$ is a covering family so the generic member must intersect D positively. The natural pullback map

$$\widetilde{f}^* \left(\bigotimes_{\sigma \in \mathbb{B}} N^{\vee}_{\widetilde{\mathscr{F}}^{\sigma}} \right)^{\otimes m} \to \widetilde{f}^* \operatorname{Sym}^{md} \Omega_{\widetilde{X}} \xrightarrow{\widetilde{f}^*} \operatorname{Sym}^{md} \Omega_{\mathscr{C}/T} = \Omega_{\mathscr{C}/T}^{\otimes md}$$

is the m-th tensor power of the product of the pullback maps

$$\varphi_{\sigma} : \widetilde{f}^* N^{\vee}_{\widetilde{\mathscr{F}}^{\sigma}} \to \widetilde{f}^* \Omega_{\widetilde{X}} \xrightarrow{\widetilde{f}^*} \Omega_{\mathscr{C}/T}.$$

Since these are maps of line bundles, we conclude that one of the maps $\{\varphi_{\sigma}\}_{\sigma \in \mathbb{B}}$ is identically zero. Dualizing, we can rephrase this as saying that for a particular $\sigma_0 \in \mathbb{B}$ there is a factorization,

$$\begin{tikzcd}
 & & \widetilde{f}^* \mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma_0}} \\
\mathscr{T}_{\mathscr{C}/T} \arrow[r, hook] \arrow[urr, "0"] \arrow[drr, dashed] & \mathscr{T}_{\mathscr{C}} \arrow[r, "\mathrm{d}\widetilde{f}"] & \widetilde{f}^* \mathscr{T}_{\widetilde{X}} \arrow[u] \\
 & & \widetilde{f}^* \mathscr{F}^{\sigma_0} \arrow[u, hook]
\end{tikzcd}$$

To prove the dashed arrow exists, we need to know that the sequence,

$$0 \to \widetilde{f}^* \mathscr{F}^{\sigma_0} \to \widetilde{f}^* \mathscr{T}_{\widetilde{X}} \to \mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma_0}}$$

is still right-exact after the application of $\widetilde{f}^*$. This is true after possibly shrinking T since $\widetilde{f}$ is quasi-finite so the preimage of the singular locus of $\widetilde{\mathscr{F}}^{\sigma_0}$ (which is the locus where $\mathscr{F}^{\sigma_0}$ fails to be a sub-bundle and hence the sequence is locally split) lies in codimension ≥ 2 and hence does not dominate T.

6.6.4 Minimal tangent foliations

We now know that $\widetilde{f}$ is tangent to some corank 1 foliation $\widetilde{\mathscr{F}}^{\sigma_0}$. Recall that this is shorthand for the relative tangent bundle factoring through $\widetilde{\mathscr{F}}^{\sigma_0}$ in the pullback map,

$$\mathscr{T}_{\mathscr{C}/T} \to \widetilde{f}^* \widetilde{\mathscr{F}}^{\sigma_0} \subset \widetilde{f}^* \mathscr{T}_{\widetilde{X}}.$$

We now consider a minimal subset $\Theta \subset \mathbb{B} \setminus \{\sigma_0\}$ so that there is a further factorization

$$\mathscr{T}_{\mathscr{C}/T} \to \widetilde{f}^* \widetilde{\mathscr{F}}_{\Theta} \subset \widetilde{f}^* \widetilde{\mathscr{F}}^{\sigma_0} \subset \widetilde{f}^* \mathscr{T}_{\widetilde{X}}.$$

We claim that, up to possibly further shrinking T, this set Θ consists exactly of those $\sigma \in \mathbb{B}$ such that the pullback map

$$\varphi_\sigma : \widetilde{f}^* \mathcal{N}^\vee_{\widetilde{\mathcal{F}}^\sigma} \to \widetilde{f}^* \Omega_{\widetilde{X}} \xrightarrow{\widetilde{f}^*} \Omega_{\mathcal{C}/T}$$

is nonzero. Indeed, if $\mathcal{T}_{\mathcal{C}/T} \to \widetilde{f}^* \mathcal{T}_{\widetilde{X}}$ factors through $\widetilde{f}^* \widetilde{\mathcal{F}}_\Theta$ it is clear these pullback maps vanish for each $\sigma \in \Theta^c$ because $\widetilde{\mathcal{F}}_\Theta \subset \widetilde{\mathcal{F}}^\sigma$ for $\sigma \in \Theta^c$. Conversely, by definition

$$\widetilde{\mathcal{F}}_\Theta := \left(\bigcap_{\sigma \in \Theta^c} \mathcal{F}^\sigma \right)^{\vee\vee}$$

If φ_σ is zero then, after possibly shrinking T, the pullback map $\mathcal{T}_{\mathcal{C}/T} \to \widetilde{f}^* \mathcal{T}_{\widetilde{X}}$ factors through each $\widetilde{f}^* \widetilde{\mathcal{F}}^\sigma$ and hence also the intersection. This description of Θ as the $\sigma \in \mathbb{B}$ so that φ_σ is nonzero will be important later.

Next we claim that Θ is ϕ-invariant. This arises from the compatibility of the p-curvature maps with pullback (Proposition 2.2.12)

$$\begin{tikzcd}
\mathrm{Frob}^*_{\mathcal{C}} \mathcal{T}_{\mathcal{C}/T} \arrow[r, "\psi_{\mathcal{T}_{\mathcal{C}/T}, p}"] \arrow[d, "\mathrm{d}\widetilde{f}"] & g^* \mathcal{T}_T \arrow[d, "\mathrm{d}\widetilde{f}"] \\
\widetilde{f}^* \mathrm{Frob}^*_{\widetilde{X}} \widetilde{\mathcal{F}}_\Theta \arrow[r, "\widetilde{f}^* \psi_{\widetilde{\mathcal{F}}_\Theta, p}"] & \widetilde{f}^* \mathcal{T}_{\widetilde{X}} / \widetilde{\mathcal{F}}_\Theta
\end{tikzcd}$$

but the map $\psi_{\mathcal{T}_{\mathcal{C}/T}, p}$ vanishes because $\mathcal{T}_{\mathcal{C}/T}$ is the relative tangent bundle of an actual morphism. By Theorem 6.3.8, on the open set of $\widetilde{X}$ isomorphic to a component of the special fiber of Y_U there is an isomorphism

$$\begin{tikzcd}
\mathrm{Frob}^*_{Y_U} \mathcal{F}_\Theta \arrow[r, "\psi_{\mathcal{F}_\Theta, p}"] \arrow[d, equal] & \mathcal{T}_M / \mathcal{F}_\Theta \arrow[d, equal] \\
\bigoplus_{\sigma \in \Theta} \mathcal{F}_\sigma^p \arrow[r] & \bigoplus_{\tau \in \Theta^c} \mathcal{F}_\tau
\end{tikzcd}$$

where the bottom map is zero except for the components $\mathcal{F}_\sigma^p \to \mathcal{F}_{\phi \circ \sigma}$ which are nonzero on the ordinary locus. Therefore, if Θ is not ϕ-stable, there exists $\sigma \in \Theta$ such that $\phi \circ \sigma \in \Theta^c$. Then we can consider the map

$$\mathrm{Frob}^* \mathcal{T}_{\mathcal{C}/T}|_{\widetilde{f}^{-1}(Y_U)} \to \widetilde{f}^* \bigoplus_{\sigma \in \Theta} \mathcal{F}_\sigma^p \to \widetilde{f}^* \mathcal{F}_\sigma^p \xrightarrow{u_\sigma h^2_{\phi \circ \sigma}} \mathcal{F}_{\phi \circ \sigma}$$

which must be zero by the compatibility of p-curvature diagram. Therefore,

$$\mathscr{T}_{\mathscr{C}/T}^{\otimes p}|_{\widetilde{f}^{-1}(Y_U)} = \mathrm{Frob}^* \mathscr{T}_{\mathscr{C}/T}|_{\widetilde{f}^{-1}(Y_U)} \to \widetilde{f}^* \mathscr{F}_\sigma^p$$

vanishes, but this is the p-th tensor power of the map $\mathscr{T}_{\mathscr{C}/T}|_{\widetilde{f}^{-1}(Y_U)} \to \widetilde{f}^* \mathscr{F}_\sigma$ which must vanish as well. The projection to $\mathscr{F}_\sigma$ vanishing means exactly that $\widetilde{f}$ is tangent to $\widetilde{\mathscr{F}}^\sigma$ i.e. that $\varphi_\sigma = 0$ contradicting $\sigma \in \Theta$. Hence Θ is ϕ-closed so we may write $\Theta = \mathbb{B}_{I^c}$ for some $I \subset \mathrm{Spec}\,(\mathscr{O}_F/p)$.

6.6.5 Descent of the morphism along the foliation

Since $\widetilde{f}$ is tangent to $\widetilde{\mathscr{F}}_\Theta$ the morphism descends along the quotient $\pi : X \to X/\widetilde{\mathscr{F}}_\Theta$ in the sense that there is a diagram,

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\widetilde{f}"] \arrow[d, "F_{\mathscr{C}/T}"'] & \widetilde{X} \arrow[d] \\
\mathscr{C}^{(p/T)} \arrow[r, dashed] \arrow[d, "F_{\mathscr{C}/T}"'] & \widetilde{X}/\widetilde{\mathscr{F}}_\Theta \arrow[d, equal] \\
\vdots \arrow[d] & \vdots \arrow[d, equal] \\
\mathscr{C}^{(p^\ell/T)} \arrow[r, dashed, "f'"] & \widetilde{X}/\widetilde{\mathscr{F}}_\Theta
\end{tikzcd}$$

where we factor out relative Frobenii of $\mathscr{C} \to T$ until $f' : \mathscr{C}^{(p^\ell/T)} \to \widetilde{X}/\widetilde{\mathscr{F}}_\Theta$ is generically an immersion on fibers. If $\widetilde{f}$ is generically an embedding then $\ell = 1$ automatically. This is likely to be true more generally but it will not affect the structure of the argument to consider any value of $\ell \geq 1$. Furthermore, notice that $\widetilde{f'} : \mathscr{C}^{(p^\ell/T)} \to \widetilde{X}/\widetilde{\mathscr{F}}_\Theta$ is quasi-finite since the other maps in the diagram are all quasi-finite.

6.6.6 Birational identification of the quotient

By Theorem 6.3.8, $X' := (\widetilde{X}/\widetilde{\mathscr{F}}_\Theta)^{\mathrm{sm}}$ is birational to an irreducible component of $(Y_U)_{\overline{\mathbb{F}}_p}$ which has a toroidal compactification isomorphic to an irreducible component of $(X_U^{\mathrm{tor}})_{\overline{\mathbb{F}}_p}$. We can pull back the tautological foliations from X_U^{tor} to give an effective foliation datum on X' using Corollary 6.5.3. In particular, the induced foliations on X' have a $\mathbb{Q}$-effective tensored conormal bundle witnessed by some symmetric form ω. Hence we obtain the structure of corank 1 foliations $\{\mathscr{F}'^\sigma\}_{\sigma \in \mathbb{B}}$ indexed by the same set $\mathbb{B}$ which are induced by the Hilbert modular tautological foliations on some open set of X' isomorphic to an open of $(Y_U)_{\overline{\mathbb{F}}_p}$. Therefore, we can repeat the above argument for the generically immersed quasi-finite covering family $f' : \mathscr{C}^{(p^\ell/T)} \to X'$. Note, since we shrunk T in previous steps so that $\widetilde{f} : \mathscr{C} \to \widetilde{X}$ does not hit the singular locus of $\widetilde{\mathscr{F}}_\Theta$, this implies that f' maps into the smooth locus of $\widetilde{X}/\widetilde{\mathscr{F}}_\Theta$.

6.6.7 The degree inequality

The final step is to prove that the canonical degree decreases after the prior construction, in particular that

$$\deg f_t'^*\omega_{X'} < \deg \widetilde{f}^*\omega_{\widetilde{X}}.$$

Our main tool is the relationship between the canonical divisors for the quotient by a foliation (Theorem 2.3.2)

$$K_{\widetilde{X}} = \pi^* K_{\widetilde{X}/\widetilde{\mathscr{F}}_\Theta} + (p-1)c_1(\widetilde{\mathscr{F}}_\Theta).$$

Restricting to the smooth locus of $\mathscr{F}_\Theta$ and pulling back along $\widetilde{f}$ we obtain,

$$\deg \widetilde{f}_t^*\omega_{\widetilde{X}} = \deg \widetilde{f}_t^*\pi^*\omega_{X'} + (p-1)\deg \widetilde{f}_t^*\widetilde{\mathscr{F}}_\Theta.$$

Moreover, $\widetilde{f}^*\pi^*\omega_{X'} = (F^\ell_{\mathscr{C}/T})^* f'^*\omega_{X'}$ and therefore,

$$\deg \widetilde{f}_t^*\omega_{\widetilde{X}} = p^\ell \deg f'^*\omega_{X'} + (p-1)\deg \widetilde{f}_t^*\widetilde{\mathscr{F}}_\Theta. \tag{6.7}$$

Let $H := \tan\{\widetilde{\mathscr{F}}^\sigma\}$ be the tangential divisor of the induced foliations. This is effective by definition. Let $\mathscr{N}_{\widetilde{\mathscr{F}}_\Theta} := \mathscr{T}_{\widetilde{X}}/\widetilde{\mathscr{F}}_\Theta$ denote the normal sheaf of $\widetilde{\mathscr{F}}_\Theta$. Since

$$\bigcap_{\sigma\in\Theta^c}\mathscr{F}^\sigma \subset \widetilde{\mathscr{F}}_\Theta = \left(\bigcap_{\sigma\in\Theta^c}\mathscr{F}^\sigma\right)^{\vee\vee}$$

and $\mathscr{N}_{\widetilde{\mathscr{F}}_\Theta}$ is torsion-free by definition, there is an exact sequence

$$0 \to \mathscr{N}_{\widetilde{\mathscr{F}}_\Theta} \to \bigoplus_{\sigma\in\Theta^c}\mathscr{N}_{\widetilde{\mathscr{F}}^\sigma} \to \operatorname{coker} \to 0$$

of coherent sheaves on $\widetilde{X}$ where coker is supported inside H. Therefore,

$$c_1(\widetilde{\mathscr{F}}_\Theta) = c_1(\mathscr{T}_{\widetilde{X}}) - c_1(\mathscr{N}_{\widetilde{\mathscr{F}}_\Theta}) = -K_{\widetilde{X}} - \sum_{\sigma\in\Theta^c} c_1(\mathscr{N}_{\widetilde{\mathscr{F}}^\sigma}) + c_1(\operatorname{coker}).$$

Here $c_1(\text{coker})$ is an effective divisor supported in H. Recall the defining isomorphism of H,

$$\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} - H) \cong \bigotimes_{\sigma \in \mathbb{B}} \mathscr{N}^{\vee}_{\widetilde{\mathscr{F}}^{\sigma}}$$

which we can rearrange to give

$$\sum_{\sigma \in \mathbb{B}} c_1(\mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma}}) - H = -K_{\widetilde{X}}$$

and plugging in for $-K_{\widetilde{X}}$ we obtain

$$c_1(\widetilde{\mathscr{F}}_{\Theta}) = \sum_{\sigma \in \Theta} c_1(\mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma}}) - H + c_1(\text{coker}). \tag{6.8}$$

Now plugging (6.8) into the canonical bundle relation

$$\pi^* K_{\widetilde{X}/\widetilde{\mathscr{F}}_{\Theta}} = K_{\widetilde{X}} - (p-1) c_1(\widetilde{\mathscr{F}}_{\Theta})$$

gives

$$\pi^* K_{\widetilde{X}/\widetilde{\mathscr{F}}_{\Theta}} = K_{\widetilde{X}} + (p-1)H - (p-1)\left[\sum_{\sigma \in \Theta} c_1(\mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma}}) + c_1(\text{coker})\right] \tag{6.9}$$

$$= pK_{\widetilde{X}} - (p-1)\left[K_{\widetilde{X}} - H + \sum_{\sigma \in \Theta} c_1(\mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma}}) + c_1(\text{coker})\right] \tag{6.10}$$

Notice that the symmetric form ω witnesses that $K_{\widetilde{X}} - H$ is $\mathbb{Q}$-effective and hence intersects the generic curve in the family $\mathscr{C}$ nonnegatively. Combining this with the calculations of (6.7) we obtain

$$\deg f'^* \omega_{X'} = \frac{1}{p^{\ell-1}} \deg \widetilde{f}^* \omega_{\widetilde{X}} - \frac{p-1}{p^{\ell-1}} \deg \widetilde{f}_t^* \left[K_{\widetilde{X}} - H + \sum_{\sigma \in \Theta} c_1(\mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma}}) + c_1(\text{coker})\right]$$

However, $\deg f_t^*[K_{\widetilde{X}} - H] \geq 0$ because $K_{\widetilde{X}} - H$ is $\mathbb{Q}$-effective and $\deg f_t^* c_1(\text{coker}) \geq 0$ because $c_1(\text{coker})$ is effective. Moreover, for $\sigma \in \Theta$ we showed that the maps

$$\varphi_{\sigma} : \widetilde{f}^* N^{\vee}_{\widetilde{\mathscr{F}}^{\sigma}} \to \Omega_{\mathscr{C}/T}$$

are each nonzero. Dualizing, the nonzero and hence injective map of line bundles,

$$\varphi^{\vee}_{\sigma} : \mathscr{T}_{\mathscr{C}/T} \to \widetilde{f}^* \mathscr{N}_{\widetilde{\mathscr{F}}^{\sigma}}$$

implies that $\deg \widetilde{f}^* \mathcal{N}_{\widetilde{\mathcal{F}}^\sigma} \geq 2 - 2g$ where g is the genus of the fiber $\mathscr{C}_t$. If $g = 0$ then each $\deg \widetilde{f}^* \mathcal{N}_{\widetilde{\mathcal{F}}^\sigma} \geq 2$ and hence

$$\deg f'^* \omega_{X'} < \frac{1}{p^{\ell - 1}} \deg \widetilde{f}^* \omega_{\widetilde{X}} \leq \deg \widetilde{f}^* \omega_{\widetilde{X}}$$

If $g = 1$ we only conclude that each $\deg \widetilde{f}^* \mathcal{N}_{\widetilde{\mathcal{F}}^\sigma} \geq 0$. However, in this case we also assumed that $K_{\widetilde{X}} - H$ is big and hence $\deg \widetilde{f}_t^* [K_{\widetilde{X}} - H] > 0$ is strictly positive. Therefore, we also obtain a strict degree inequality

$$\deg f'^* \omega_{X'} < \frac{1}{p^{\ell - 1}} \deg \widetilde{f}^* \omega_{\widetilde{X}} \leq \deg \widetilde{f}^* \omega_{\widetilde{X}}$$

in the $g = 1$ case.

6.6.8 Contradiction by infinite descent

Because X' is birational to a Hilbert modular variety with $\mathbb{Q}$-effective $K - \tan$ we can repeat the argument to obtain a sequence of generically immersed families of curves,

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\widetilde{f}"] \arrow[d, "F^{\ell(1)}_{\mathscr{C}/T}"'] & \widetilde{X} \arrow[d] \\
\mathscr{C}^{(p^{\ell(1)}/T)} \arrow[r, "f^{(1)}"'] \arrow[d, "F^{\ell(2)-\ell(1)}_{\mathscr{C}/T}"'] & X^{(1)} \arrow[d] \\
\mathscr{C}^{(p^{\ell(2)}/T)} \arrow[r, "f^{(2)}"] \arrow[d] & X^{(2)} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

The degree inequality implies that

$$\deg f^{(n)*} \omega_{X^{(n)}} < \deg f^{(n-1)*} \omega_{X^{(n-1)}} < \cdots < \deg f^{(0)*} \omega_{X^{(0)}} = \deg \widetilde{f}^* \omega_{\widetilde{X}} = A.$$

But these numbers are positive since $K_{X^{(n)}}$ is $\mathbb{Q}$-effective for each n by our assumptions. Therefore, A cannot strictly decrease an infinite number of times which contradicts the existence of the initial covering family of X. This completes the proof by infinite descent.

6.6.9 "Boundedness" of Families of Curves

Recall that families of curves on X_U^{tor} are unbounded because we can pullback by partial Frobenius to obtain a family of curves of the same genus but with degree growing without bound. However, the techniques used here allow us to show that this is the only source of unboundedness. Precisely, we will show that every family of curves of fixed genus g is pulled back via a finite sequence of partial Frobenii from a bounded family of curves. We will now sketch out how this partial boundedness argument works using the same ideas as the proof of Theorem 6.6.1. The additional steps largely follow the arguments of [GRVR18] and [RT18] in characteristic zero.

Theorem 6.6.2. *Let $U \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$ be a compact open subgroup and X_U^{tor} a choice of toroidal compactification of Y_U. Suppose that $K_{X_U^{\text{tor}}} - (d-1)E$ is big. Then for any prime $p \in \operatorname{Spec}\mathbb{Z}$ unramified in F and coprime to the level of U, each geometric component X of the reduction $(X_U^{\text{tor}})_{\mathbb{F}_p}$ has a proper Zariski closed subset Z so that any family of curves of genus g on X not contained in Z is pulled back via a finite sequence of partial Frobenii from a bounded family of curves.*

Proof. Since $D = K_{X_U^{\text{tor}}} - (d-1)E$ is big, the stable base locus $\mathbb{B}(D)$ is a proper closed subset of X. We will take $Z := \mathbb{B}(D) \cup \operatorname{Sing}(\{\mathscr{F}^\sigma\}_{\sigma\in\mathbb{B}})$ to be the union of the stable base locus of D and the union of the singular loci of the tautological foliations. Note that $\operatorname{Sing}(\{\mathscr{F}^\sigma\}_{\sigma\in\mathbb{B}}) \subset E$ lies in the boundary. Consider a family of curves of genus g

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{f} & X \\ \Big\downarrow{\scriptstyle g} & & \\ T & & \end{array}$$

dominating a geometric component $X \subset (X_U^{\text{tor}})_{\overline{\mathbb{F}}_p}$. At the cost of taking a modification $\widetilde{X} \to X$ (or if we want $\widetilde{X}$ smooth, we cannot ensure that $j : \widetilde{X} \to X$ is proper) we can replace f with a new family $\widetilde{f} : \mathscr{C}^{(p^\ell/T)} \to X$ which is generically an immersion on fibers and does not meet the singular locus of the tautological foliations (possibly after shrinking the parameter space T). Now either f' is tangent to one of the tautological foliations or the pullback map

$$\widetilde{f}^*\left(\bigotimes_{\sigma\in\mathbb{B}} N^\vee_{\widetilde{\mathscr{F}}^\sigma}\right)^{\otimes m} \to \widetilde{f}^*\operatorname{Sym}^{md}\Omega_{\widetilde{X}} \xrightarrow{\widetilde{f}^*} \operatorname{Sym}^{md}\Omega_{\mathscr{C}/T} = \Omega_{\mathscr{C}/T}^{\otimes md}$$

is nonzero. In the latter case, this proves that:

$$\deg \widetilde{f}_t^* D = \deg \widetilde{f}_t^* \left(\bigotimes_{\sigma \in \mathbb{B}} N^\vee_{\widetilde{\mathscr{F}}^\sigma} \right) \leq \deg \Omega^{\otimes d}_{\mathscr{C}/T}|_{\mathscr{C}_t} = d(2g-2).$$

In particular, $\deg \widetilde{f}_t^* D \leq d(2g-2)$ is bounded uniformly in terms of the genus g. This implies that the family $\widetilde{f}$ is bounded since away from the stable base locus $\mathbb{B}(D)$, for large enough m, the linear series $|mD|$ maps these curves to a family of curves on a fixed (in terms of m) projective space with degree bounded by $md(2g-2)$. This family of curves is bounded by Hilbert scheme arguments.

Alternatively, suppose that $\deg \widetilde{f}_t^* D > d(2g-2)$. Then the previous paragraph implies the pullback map vanishes so $\widetilde{f}$ is tangent to one of the tautological foliations. In this case, $\widetilde{f}$ is pulled back from a new family along the partial Frobenius i.e. the quotient map $\widetilde{X} \to \widetilde{X}/\widetilde{\mathscr{F}}^\sigma$ for some $\sigma \in \mathbb{B}$. As in § 6.6.4 we obtain a minimal subset $\Theta \subset \mathbb{B} \setminus \{\sigma_0\}$ so that $\widetilde{f}$ is tangent to $\widetilde{\mathscr{F}}_\Theta$ which is ϕ-stable and consists of exactly those $\sigma \in \mathbb{B}$ so that the pullback map

$$\widetilde{f}^* \mathscr{N}^\vee_{\widetilde{\mathscr{F}}^\sigma} \to \widetilde{f}^* \Omega_{\widetilde{X}} \xrightarrow{\widetilde{f}^*} \Omega_{\mathscr{C}/T}$$

is nonzero. Since $\widetilde{f}$ does not meet the singular locus of the foliation, the degree equality arising from (6.9) in § 6.6.7 shows that

$$\deg f'^* \omega_{X'} = \frac{1}{p^{\ell-1}} \deg \widetilde{f}^* \omega_{\widetilde{X}} - \frac{p-1}{p^{\ell-1}} \deg \widetilde{f}_t^* \left[K_{\widetilde{X}} - H + \sum_{\sigma \in \Theta} c_1(\mathscr{N}_{\widetilde{\mathscr{F}}^\sigma}) + c_1(\text{coker}) \right]$$

Now $c_1(\text{coker})$ is effective and hence $\deg \widetilde{f}_t^* c_1(\text{coker}) \geq 0$. Moreover, for $\sigma \in \Theta$ we showed that the maps

$$\varphi_\sigma : \widetilde{f}^* \mathscr{N}^\vee_{\widetilde{\mathscr{F}}^\sigma} \to \widetilde{f}^* \Omega_{\widetilde{X}} \xrightarrow{\widetilde{f}^*} \Omega_{\mathscr{C}/T}$$

are each nonzero. Dualizing, the nonzero and hence injective map of line bundles,

$$\varphi_\sigma^\vee : \mathscr{T}_{\mathscr{C}/T} \to \widetilde{f}^* \mathscr{N}_{\widetilde{\mathscr{F}}^\sigma}$$

implies that $\deg \widetilde{f}^* \mathscr{N}_{\widetilde{\mathscr{F}}^\sigma} \geq 2-2g$. Therefore,

$$\deg f'^* \omega_{X'} \leq \frac{1}{p^{\ell-1}} \deg \widetilde{f}^* \omega_{\widetilde{X}} - \frac{p-1}{p^{\ell-1}} \deg \widetilde{f}_t^* \left[\widetilde{D} + \sum_{\sigma \in \Theta} (2-2g) \right]$$

where $\widetilde{D} := K_{\widetilde{X}} - H$. Therefore, as long as

$$d(2g-2) < \deg \widetilde{f_t}^* \widetilde{D}$$

we obtain a strict inequality,

$$\deg f'^* \omega_{X'} < \frac{1}{p^{\ell-1}} \deg \widetilde{f}^* \omega_{\widetilde{X}}.$$

By the pullback functoriality in § 6.5.1,

$$j^* \mathscr{O}_X(D) \to \mathscr{O}_{\widetilde{X}}(\widetilde{D})$$

we see that $\deg \widetilde{f_t}^* \widetilde{D} \geq \deg \widetilde{f_t}^* D > d(2g-2)$ verifying the needed assumption. Hence we can descend along the partial Frobenius only finitely many times since the sequence

$$\deg f^{(n)*} \omega_{X^{(n)}} < \deg f^{(n-1)*} \omega_{X^{(n-1)}} < \cdots < \deg f^{(0)*} \omega_{X^{(0)}} = \deg \widetilde{f}^* \omega_{\widetilde{X}} = A$$

of degrees must be strictly decreasing. Note these numbers are positive since $\mathbb{B}(K_X) \subset \mathbb{B}(D)$ and we assume that f does not map into $\mathbb{B}(D)$. Thus after finitely many stages, we must obtain a family satisfying $\deg f^{(n)*} \omega_{X^{(n)}} \leq d(2g-2)$ so that the process terminates. This family is bounded, completing the proof. □

The same argument also applies to product-quotient surfaces and more generally any surface where the procedure of chapter 5 either terminates successfully or runs indefinitely. We leave the details to the reader.

Chapter 7

Jet Bundles and Higher-Order Obstructions

> If I correctly understand the sense of this succinct observation, our poet suggests here that human life is but a series of footnotes to a vast obscure unfinished masterpiece.
>
> Charles Kinbote

7.1 Jet Obstructions

Here we describe another approach to obstructing unirationality. The idea is very simple: recall that in characteristic zero we may use forms to obstruct unirationality; however, in positive characteristic, the issue is that degree p purely inseparable maps can kill first-order derivatives. The fundamental observation is that they cannot kill p^{th}-order (divided) derivatives. Therefore, if we can find higher-order forms on our varieties, then these will pull back nontrivially even along certain inseparable maps to produce obstructions. This idea can be formalized using various notions of jet bundles. We plan to extend and combine these techniques with the foliation–descent methods developed in this thesis to develop more applicable higher-jet unirationality obstructions.

Definition 7.1.1. Let $X \to S$ be a smooth scheme. Then $J^n_{X/S} := \pi_{1*}\pi_2^*\mathscr{O}_X$ where the projections are along the n^{th}-order thickening of the diagonal $\Delta^n_{X/S} := V(\mathscr{I}_\Delta^{n+1}) \subset X \times_S X$. This satisfies the following universal property,

$$\mathscr{H}om_{\mathscr{O}_X}(J^n_{X/S}, \mathscr{E}) = \mathscr{D}iff_X^{\le n}(\mathscr{O}_X, \mathscr{E})$$

where $\mathscr{D}\!\mathit{iff}_X^{\leq n}(\mathscr{O}_X, \mathscr{E})$ is the sheaf of differential operators $D : \mathscr{O}_X \to \mathscr{E}$ of order $\leq n$ (see [DG67, IV$_4$, §16.8]).

We begin by defining a notion of bigness for lifts of jets.

Definition 7.1.2. Let $\widetilde{J}^n_X := \ker(J^n_X \to \mathscr{O}_X)$ using the natural projection $J^n_X \to J^0_X = \mathscr{O}_X$.

Definition 7.1.3. Let X be a smooth proper variety. Define the *jet-amplitude* of X to be

$$\operatorname{jetamp}(X) := \sup\left\{\frac{n}{r} : \exists\, \omega \in H^0(X, \widetilde{J}^n_X) \text{ such that } \omega|_{\widetilde{J}^r_X} \neq 0\right\}$$

Theorem 7.1.4 (C)**.** *Let X be a smooth proper variety over a perfect field k. Suppose that $\log_p \operatorname{jetamp}(X) > \ell$. Then there does not exist a unirational parametrization $\mathbb{P}^n \dashrightarrow X$ whose inseparable degree is p^ℓ.*

Proof. Since jetamp only increases along separable maps, it suffices to consider a purely inseparable unirational parametrization $f : \mathbb{P}^n \dashrightarrow X$ of degree $q := p^\ell$. Since $\operatorname{jetamp}(X) > q$, there are integers $r, s \geq 0$ and a section $\omega \in H^0(X, \widetilde{J}^s_X)$ such that $\omega|_{\widetilde{J}^r} \neq 0$ and $s > qr$. By decreasing r we may assume that $\omega|_{\widetilde{J}^{r-1}} = 0$, so $\omega|_{\widetilde{J}^r} \in H^0(X, \operatorname{Sym}^r(\Omega_X))$. Replacing ω by its image in $H^0(X, \widetilde{J}^{qr}_X)$, we may assume that $s = qr$. There is a factorization of rational maps

$$\begin{tikzcd} X^{(1/q)} \arrow[r, dashed, "f'"] \arrow[dr, "F"'] & \mathbb{P}^n \arrow[d, dashed, "f"] \\ & X \end{tikzcd}$$

where $F : X^{(1/q)} \to X$ is the ℓ^{th}-relative Frobenius. Using extension in codimension ≥ 2, there are well-defined pullback maps

$$\begin{tikzcd} H^0(X, \widetilde{J}^s_X) \arrow[r] \arrow[d, "f^*"] & H^0(X, \widetilde{J}^r_X) \arrow[d, "f^*"] \\ H^0(\mathbb{P}^n, \widetilde{J}^s_{\mathbb{P}^n}) \arrow[r] \arrow[d, "f'^*"] & H^0(\mathbb{P}^n, \widetilde{J}^r_{\mathbb{P}^n}) \arrow[d, "f'^*"] \\ H^0(X^{(1/q)}, \widetilde{J}^s_{X^{(1/q)}}) \arrow[r] & H^0(X^{(1/q)}, \widetilde{J}^r_{X^{(1/q)}}) \end{tikzcd}$$

where the composition along the column is F^*. Therefore, it suffices to show that $F^*\omega \neq 0$. To do this, we may work étale-locally. Indeed, if $c : U \to X$ is an étale chart then it suffices to show that $F_U^*\omega|_U \neq 0$ since the diagram

$$\begin{tikzcd} U^{(1/q)} \arrow[r, "F_U"] \arrow[d] & U \arrow[d] \\ X^{(1/q)} \arrow[r, "F"] & X \end{tikzcd}$$

commutes. Since X is smooth, there are étale charts $U \to X$ with $g : U \to \mathbb{A}^n_k$ étale. The diagram,

$$\begin{tikzcd} \widetilde{J}^s_{\mathbb{A}^n} \arrow[r] \arrow[d, "F^*"] & \widetilde{J}^r_{\mathbb{A}^n} \arrow[d, "F^*"] \arrow[dl, dashed, "t"] \\ \widetilde{J}^s_{\mathbb{A}^n} \arrow[r] & \widetilde{J}^r_{\mathbb{A}^n} \end{tikzcd}$$

has a commuting lift t. Indeed, $J^r_{\mathbb{A}^n}$ is represented by the algebra,

$$k[x_1, \dots, x_n][\mathrm{d}x_1, \dots, \mathrm{d}x_n]/(\mathrm{d}x_1, \dots, \mathrm{d}x_n)^{r+1}$$

and $F^*\mathrm{d}x_i = \mathrm{d}x_i^q = (\mathrm{d}x_i)^q$. Therefore, the kernel of $\widetilde{J}^s_{\mathbb{A}^n} \to \widetilde{J}^r_{\mathbb{A}^n}$ is the space of polynomials $h(\mathrm{d}x_1, \dots, \mathrm{d}x_n)$ with coefficients in $k[x_1, \dots, x_n]$ of minimal degree $\geq r+1$ and hence $F^*h = h(\mathrm{d}x_1^q, \dots, \mathrm{d}x_n^q) = 0$ in $J^s_{\mathbb{A}^n}$ since it has minimal degree $\geq q(r+1) \geq s+1$. Furthermore, consider a symmetric form $\eta \in H^0(\mathbb{A}^n, \mathrm{Sym}^r(\Omega))$ write

$$\eta = \sum_{|\alpha|=r} f_\alpha(x_1, \dots, x_n)(\mathrm{d}x)^\alpha$$

then we see

$$t(\eta) = \sum_{|\alpha|=r} f_\alpha(x_1^q, \dots, x_n^q)(\mathrm{d}x)^{q\alpha} \in H^0(\mathbb{A}^n, \mathrm{Sym}^s(\Omega))$$

and the monomials $(\mathrm{d}x)^{q\alpha}$ are distinct as α ranges over the multi-indices of weight r. Therefore, $t(\eta) = 0$ if and only if all $f_\alpha(x_1^q, \dots, x_n^q) = 0$ if and only if all $f_\alpha = 0$. Hence t is injective. Since g is étale, we get isomorphisms $g^*\widetilde{J}^r_{\mathbb{A}^n} \xrightarrow{\sim} \widetilde{J}^r_U$ for all r so the diagram pulls back to

$$\begin{tikzcd} \widetilde{J}^s_U \arrow[r] \arrow[d, "F_U^*"'] & \widetilde{J}^r_U \arrow[d, "F_U^*"] \arrow[dl, "t_U"'] \\ \widetilde{J}^s_U \arrow[r] & \widetilde{J}^r_U \end{tikzcd}$$

such that the restriction $t_U : \mathrm{Sym}^r(\Omega_U) \to \mathrm{Sym}^s(\Omega_U)$ is injective. Therefore $F_U^*\omega|_U = t(\omega|_{\widetilde{J}^r_U}) \neq 0$ because we assumed that $\omega|_{\widetilde{J}^r} \in H^0(X, \mathrm{Sym}^r(\Omega_X))$ and is nonzero. $\square$

Unfortunately it is quite difficult to compute jetamp(X). For "generic" complete intersections of large degree, it is known that jetamp$(X) = \infty$. However, for any particular example it is very hard to get one's hands on this thing. There is a trivial bound as follows.

Proposition 7.1.5. *Let X be a smooth projective variety. Then,*

$$\operatorname{jetamp}(X) \geq \left(\frac{\underline{\hat{h}}^0(\Omega_X)}{\hat{h}^1(\Omega_X)}\right)^{\frac{1}{2\dim X}}$$

where $\hat{h}^i$ are the so-called asymptotic cohomology

$$\hat{h}^i(X, \mathscr{E}) := \limsup_{m\to\infty} \frac{h^i(X, \operatorname{Sym}^m(\mathscr{E}))}{m^{\dim X + \operatorname{rank}\mathscr{E} - 1}}$$

and likewise $\underline{\hat{h}}^i$ are the lower asymptotic cohomology

$$\underline{\hat{h}}^i(X, \mathscr{E}) := \liminf_{m\to\infty} \frac{h^i(X, \operatorname{Sym}^m(\mathscr{E}))}{m^{\dim X + \operatorname{rank}\mathscr{E} - 1}}$$

We first record the elementary estimate used in the proof.

Lemma 7.1.6. *Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. Suppose that*

$$\liminf_{n\to\infty} \frac{f(n)}{n^k} = a \quad \text{and} \quad \limsup_{n\to\infty} \frac{f(n)}{n^k} = b$$

then[1]

$$\frac{b}{k+1} \geq \limsup_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} f(i) \geq \liminf_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} f(i) \geq \frac{a}{k+1}$$

[1] Notice that the outer inequalities are not usually equalities. For example let

$$f(n) = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

then the hypotheses are satisfied with $k = 0$ and $a = 0$ and $b = 1$ but

$$\sum_{i=1}^{n} f(i) = \lceil \frac{n}{2} \rceil$$

and therefore the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i) = \frac{1}{2}$$

exists so the inequalities in the conclusion of the lemma are strict.

Proof. The assumptions give for all $\epsilon > 0$ there is n_ϵ such that $n \geq n_\epsilon$ implies

$$(b+\epsilon)n^k \geq f(n) \geq (a-\epsilon)n^k$$

summing this from $i = 1$ to n, we get,

$$(b+\epsilon)\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k \geq \frac{1}{n^{k+1}}\sum_{i=n_\epsilon}^{n} f(i) + \frac{1}{n^{k+1}}\sum_{i=1}^{n_\epsilon} i^k \geq (a-\epsilon)\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k$$

Therefore,

$$\frac{b+\epsilon}{k+1}(1+O(n^{-1})) \geq \frac{1}{n^{k+1}}\sum_{i=1}^{n} f(i) + \frac{1}{n^{k+1}}\sum_{i=1}^{n_\epsilon}(i^k - f(i)) \geq \frac{a-\epsilon}{k+1}(1+O(n^{-1}))$$

Taking the limit $n \to \infty$ and then $\epsilon \to 0$ gives the bounds

$$\frac{b}{k+1} \geq \limsup_{n\to\infty} \frac{1}{n^{k+1}}\sum_{i=1}^{n} f(i) \geq \liminf_{n\to\infty} \frac{1}{n^{k+1}}\sum_{i=1}^{n} f(i) \geq \frac{a}{k+1}$$

□

Proof of Proposition 7.1.5. This arises from the exact sequence

$$0 \longrightarrow \mathrm{Sym}^n(\Omega_X) \longrightarrow J_X^n \longrightarrow J_X^{n-1} \longrightarrow 0$$

from which we the long exact sequence in cohomology produces an inequality of dimensions:

$$h^0(X, J_X^r) \geq \sum_{i=1}^{r} \left[h^0(X, \mathrm{Sym}^i(\Omega_X)) - h^1(X, \mathrm{Sym}^i(\Omega_X))\right].$$

Fix a rational number $s \geq 1$ so that sr is an integer, then the long exact sequence also gives an inequality,

$$\dim \mathrm{im}\,(H^0(X, J_X^{rs}) \to H^0(X, J_X^r)) \geq h^0(X, J_X^r) - \sum_{m=r+1}^{rs} \left[h^1(X, \mathrm{Sym}^m(\Omega_X))\right].$$

Therefore,

$$\dim \mathrm{im}\,(H^0(X, J_X^{rs}) \to H^0(X, J_X^r)) \geq \sum_{i=1}^{r} h^0(X, \mathrm{Sym}^i(\Omega_X)) - \sum_{i=1}^{rs} h^1(X, \mathrm{Sym}^i(\Omega_X)).$$

Applying Lemma 7.1.6, for any $\epsilon > 0$ and $r \gg_\epsilon 0$ we have

$$\sum_{i=1}^{r} h^0(X, \mathrm{Sym}^i(\Omega_X)) - \sum_{i=1}^{rs} h^1(X, \mathrm{Sym}^i(\Omega_X)) \geq \frac{1}{2\dim X}\left[(\underline{\hat{h}}^0 - \epsilon) r^{2\dim X} - (\hat{h}^1 + \epsilon)(rs)^{2\dim X}\right]$$

Thus,

$$s^{2\dim X} \leq \frac{\underline{\hat{h}}^0 - \epsilon}{\hat{h}^1 + \epsilon} \implies \mathrm{jetamp}(X) \geq s$$

meaning that

$$\mathrm{jetamp}(X) \geq \sup_{\epsilon > 0}\left(\frac{\underline{\hat{h}}^0 - \epsilon}{\hat{h}^1 + \epsilon}\right)^{\frac{1}{2\dim X}} = \left(\frac{\underline{\hat{h}}^0}{\hat{h}^1}\right)^{\frac{1}{2\dim X}}.$$

□

Example 7.1.7. Let X be a smooth projective surface in characteristic 2. Suppose

$$h^0(X, \mathrm{Sym}^2(\Omega_X)) > h^1(X, \mathrm{Sym}^3(\Omega_X)) + h^1(X, \mathrm{Sym}^4(\Omega_X)).$$

Then X admits no dominant purely inseparable rational map $\mathbb{P}^2 \dashrightarrow X$ of degree 2; in particular, X is not a Zariski surface.

Appendix A

Notation

This appendix records the notation used most frequently throughout the thesis. It is not exhaustive, but it collects the symbols that recur most often without comment.

General algebraic geometry

- k a base field, $\bar{k}$ a fixed algebraic closure, and $\overline{\mathbb{Q}}$ an algebraic closure of $\mathbb{Q}$
- $\mathbb{F}_q$ the finite field with q elements
- Ω_X and $\mathscr{T}_X$ the cotangent and tangent sheaves of a variety X
- ω_X the canonical line bundle of a smooth variety X
- $\operatorname{Sym}^m \Omega_X$ the sheaf of symmetric differentials of degree m
- $\mathbb{P}_X(\mathscr{E}) = \mathbf{Proj}_X(\operatorname{Sym}^\bullet \mathscr{E})$ the projectivization of a coherent sheaf $\mathscr{E}$ on X
- $\operatorname{Jac}(C)$ the Jacobian of a smooth projective curve C
- $\operatorname{Prym}(D/C)$ the Prym variety of an étale double cover $D \to C$
- $\pi_1^{\text{ét}}(X)$ the étale fundamental group of X
- $\operatorname{Tors}(G)$ the normal subgroup of G normally generated by all elements of finite order
- $\langle\langle S \rangle\rangle_G$ the normal closure of a subset S in a group G

Foliations and positive characteristic

- $\mathrm{Frob}_X : X \to X$ the absolute Frobenius of a scheme in characteristic p
- $X^{(p^e/S)}$ the e-fold Frobenius twist of X relative to a base scheme S of characteristic $p > 0$
- $X^{(p^e)}$ the e-fold Frobenius twist relative to k of a variety X/k over a field of characteristic $p > 0$
- $F_X^e : X \to X^{(p^e)}$ the e-fold relative Frobenius; for $e = 1$ we write F_X
- $\mathscr{F} \subset \mathscr{T}_X$ a foliation on a normal variety X
- $\mathrm{Sing}(\mathscr{F})$ the singular locus of a foliation where $\mathscr{F} \subset \mathscr{T}_X$ is not a sub-bundle
- $f^+\mathscr{F}$ the pullback foliation along a morphism $f : X \to Y$
- $\psi_{\mathscr{F},p} : \mathrm{Frob}^* \mathscr{F} \to \mathscr{T}_X/\mathscr{F}$ the p-curvature of an involutive subsheaf $\mathscr{F}$
- $X/\mathscr{F}$ the Ekedahl quotient of a smooth variety by a p-closed foliation
- $\mathscr{N}_{\mathscr{F}}$ the normal sheaf of a foliation (the reflexive hull of $\mathscr{T}_X/\mathscr{F}$)
- $\tan\{\mathscr{F}_i\}$ the tangential divisor of a collection of foliations in general position

Hurwitz curves and related notation

- $\Delta(a, b, c)$ the triangle group with signature (a, b, c)
- $K_+ := \mathbb{Q}(\zeta_7)^+ = \mathbb{Q}(\cos(2\pi/7))$
- $\mathscr{Q}_{\mathrm{Hur}}$ the Hurwitz order in the quaternion algebra used to realize certain Hurwitz curves as Shimura curves
- $\mathscr{X}^{\mathrm{Hur}}(\mathfrak{a})$ the Shimura curve attached to the Hurwitz order at level $\mathfrak{a}$

Hilbert modular varieties

- F a totally real number field of degree $d := [F : \mathbb{Q}]$ with ring of integers $\mathscr{O}_F$
- $\Sigma := \mathrm{Hom}(F, \mathbb{R})$ the set of real embeddings of F; we write $\alpha_\sigma := \sigma(\alpha)$
- $D := \mathrm{disc}_{F/\mathbb{Q}}$ the absolute discriminant and $\mathfrak{d} := \mathfrak{d}_{F/\mathbb{Q}}$ the different ideal

- $\mathscr{C}\ell^+(F)$ the narrow class group of F
- $\mathfrak{c}$ a fractional ideal of F serving as the polarization module
- $\mathbb{A}_F^\infty$ (resp. $\mathbb{A}^\infty$) the finite adeles of F (resp. $\mathbb{Q}$)
- $\mathfrak{H} \subset \mathbb{C}$ the upper half-plane and $\mathfrak{H}^\pm := \mathbb{C} \setminus \mathbb{R}$
- $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ the Deligne torus
- $G^{\mathrm{Hilb}} := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$ the Hilbert modular group
- G^{PEL} the PEL variant of the Hilbert modular group with modified center
- X^{Hilb} and X^{PEL} the corresponding conjugacy classes, both identified with $(\mathfrak{H}^\pm)^\Sigma$
- $X^+ = \mathfrak{H}^\Sigma$ a connected component of the symmetric domain
- $\mathrm{Sh}_K(G,X)$ the Shimura variety attached to a Shimura datum (G,X) at level K
- $U(N) := \ker(\mathrm{GL}_2(\widehat{\mathscr{O}}_F) \to \mathrm{GL}_2(\mathscr{O}_F/N))$ the principal level-N subgroup
- $Y^{\mathfrak{c}}(N)$ the Hilbert modular scheme parametrizing $\mathfrak{c}$-polarized HBAVs with full level-N structure
- $Y^{\mathfrak{c}}_{00}(N)$ the variant with $\Gamma_{00}(N)$-level structure
- $Z^{\mathfrak{c}}(N)$ the auxiliary scheme parametrizing $\mathscr{O}_F$-linear isomorphisms $\mathfrak{c} \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mu_N$
- $G_{U,N}$ the finite quotient acting on $Y^{\mathfrak{c}}(N)$ and $Y_U := \coprod_{\mathfrak{c} \in \mathscr{C}\ell^+(F)} Y^{\mathfrak{c}}(N)/G_{U,N}$ the resulting Hilbert modular variety
- p a rational prime unramified in F and coprime to the level
- $p\mathscr{O}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ with inertia degrees $f_i := f(\mathfrak{p}_i/p)$
- $\kappa := \mathbb{F}_{p^f}$ where $f := \mathrm{lcm}\{f_1, \dots, f_r\}$, and $W(\kappa)$ its Witt ring
- ϕ the Frobenius of $W(\kappa)$
- $\mathbb{B} := \mathrm{Hom}(F, W(\kappa)[1/p]) = \bigsqcup_{\mathfrak{p}|p} \mathbb{B}_{\mathfrak{p}}$ the set of p-adic embeddings; ϕ acts by $\sigma \mapsto \phi \circ \sigma$
- for $I \subset \mathrm{Spec}(\mathscr{O}_F/p)$ we write $\mathfrak{p}_I := \prod_{\mathfrak{p} \in I} \mathfrak{p}$ and $\mathbb{B}_I := \bigcup_{\mathfrak{p} \in I} \mathbb{B}_{\mathfrak{p}}$

- $\mathscr{M}$ a Hilbert modular scheme carrying a universal HBAV $\pi : \mathscr{A} \to \mathscr{M}$; $M := \mathscr{M} \times_{W(\kappa)} \kappa$ its special fiber
- $\mathscr{M}_0^{\mathfrak{c}}(\mathfrak{p}_I)$ the Iwahori $\Gamma_0(\mathfrak{p}_I)$-level moduli space; $M_0^{\mathfrak{c}}(\mathfrak{p}_I)$ its special fiber
- M^{ord} the ordinary locus, and $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\mathrm{m}}$, $M_0^{\mathfrak{c}}(\mathfrak{p}_I)^{\text{ét}}$ the distinguished fully multiplicative and fully étale components of the Iwahori special fiber
- $\underline{\omega} := \pi_*\Omega^1_{\mathscr{A}/\mathscr{M}}$ the Hodge bundle, $\underline{\mathrm{Lie}} := \mathrm{Lie}(\mathscr{A}/\mathscr{M}) \cong \underline{\omega}^\vee$, and ω_σ the σ-isotypic summand of $\underline{\omega}$
- $\underline{\mathfrak{c}}$ the equivariant sheaf $\mathfrak{c} \otimes \mathscr{O}_{\mathscr{M}}$ and δ_σ the σ-isotypic part of $\underline{\mathfrak{c}\mathfrak{d}}^{-1}$
- $\mathrm{KS} : \mathscr{T}_{\mathscr{M}} \xrightarrow{\sim} \underline{\mathrm{Lie}}^{\otimes 2} \otimes \underline{\mathfrak{d}\mathfrak{c}}$ the Kodaira–Spencer isomorphism
- X_U^{tor} a toroidal compactification of Y_U with reduced boundary divisor E
- $\mathscr{F}_\sigma \cong \omega_\sigma^{-2} \otimes \delta_\sigma$ the rank-1 tautological foliation indexed by $\sigma \in \mathbb{B}$
- $\mathscr{F}_\Theta := \bigoplus_{\sigma\in\Theta} \mathscr{F}_\sigma$ for $\Theta \subset \mathbb{B}$; lower indices are used for the rank-1 tautological summands and their direct sums
- $\mathscr{F}^\sigma := \mathscr{F}_{\mathbb{B}\setminus\{\sigma\}}$ the complementary corank-1 foliation; more generally, upper indices indicate complementary foliations obtained by omitting the indicated lower-index summands
- h_σ the partial Hasse invariant

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