Lecture 1

Benjamin Church

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1 Holomorphic Functions

Definition: We say that a complex function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* at $z \in \mathbb{C}$ if the limit,

$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w}$$

exists in which case we call its value $f'(z)$ the *complex derivative* of f at z.

Remark 1.1. Notice that the limit here means that $|h| \to 0$ so h can go to zero from any direction.

Example 1.1. Consider $f = z^2$. Then,

$$f'(z) = \lim_{w \to 0} \frac{(z+w)^2 - z^2}{w} = \lim_{w \to 0} \frac{z^2 + 2zw + w^2 - z^2}{w} = \lim_{w \to 0} \frac{2zw + w^2}{w} = \lim_{w \to 0} (2z + w) = 2z$$

as expected.

Example 1.2. Consider $f(z) = \overline{z}$. Then,

$$f'(z) = \lim_{w \to 0} \frac{\overline{z+w} - \overline{z}}{w} = \lim_{w \to 0} \frac{\overline{z} + \overline{w} - \overline{z}}{w} = \lim_{w \to 0} \frac{\overline{w}}{w}$$

However, suppose we send w to zero along the real axis i.e. $w = t$ for $t \in \mathbb{R}$ and take,

$$f'(z) = \lim_{t \to 0} \frac{\overline{t}}{t} = \lim_{t \to 0} \frac{t}{t} = 1$$

However, if we send w to zero along the imaginary axis i.e. $w = it$ $t \in \mathbb{R}$ and take,

$$f'(z) = \lim_{t \to 0} \frac{\overline{it}}{t} = \lim_{t \to 0} \frac{-it}{it} = -1$$

Oh no. These do not agree so the limit cannot exist. Therefore $f(z) = \overline{z}$ is not holomorphic anywhere.

Theorem 1.3 (Cauchy)**.** Let $\gamma : [0,1] \to \mathbb{C}$ be a closed curve ($\gamma(0) = \gamma(1)$) in the complex plane and $f : \mathbb{C} \to \mathbb{C}$ be holomorphic everywhere on the region bounded by γ. Then,

$$\oint_\gamma f(z) \, \mathrm{d}z = 0$$

Proof. Look up the proof of Green's theorem and the Cauchy-Riemann equations. It is a good exercise to try and prove Cauchy's theorem from these facts. □

Remark 1.2. The integral,

$$\oint_\gamma f(z) \, \mathrm{d}z$$

can be defined as follows. Parametrize the loop as $\gamma(t)$ for $t \in [0,1]$ and take "by the chain rule",

$$\oint \gamma f(z) \, \mathrm{d}z = \int_0^1 f(\gamma(t)) \gamma'(t) \, \mathrm{d}t$$

this may serve as a definition of the loop integral.

Example 1.4. Let's take $f(z) = z^2$ and consider a loop tracing out a circle of radius r around the origin. Explicitly,

$$\gamma(t) = re^{2\pi it}$$

Then we can compute,

$$\oint_\gamma f(z)\,\mathrm{d}z \;= \int_0^1 (re^{2\pi i})^2 (re^{2\pi it}) \cdot (2\pi i)\,\mathrm{d}t \;= (2\pi i) r^3 \int_0^1 e^{3\cdot(2\pi it)}\,\mathrm{d}t \;= 0$$

Think about why this integral is zero!

2 Meromorphic Functions

Example 2.1. Consider the function $f(z) = \frac{1}{z}$. It is not difficult to show that f is holomorphic everywhere except at $z = 0$ where it blows up. We say f has a pole at $z = 0$. Let's compute the loop integral for the same circular path γ,

$$\oint_\gamma f(z)\,\mathrm{d}z \;= \int_0^1 \frac{1}{re^{2\pi it}}(re^{2\pi it}(2\pi i)\,\mathrm{d}t \;= \int_0^1 (2\pi i)\,\mathrm{d}t \;= 2\pi i$$

Interesting! We might hypothesize that each pole in the interior of γ contributes a factor of $2\pi i$ to the loop integral. Indeed this is true if we include the "residue" at the pole.

Definition: We say a function $f : \mathbb{C} \to \mathbb{C}$ has a *pole* of order n at z_0 if closed to z_0 we can write $f = (z - z_0)^{-n} u(z)$ where $u(z)$ is some nonvanishing holomorphic function near z_0.

Remark 2.1. We say the pole is *simple* if its order is 1. For example,

$$f(z) = \frac{1}{z}$$

has a simple pole at $z = 0$.

Definition: Let f have a simple pole at z_0. Then the residue of f at z_0 is,

$$\mathrm{res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0) f(z)$$

Since, by definition, closed to z_0 we can write,

$$f(z) = (z - z_0)^{-1} u(z)$$

and u is holomorphic (hence continuous in the complex plane) we see that,

$$\mathrm{res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0)(z - z_0) u(z) = \lim_{z \to z_0} u(z) = u(z_0)$$

Definition: We say a function $f : \Omega \to \mathbb{C}$ for $\Omega \subset \mathbb{C}$ is *meromorphic* if there is a set of isolated poles $P \subset \Omega$ such that f is holomorphic on $\Omega \setminus P$ and f has a pole at each point $p \in P$.

Remark 2.2. It is equivalent to say that a meromorphic function f is a ratio of two holomorphic functions g, h i.e.

$$f(z) = \frac{g(z)}{h(z)}$$

Think about how you would prove this?

Theorem 2.2 (Residue)**.** Let $\gamma : [0, 1] \to \mathbb{C}$ be a closed curve bouding a region $D \subset \mathbb{C}$ and let f be meromorphic on D. Then,

$$\oint_\gamma f(z)\,\mathrm{d}z \;= 2\pi i \sum_{p \in D} \mathrm{res}_p(f)$$