Lecture 3

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1 Meromorphic Functions: a reprise

There was some confusion about meromorphic functions. Here I hope to clear up some of the confusion. The definition we have previously was:

Definition: We say a function $f : \Omega \to \mathbb{C}$ for $\Omega \subset \mathbb{C}$ is *meromorphic* if there is a set of isolated poles $P \subset \Omega$ such that f is holomorphic on $\Omega \setminus P$ and f has a pole at each point $p \in P$.

where having a pole means the quite specific statement defined as follows.

Definition: We say a function $f : \mathbb{C} \to \mathbb{C}$ has a *pole* of order n at z_0 if near z_0 we can write $f(z) = (z - z_0)^{-n}\, u(z)$ where $u(z)$ is some nonvanishing holomorphic function.

Definition: We say a function $f : \Omega \to \mathbb{C}$ for $\Omega \subset \mathbb{C}$ is *meromorphic* if there is a set of isolated poles $P \subset \Omega$ such that f is holomorphic on $\Omega \setminus P$ and f has a pole at each point $p \in P$.

We see that a function with a jump discontinuity cannot be meromorphic because, in that case, near the jump, the function would not blow up like $\sim \frac{1}{z^n}$. We can give an alternative definition which may be more intuitive.

Definition: A function f is *meromorphic* if it is a ratio of two holomorphic functions g, h i.e.

$$f(z) = \frac{g(z)}{h(z)}$$

where h is not identically zero i.e. $h \neq 0$ (h is not the constant zero function).

Perhaps a more sophisticated definition of a meromorphic functions involves the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which is the complex plane closed into a sphere with the point at infinity. We will think about this space much more when we talk about projective space. Then we could make the following definition.

Definition: A *meromorphic* function on $\mathbb{C}$ is a holomorphic map $f : \mathbb{C} \to \hat{\mathbb{C}}$.

This corresponds to our previous notion that whenever f is undefined (i.e. is sent to ∞) then f must "blow up" nearby. This corresponds to the fact that for a continuous function $f : \mathbb{C} \to \hat{\mathbb{C}}$ to hit ∞ it must, nearby, pass through arbitrarily large values in $\mathbb{C} \subset \hat{\mathbb{C}}$. Thus we can't have a function which is nice and bounded and suddently jumps up to ∞ since this corresponds to a discontinuity viewed as a map $\mathbb{C} \to \hat{\mathbb{C}}$. Now let's try to prove that these definitions agree.

Theorem 1.1. Let $f : \mathbb{C} \setminus P \to \mathbb{C}$ be a holomorphic function where $P \subset \mathbb{C}$ is an isolated set of points. Then the following are equivalent:

1. near each $p \in P$ we can write $f(z) = (z-p)^{-n}\, u(z)$ where $u(z)$ is holomorphic and nonvanishing near p.

2. $f = \frac{g}{h}$ where $g, h : \mathbb{C} \to \mathbb{C}$ are entire functions and h is nonvanishing outside of P.

3. there exists a holomorphic map $\tilde{f} : \mathbb{C} \to \hat{\mathbb{C}}$ extending f i.e. $\tilde{f}$ restricts to f when $\tilde{f}(z) \neq \infty$,

$$\tilde{f}|_{\mathbb{C} \setminus P} = f \text{ viewed as a map } \mathbb{C} \setminus P \to \mathbb{C} = \hat{\mathbb{C}} \setminus \{\infty\} \subset \hat{\mathbb{C}}$$

Proof. This is actually surprisingly tricky. It relies on the following fact. Let $P \subset \mathbb{C}$ be an isolated set of points and at each point $p \in P$ let n_p be an integer. Then there exists an entire (meaning everywhere holomorphic) function $f : \mathbb{C} \to \mathbb{C}$ which has a zero of at least order n_p at each $p \in P$. Can you see how this shows the equivalence of (1) and (2)? How about the last property? □

2 The Weirestrass ℘ Function

Our plan to find a holomorphic doubly periodic (elliptic) function has been thwarted as soon as it was devised. Since we cannot find interesting holomorphic examples of elliptic functions we now ask for the next best thing. We want to consider elliptic meromorphic functions. That is, meromorphic functions $f : \mathbb{C}/\Lambda \to \mathbb{C}$. It turns out that now were are in luck. Notice the following fact. If $f : \mathbb{C} \to \mathbb{C}$ is any function then,

$$g(z) = \sum_{\omega \in \Lambda} f(z + \omega)$$

is invariant under shifts by any $\omega \in \Lambda$ since this simply reorders the summation. However, in general there is no reason such a function should converge. In fact. using Liouville's theorem, we know that if f is any holomorphic function that g cannot be a convergent holomorphic function. Thus we should try with a simple meromorphic f instead. In order for the sum to converge, we need that f goes to zero sufficiently quickly (faster that $1/z$ because harmonic series do not converge). Therefore, we might try the meromorphic function is z^{-2} and try,

$$f(z) = \sum_{\omega \in \Lambda} \frac{1}{(z + \omega)^2}$$

which is doubly periodic because if $\omega \in \Lambda$ then,

$$f(z + \omega) = \sum_{\omega' \in \Lambda} \frac{1}{(z + \omega + \omega')^2} = \sum_{\omega'' \in \Lambda} \frac{1}{(z + \omega'')^2} = f(z)$$

However, this definition has one major flaw. That sum still does not converge! This is because,

$$f(z) = \sum_{n,m \in \mathbb{Z}} \frac{1}{(z + n\omega_1 + m\omega_2)^2} \sim \sum_{r \in \mathbb{Z}^+} \frac{2\pi r}{(z + r)^2} \sim 2\pi \sum_{r \in \mathbb{Z}^+} \frac{r}{r^2} \to \infty$$

where I have reordered the sum to sum over circles with radius r in the $\mathbb{Z}^2$ plane each circle having approximately $2\pi r$ lattice points. To fix this, we use a clever subtraction to remove the divergent part of the sum and arrive at the following definition by Weierstrass.

Theorem 2.1. Let $\Lambda \subset \mathbb{C}$ be a Lattice. The Weierstrass ℘-function, defined as,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left[\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right]$$

is a well-defined meromorphic function $\wp : \mathbb{C} \to \mathbb{C}$ with double poles on Λ which is doubly periodic with periods ω_1, ω_2 generating Λ i.e. ℘ is elliptic with period lattice Λ.

Proof. We can write,

$$\frac{1}{(z+\omega)^2} = -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z+\omega}\right) = -\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}z}\left[\sum_{n=0}^{\infty}\left(-\frac{z}{\omega}\right)^n\right] = -\frac{1}{\omega}\sum_{n=0}^{\infty}(-1)^n\left[\frac{nz^{n-1}}{\omega^n}\right] = \frac{1}{\omega^2}\sum_{n=1}^{\infty}(-1)^{n+1}n\cdot\left(\frac{z}{\omega}\right)^{n-1}$$
$$= \frac{1}{\omega^2}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{z^n}{\omega^n}\right]$$

and this series is uniformly convergent for $|z| < |\omega|$. Thus,

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\sum_{n=1}^{\infty}(-1)^n(n+1)\frac{z^n}{\omega^n}$$

and therefore,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^\times}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] = \frac{1}{z^2} + \sum_{\omega\in\Lambda^\times}\left[\frac{1}{\omega^2}\sum_{n=1}^{\infty}(-1)^n(n+1)\frac{z^n}{\omega^n}\right]$$
$$= \frac{1}{z^2} + \sum_{n=1}^{\infty}(-1)^n(n+1)\left[\sum_{\omega\in\Lambda^\times}\frac{1}{\omega^{n+2}}\right]z^n$$

converges uniformly when $|z| < |\omega|$ for all $\omega \in \Lambda^\times$ since $n+1 > 2$. If we do not have $|z| < |\omega|$ for all $\omega \in \Lambda^\times$ then we can split up the sum as follows,

$$\wp(z) = \frac{1}{z^2}\sum_{|\omega|\leq|z|}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] + \sum_{|\omega|>|z|}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$

The first sum is finite since there are finitely many lattice points of bounded norm so it can have no convergence issues and the second sum converges by the same argument used above. Furthermore,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\omega\in\Lambda^\times}\left(-\frac{2}{(z+\omega)^3}\right) = -2\sum_{\omega\in\Lambda}\frac{1}{(z+\omega)^3}$$

which is doubly periodic with periods ω_1 and ω_2. Now, since the derivative of $\wp(z+\omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}z}\wp(z+\omega) = \wp'(z+\omega) = \wp'(z)$$

equals the derivative of $\wp$ they can only differ by a constant. In particular,

$$\wp(z+\omega_1) = \wp(z) + c_1$$
$$\wp(z+\omega_2) = \wp(z) + c_2$$

In particular,

$$\wp(\tfrac{1}{2}\omega_1) = \wp(-\tfrac{1}{2}\omega_1) + c_1$$
$$\wp(\tfrac{1}{2}\omega_2) = \wp(-\tfrac{1}{2}\omega_2) + c_2$$

However $\wp(z)$ is an even function so $c_1 = c_2 = 0$. Thus, $\wp(z+\omega_1) = \wp(z)$ and $\wp(z+\omega_2) = \wp(z)$ so $\wp$ is doubly periodic. □

Exercise 1. Show that $\wp$ is doubly periodic directly from manipulating the definition.

2.1 The Defining Differential Equation for $\wp$

We have worked out the expansion for $\wp(z)$ near 0,

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (-1)^n (n+1) \left[\sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{n+2}} \right] z^n$$

Notice that for n odd the sum,

$$\sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{n+2}} = 0$$

because the lattice is symmetric in $\omega \mapsto -\omega$ but this sum is odd under such an operation. Therefore,

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) \left[\sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{2(k+1)}} \right] z^{2k}$$

so if we define the following invariants associated to the lattice Λ,

$$G_k(\Lambda) = \sum_{\omega \in \Lambda^\times} \frac{1}{\omega^{2k}}$$

called the Eisenstein series, then we can write,

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, G_{k+1}(\Lambda)\, z^{2k}$$

Writing out the leading terms explicitly,

$$\wp(z) = \frac{1}{z^2} + 3G_2(\Lambda) z^2 + 5G_3(\Lambda) z^4 + O(z^6)$$

Next, the derivative has the following series expansion,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty} (2k+1)(2k)\, G_{k+1}(\Lambda)\, z^{2k-1}$$

which follows from differentiating the power series expansion of $\wp$ term-by-term. Similarly, the leading terms are,

$$\wp'(z) = -\frac{2}{z^3} + 6G_2(\Lambda) z + 20G_3(\Lambda) z^3 + O(z^5)$$

Thus, compute,

$$\begin{aligned} \wp'(z)^2 &= \left(-\frac{2}{z^3} + 6G_2(\Lambda) z + 20G_3(\Lambda) z^3 + O(z^5) \right)^2 \\ &= \frac{4}{z^6} - 24G_2(\Lambda)\frac{1}{z^2} - 80G_3(\Lambda) + O(z^2) \end{aligned}$$

Similarly, compute,

$$\begin{aligned} \wp(z)^3 &= \left(\frac{1}{z^2} + 3G_2(\Lambda) z^2 + 5G_3(\Lambda) z^4 + O(z^6) \right)^3 \\ &= \frac{1}{z^6} + 9G_2(\Lambda)\frac{1}{z^2} + 15G_3(\Lambda) + O(z^2) \end{aligned}$$

Therefore,

$$\begin{aligned} \wp'(z)^2 - 4\wp(z)^3 &= -24G_2(\Lambda)\frac{1}{z^2} - 36G_2(\Lambda)\frac{1}{z^2} - 80G_3(\Lambda) - 60G_3(\Lambda) + O(z^2) \\ &= -60G_2(\Lambda)\frac{1}{z^2} - 140G_3(\Lambda) + O(z^2) \end{aligned}$$

Which implies that,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda) = O(z^2)$$

Therefore the function,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda)$$

has no poles at zero and thus is holomorphic near 0. Furthermore, it is doubly periodic since it is made up of $\wp$ and $\wp'$ both of which are doubly periodic. Therefore, it is actually holomorphic everywhere by shifting and thus constant by Liouville's theorem. However, it vanishes at $z = 0$ and thus must be the constant zero function. This shows that $\wp$ satisfies the following differential equation,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2(\Lambda)\wp(z) + 140G_3(\Lambda) = 0$$

We can interpret this differential equation as telling us that the image of the elliptic Weierstrass $\wp$-function lies on a special type of object called an elliptic curve.

If we define $g_2 = 15G_2(\Lambda)$ and $g_3 = 35G_3(\Lambda)$ then the Weirestrass $\wp$ function satisfies the differential equation,

$$(\wp'(z)/2)^2 = \wp(z)^3 - g_2\wp(z) - g_3$$

This implies that, the image of the map $\mathbb{C} \to \mathbb{C} \times \mathbb{C}$ defined by $z \mapsto (\wp(z), \wp'(z)/2)$ lies within the surface,

$$E = \{(x, y) \in \mathbb{C} \times \mathbb{C} | y^2 = x^3 - g_2 x - g_3\}$$

This is a special type of algebraic surface (a surface defined by a polynomial equation) called an elliptic curve. However, there is an issue since $\wp(z)$ has a pole at $z = 0$ and thus the map $\mathbb{C} \to \mathbb{C} \times \mathbb{C}$ is not everywhere well-defined. Next time we will "complete" or "compactify" $\mathbb{C} \times \mathbb{C}$ to add "points at infinity" such that this function becomes everywhere defined. In particular, since $\wp$ and $\wp'$ are doubly periodic, this gives a map $\mathbb{C}/\Lambda \to E$.

Exercise 2. Derive the series expansion,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty} (2k+1)(2k)\, G_{k+1}(\Lambda)\, z^{2k-1}$$

directly from the formula,

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^3}$$