A NON-ISOTRIVIAL SMOOTH FANO FIBRATION OVER $\mathbb{P}^1$

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1. Introduction

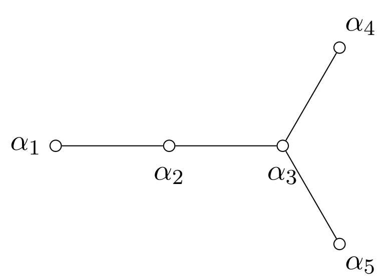
In this note we consider a particularly interesting example of a Fano fibration $f : X \to \mathbb{P}^1$ which is smooth but non isotrivial suggested by Kuznetsov. We will verify that a general pencil of codimension 4 slices of the Spinor 10-fold form produce a family with the following properties:

(a) $f : X \to \mathbb{P}^1$ is smooth and projective
(b) the fibers are Fano 6-folds of Picard rank 1
(c) two general fibers of f are not isomorphic
(d) f is birationally isotrivial, in fact all fibers are rational
(e) the degree of the Chow-Mumford bundle of f is positive.

In the following sections, we will define the Spinor 10-fold and verify the above facts. Then we will show how to compute the Chow-Mumford bundle and consider K-stability of this family (TODO).

2. Spinor 10-fold

Consider the reductive group $G = \mathrm{Spin}(10)$ corresponding to the simply connected form of the root system D_5 whose Dynkin diagram is



Let P_4 be the maximal parabolic corresponding to omitting the simple root α_4 (or we can symmetrically consider P_5) and let $X_{10} = G/P_4$ which is a smooth proper 10-fold. There is a 16-dimensional representation S_{16} called the half-spinor representation of G. Under this representation P_4 maps to the maximal parabolic omitting the first index of GL_{16}. This given an embedding $X_{10} \hookrightarrow \mathbb{P}^{15}$. Here are some facts we need:

Lemma 2.1. [Kuz18, Section 3.1] *Let $\mathcal{O}_{X_{10}}(1)$ denote the very ample line bundle induced by the half-spinor embedding $X_{10} \hookrightarrow \mathbb{P}^{15}$ then:*

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(1) $\mathcal{O}_{X_{10}}(1)$ *generates* $\operatorname{Pic} X_{10}$ *and hence is the line bundle corresponding to the fundamental weight of the omitted index*

$$\mathcal{O}_{X_{10}}(1) = \mathcal{O}_{X_{10}}(\omega_P)$$

(2) $\omega_{X_{10}} \cong \mathcal{O}_{X_{10}}(-8)$

Lemma 2.2. [Kuz18, Corollary 7.9] *For any* $1 \le k \le 5$ *a smooth linear section* $X_{10} \cap V$ *of codimension* k *of* X_{10} *is rational.*

This verifies fact (d).

2.1. **Projective Duality, Defect, and Slicing.** The most important fact making the construction work is the projective duality of $X_{10} \hookrightarrow \mathbb{P}^{15}$. Recall that for a subvariety $Y \subsetneq \mathbb{P}^N$ we define $Y^\vee \hookrightarrow \check{\mathbb{P}}^N$ as the set of hyperplanes H so that H is tangent to Y meaning there is a point $y \in H \cap Y$ where $T_y Y \subset T_y H$. If Y is smooth, this is the set of hypersurfaces so that $H \cap Y$ is singular. Bertini's theorem proves that $Y^\vee \hookrightarrow \check{\mathbb{P}}^N$ is irreducible and has codimension ≥ 1.

Definition 2.3. The *defect* of an embedding $Y \hookrightarrow \check{\mathbb{P}}^N$ is the number

$$\delta = \operatorname{codim}_{\check{\mathbb{P}}^N} Y^\vee - 1$$

Usually we have $\delta = 0$ unless something very special happens with the embedding. Indeed, there is the following classification in low dimensions of embeddings with positive defect:

Theorem 2.4. [LS87] *Let* $\dim X \le 6$ *and* X *admit an embedding in* $\mathbb{P}^N$ *with defect* $\delta > 0$. *Then* X *is one of the following,*

(1) $\mathbb{P}^n$
(2) *a scroll over a curve or surface*
(3) *Plucker embedding* $\operatorname{Gr}(2,5) \hookrightarrow \mathbb{P}^9$
(4) *a smooth hyperplane slice of the above Plucker embedding*

Furthermore, flag varieties have embeddings with positive defect in exactly the following cases:

Theorem 2.5. [KM87] *Let* $X = G/P$ *where* P *is a parabolic subgroup and consider an embedding* $X \hookrightarrow \mathbb{P}^N$ *with defect* δ. *Then* $\delta > 0$ *if and only if* $X \hookrightarrow \mathbb{P}^N$ *is isomorphic to one of the following*

(1) *a linear embedding* $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ *with* $\delta = n$
(2) *the Pücker embedding of the Grassmanian* $\operatorname{Gr}(2, 2m+1) \hookrightarrow \mathbb{P}^{m(2m+1)-1}$ *with* $\delta = 2$
(3) *the Spinor* 10*-fold* $X_{10} \hookrightarrow \mathbb{P}^{15}$ *with* $\delta = 4$
(4) $X_1 \times X_2$ *with the Segre embedding where* X_1 *is one of the above and* $\delta = \delta(X_1) - \dim X_2$.

Lemma 2.6. *Let* $Y \hookrightarrow \mathbb{P}^N$ *be a smooth variety embedded in projective space with defect* δ. *The locus,*

$$S_k := \{V \subset \mathbb{P}^N \mid V \cap Y \text{ is singular}\} \subset \operatorname{Gr}(N-k+1, N+1)$$

of those planes $\mathbb{P}^{N-k} \subset \mathbb{P}^N$ *whose intersection with* Y *is singular, has codimension* $\ge \delta - k + 2$.

Proof. By definition, $y \in Y$ is a singular point of $Y \cap V$ exactly when

(1) $y \in V$
(2) the linear forms $\ell_1, \dots, \ell_k \in H^0(\mathbb{P}^N, \mathcal{O}(1))$ cutting out V are dependent in the cotangent space of Y at y

$$\mathcal{O}(1) \otimes \mathfrak{m}_y/\mathfrak{m}_y^2 \cong \mathfrak{m}_y/\mathfrak{m}_y^2$$

therefore, there exists a nonzero linear form $a_1\ell_1 + \cdots + a_k\ell_k$ vanishing at y whose image in $\mathfrak{m}_y/\mathfrak{m}_y^2$ is zero. This cuts out a hyperplane H such that

(1) $V \subset H$
(2) $H \cap Y$ is singular at y.

Therefore,

$$S_k \subset \{V \subset \mathbb{P}^N \mid V \subset H \text{ for some } H \in S_1\} \subset \mathrm{Gr}(N-k+1, N+1)$$

This larger space is what we call the *saturation* of S_1. It is the image under the first projection of the incidence correspondence

$$\mathcal{X} \subset \mathrm{Gr}(N-k+1, N+1) \times S_1$$

of (V, H) such that $V \subset H$. Now the second projection realizes $\mathcal{X}$ as a grassmannian bundle of $N-k$ planes inside $\mathbb{P}^{N-1}$. Therefore,

$$\dim \mathcal{X} = \dim \mathrm{Gr}(N-k+1, N) + \dim S_1 = (N-k+1)(k-1) + \dim S_1$$

and thus

$$\begin{aligned}
\mathrm{codim}_{\mathrm{Gr}(N-k+1,N+1)} S_k &\geq \mathrm{codim}_{\mathrm{Gr}(N-k+1,N+1)} \pi_2(\mathcal{X}) \\
&\geq \dim \mathrm{Gr}(N-k+1, N+1) - \dim \mathcal{X} \\
&= (N-k+1)k - (N-k+1)(k-1) - \dim S_1 \\
&= (N-k+1) - \dim S_1 = \delta - k + 2
\end{aligned}$$

because by definition $\delta := N - \dim S_1 - 1$. □

Corollary 2.7. *Let $Y \hookrightarrow \mathbb{P}^N$ be a smooth variety embedded in projective space with defect δ. A generic PGL_{N+1} translate of any curve $C \hookrightarrow \mathrm{Gr}(N-k+1, N+1)$ parametrizing codimension k linear spaces with $1 \leq k \leq \delta$ produces a smooth family of slices*

$$\mathcal{Y} \to C$$

where $\mathcal{Y}_t := Y \cap V_t$ where for $t \in C$ then V_t is the linear space corresponding to the point t on the translate of C in $\mathrm{Gr}(N-k+1, N+1)$.

Proof. Since $\mathrm{codim}\, S_k, \mathrm{Gr}(N-k+1, N+1) \geq \delta - k + 2$ if $k \leq \delta$ then S_k has only excess intersection with any curve C. By Kleiman's Bertini, since the Grasmannian is a PGL_{N+1}-homogeneous space, the generic translate of C does not intersect S_k. Hence every parametrized slice is smooth. In particular, the incidence correspondence $\mathcal{Y} \to C$ is a smooth morphism. □

This proves (a) taking the family of slices over a generic rational curve in $\mathrm{Gr}(11, 16)$ slicing X_{10}.

2.2. **Linear Slices of the Spinor 10-fold.** Consider taking r hyperplane slices of X_{10} to get a variety Y. Note that Y is Fano for $k < 8$ since $\omega_{X_{10}} = \mathcal{O}_{X_{10}}(-8)$ and $\operatorname{Pic} Y = \mathbb{Z}\langle \mathcal{O}_Y(1)\rangle$ by the Grothendieck-Lefschetz theorem. This verifies (b). The normal bundle sequence gives an exact sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{O}_Y(1)^r \to 0$$

Furthermore, the composition

$$H^0(X, \mathcal{O}_X(1))^r \to H^0(Y, \mathcal{O}_Y(1))^r \to H^1(Y, \mathcal{T}_Y)$$

of the restriction with the connecting map from the normal bundle sequence gives the Kodaira-Spencer map κ for the family of slices over the Grassmannian. Furthermore, to compute the restrictions of bundles to Y we use the Kozul resolution

$$0 \to \mathcal{O}_X(-r) \to \mathcal{O}_X(-(r-1))^r \to \cdots \to \mathcal{O}_X(-1)^r \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

Now we need the following facts which are proved in the next section:

Lemma 2.8. *As a G-module there is a canonical isomorphism in the derived category*

$$R\Gamma(X, \mathcal{O}_X(k)) = \begin{cases} V_{\omega_4}[0] & k = 1 \\ \mathbb{C}[0] & k = 0 \\ 0 & 0 > k > -8 \\ \mathbb{C}[-10] & k = -8 \\ V_{\omega_5}[-10] & k = -9 \end{cases}$$

where V_λ is the irreducible representation of G with highest weight λ and ω_i is the fundamental weight corresponding to the simple root α_i.

Remark 2.9. $V_{\omega_4} = V_{\omega_5}^\vee$ are 16-dimensional.

Corollary 2.10. *For $r \le 8$ the canonical map*

$$V_{\omega_4}[0] = R\Gamma(X, \mathcal{O}_X(1)) \to R\Gamma(Y, \mathcal{O}_Y(1))$$

is surjective on cohomology and gives

$$R\Gamma(Y, \mathcal{O}_Y(1)) = (V_{\omega_4}/\mathbb{C}^r)[0]$$

Proof. Indeed, $\mathcal{O}_Y(1)$ is computed by the complex

$$[\mathcal{O}_X(-r) \to \mathcal{O}_X(-(r-1))^r \to \cdots \to \mathcal{O}_X(-1)^r \to \mathcal{O}_X] \otimes \mathcal{O}_X(1)$$

supported in degrees $[-r, 0]$. None of these terms have any higher cohomology so there is an exact sequence,

$$0 \to H^0(X, \mathcal{O}_X^r) \to H^0(X, \mathcal{O}_X(1)) \to H^0(Y, \mathcal{O}_Y(1)) \to 0$$

□

To go further we need to understand the cohomology of the twisted tangent bundle pf X_{10} whose proof we defer to the next section.

Lemma 2.11. *There are canonical isomorphims as G-modules*

$$R\Gamma(X, \mathcal{T}_X(k)) = \begin{cases} V_{\omega_2}[0] & k = 0 \\ 0 & 0 > k > -8 \\ \mathbb{C}[-9] & k = 8 \\ 0 & k = 9 \\ V_{\omega_3}[-10] & k = 10 \end{cases}$$

Remark 2.12. V_{ω_2} is the adjoint representation.

Corollary 2.13. *The canonical map*

$$R\Gamma(X, \mathcal{T}_X) \to R\Gamma(Y, \mathcal{T}_X|_Y)$$

is an isomorphism if $r < 8$.

Proof. Indeed, $\mathcal{T}_X|_Y$ is computed by the complex

$$[\mathcal{O}_X(-r) \to \mathcal{O}_X(-(r-1))^r \to \cdots \to \mathcal{O}_X(-1)^r \to \mathcal{O}_X] \otimes \mathcal{T}_X$$

supported in degrees $[-r, 0]$. None of these terms have any higher cohomology and the terms of negative degree have no cohomology at all. This proves the claim. □

Corollary 2.14. *If $r < 8$ then there is a diagram*

$$\begin{array}{ccccccccc} & & & & H^0(X, \mathcal{T}_X) & & H^0(X, \mathcal{O}_X(1))^r & & \\ & & & & \| & & \downarrow\!\!\!\downarrow & \searrow^{\kappa} & \\ 0 & \longrightarrow & H^0(Y, \mathcal{T}_Y) & \longrightarrow & H^0(Y, \mathcal{T}_X|_Y) & \longrightarrow & H^0(Y, \mathcal{O}_Y(1))^r & \longrightarrow & H^1(Y, \mathcal{T}_Y) \to 0 \end{array}$$

therefore

$$h^0(Y, \mathcal{T}_Y) \geq r(\dim V_{\omega_4} - r) - \dim V_{\omega_2} = r(16 - r) - \binom{10}{2} = -r^2 + 16r - 45$$

which is nonzero for $r \geq 4$. The bound gives ≥ 3 for $r = 4$.

Hence for $r \geq 4$ the Kodaira-Spencer map κ for the family of linear slices over the Grassmannian is nonzero. This verifies fact (c) after restricting to a generic $\mathbb{P}^1 \hookrightarrow \mathrm{Gr}$.

3. Borel-Weil-Bott

3.1. **Review of Root Systems Theory of Reductive Groups.** Let G be a reductive group. Fix inclusions $T \subset B \subset P \subset G$ where T is a maximal torus, B a borel subgroup, and P a parabolic. The main lattices of import live in either the character lattice

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$$

and the cocharacter lattice

$$X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$$

For a representation $\rho : G \to \mathrm{GL}(V)$, we say the *weights* of V are the nonzero elements $\alpha \in X^*(T)$ such that

$$V_\alpha = \{v \in V \mid \forall t \in T : \rho(t) \cdot v = \alpha(t) \cdot v\}$$

For the adjoint representation, the weights have a special name: roots $\Phi \subset X^*(T)$. We make the convention of using the "lower Borel" (so that dominant weights correspond to ample line bundles) so that we call the weights of the adjoint torus action on the Lie algebra $\mathfrak{b} \subset \mathfrak{g}$ of the Borel form a subset $\Phi^- \subset \Phi$ called the *negative roots*. The complement $\Phi^+ \subset \Phi$ are the *positive roots*. It turns out that $\Phi^- = -\Phi^+$.

Definition 3.1. For real weights $\mu, \lambda \in X^*(T)_{\mathbb{R}}$ we say that μ is *higher* than λ if $\mu - \lambda$ is a convex (positive real) combination of elements of Φ^+.

Inside Φ^+ there is a distinguished subset $\Delta \subset \Phi^+$ of *simple* roots which are the $\alpha \in \Phi^+$ that cannot be expressed as a sum of positive roots.

Lemma 3.2. *The simple roots satisfy*

(1) $\#\Delta = \operatorname{rank} X^*(T)$
(2) *every $\alpha \in \Phi$ can be written as a unique integer linear combination of elements of Δ*
(3) *whenever $\alpha \in \Phi$ is written as a linear combination of elements of Δ the coefficients are all positive (in which case $\alpha \in \Phi^+$) or all negative (in which case $\alpha \in \Phi^-$) giving alterative descriptions of the positive and negative roots.*

The Weyl group, $W = N_G(T)/T$ is a finite reflection group acting on $X^*(T)$ and preserving the sets of positive and negative roots. For a root $\alpha : T \to \mathbb{G}_m$ there is a canonically associated *coroot* $\alpha^\vee : \mathbb{G}_m \to T$ defined as the unique cocharacter satisfying

(1) $\langle \alpha, \alpha^\vee \rangle = 2$
(2) $\alpha^\vee : \mathbb{G}_m \to T$ factors through $T \cap D(Z_G((\ker \alpha)^0_{\text{red}}))$ where $D(-)$ is the derived subgroup.

This gives a set $\Phi^\vee \subset X_*(T)$ of *coroots* and likewise a set of positive and negative coroots.

We choose a basis $\{\omega_\alpha\}_{\alpha \in \Delta}$ of $X^*(T)_{\mathbb{R}}$ consisting of *fundamental weights* which are defined so that

$$\left\langle \omega_\alpha, \beta^\vee \right\rangle = \delta_{\alpha\beta} \quad \alpha, \beta \in \Delta$$

Definition 3.3. A weight $\mu \in X^*(T)_{\mathbb{R}}$ is *dominant* if $\langle \mu, \alpha^\vee \rangle \geq 0$ for every positive (equivalently every simple) root $\alpha \in \Phi^+$. Equivalently, μ is a convex real combination of fundamental weights.

The Weyl group is generated by reflections

$$r_\alpha(\beta) := \beta - \left\langle \beta, \alpha^\vee \right\rangle \alpha$$

In fact, it is generated by the *simple* reflections, those r_α for $\alpha \in \Delta$. For an element $w \in W$, the length $\ell(w)$ of w is the minimal number of simple reflections needed to express w. The Weyl group acts simply transitively on the Weyl chambers – the connected components of

$$X^*(T)_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Phi} \{\lambda \in X^*(T)_{\mathbb{R}} \mid \left\langle \mu, \alpha^\vee \right\rangle = 0\}.$$

The fundamental chamber consist of weights μ such that $\langle \mu, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$ which is the set of dominant weights. For any weight μ there is always $w \in W$ such that $w \cdot \mu$ is dominant. The dominant weight $[\lambda] := w \cdot \lambda$ is unique (although $w \in W$ is not). We let the index $\operatorname{ind}(\lambda)$ of λ be the minimal $\ell(w)$ over $w \in W$ realizing $w \cdot \lambda = [\lambda]$. A weight on the boundary of a Weyl chamber is called *singular*. Explicitly, μ is singular if $\langle \mu, \alpha^\vee \rangle = 0$ for some $\alpha \in \Phi$. We also define

$$\rho := \sum_{\alpha \in \Delta} \omega_\alpha = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

Notice that $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta$ so ρ is dominant and nonsingular. Furthermore if μ is any dominant weight then $\mu + \rho$ is dominant and nonsingular.

The weights of the adjoint representation on the Lie algebra $\mathfrak{p} \subset \mathfrak{g}$ of the parabolic $P \subset G$ are $\Phi^- \cup \Phi_P^+$ so that

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi_P^+} \mathfrak{g}_\alpha$$

where Φ_P^+ is the set of positive roots which are weights for this action. This is a closed set of positive roots. We set $\Phi_P = \Phi_P^+ \cup \Phi_P^-$ where $\Phi_P^- = -\Phi_P^+$.

3.2. **Cohomology Computations.** To prove the cohomology lemmas, we need to apply Borel-Weil-Bott. Here we actually need a more general version due to Bott that computes the cohomology of all homogeneous vector bundles.

Theorem 3.4 (Bott)**.** *Let G be a reductive group and $P \subset G$ a parabolic. Chose a Borel subgroup $B \subset P$. Let V be an irreducible representation of P with highest weight $\lambda \in \mathfrak{g}^*$ and $E = G \times^P V$ the corresponding homogeneous vector bundle. Then there is a canonical isomorphism*

$$H^q(X, E) \cong H^q(G/B, \mathcal{O}_{G/B}(\lambda))$$

which is computed as follows:

1. *if $\lambda + \rho$ is singular then $H^q(X, E) = 0$ for all q*
2. *otherwise $H^q(X, E) = 0$ for all $q \neq \ell := \mathrm{ind}(\lambda + \rho)$ and $H^\ell(X, E) = V_{\lambda'}$ is the irreducible G-module of highest weight $\lambda' := [\lambda + \rho] - \rho$.*

See [Bot57] and for this formulation [AF10, Theorem 2.4.6].

Lemma 3.5. *The tangent bundle of $X = G/P$ is the homogenous vector bundle associated to the P-representation $\mathfrak{g}/\mathfrak{p}$. Hence, its weights are $\Phi_X^+ := \Phi^+ \setminus \Phi_P^+$.*

Recall that if V is an irreducible P-representation and $U \subset P$ is the unipotent radical then $V^U \subset V$ is a P-subrepresentation because U is normal. However, V^U is nonempty by the Lie–Kolchin theorem. By irreducibility, $V^U = V$ so V factors though a representation of the Levi factor $L := P/U$. Because L is reductive, the representation V is determined by its highest weight in $X^*(T)$. If V is not irreducible, what makes life tricky is that L is not semisimple, it has an abelian factor. Therefore, we cannot just look the set of highest weights of V to determine its irreducible Jordan-Hölder factors. If P is a maximal parabolic then it corresponds to a unique omitted simple root $\{\alpha_P\} = \Delta \setminus \Delta_P$ where $\Delta_P = \Phi_P \cap \Delta$. The abelian part $Z(L) \cap T = \bigcap_{\alpha \in \Phi_P} \ker \alpha$ is spanned by the fundamental coweight $\omega_P^\vee$ dual to α_P. Note also that a character $\lambda \in X^*(T)$ extends to P iff it extends to L iff it kills $D(L) \cap T$ and this is generated by $\alpha^\vee$ for $\alpha \in \Phi_P$. Therefore, $X^*(P) = \bigcap_{\alpha \in \Phi_P} \ker \langle -, \alpha^\vee \rangle$ which is spanned by the fundamental weights $\{\omega_\alpha\}_{\alpha \in \Delta \setminus \Delta_P}$ in particular by ω_P if P is maximal. These characters $\mathbb{Z}\omega_P$ correspond to homogeneous line bundles on $X = G/P$.

Let V be a P-representation, to compute its Jordan-Hölder factors we partition the weights by *grade* – the coefficient of α_P when expressed in the basis of simple roots. The decomposition relies on nice properties for the weight spaces of irreps for semisimple Lie groups arising from what I call the "uninterrupted string property" for $\mathfrak{sl}_2$-reps. Precisely, if V is an irrep for $\mathfrak{sl}_2$ its weight spaces V_n are indexed by an integer $n \in \mathbb{Z}$, the degree of the corresponding character $T \to \mathbb{G}_m$, and they are symmetric about the origin and consist of sums of contiguous blocks. In particular, if V_a is nontrivial then V_k is nontrivial for all $-a \le k \le a$. Hence an *uninterrupted string*. This is consequential for an irrep V of any semisimple group G because any coroot $\alpha^\vee$ factors through the maximal torus of a copy of SL_2 and the intersection pairing $\langle -, \alpha^\vee \rangle$ measures the induced weight of V viewed as an SL_2-representation. In particular, if $V_{\mu+k\alpha}$ is nontrivial then so are $V_{\mu+k'\alpha}$ for $-k \le k' \le k$ by the uninterrupted string property.

Lemma 3.6. *Let P be a maximal parabolic and $L = P/U$ the Levi. The Jordan-Hölder factors of V are exactly the L-representations of highest weight λ where λ varies over the weights of V that are highest among all weights of the same grade.*

Proof. Among weights of the same grade, there is a unique irreducible representation for each highest weight. Indeed, these are the weights so that $\omega_P^\vee \mathbb{Z}$ acts by a fixed character so it is equivalent to a representation of $L/D(L)$. Since $L/D(L)$ is semisimple its representations split into irreps determined by their highest weight. □

Often, there will be a unique highest weight of each fixed grade in which case the grade filtration on V is a Jordan-Hölder filtration. Indeed, this occurs for $V = \mathfrak{g}/\mathfrak{p}$. Since $\mathfrak{g}$ is semisimple, the root spaces of the adjoint representation, and hence of V, are all 1-dimensional so there is a most a single irrep of $L/D(L)$ at each grade. Indeed, if we have two irreps in the decomposition then two of their weight spaces must collide to give a weight space of higher multiplicity by the "uninterrupted string property" for $\mathfrak{sl}_2$-reps.

Lemma 3.7. *For $G = \mathrm{Spin}(10)$ and $P = P_4$ the P-module $\mathfrak{g}/\mathfrak{p}$ has only grade 1 and is irreducible of highest weight ω_2.*

Proof. This calculation is done in Magma for the root system D_5 by listing the positive roots in $\Phi^+ \setminus \Phi_P^+$ and stratifying them according to the coefficient of α_P based on simple roots. We find that only the coefficient 1 appears and these weights have a unique highest element. There are 9 roots in Φ_P^+ and those have grade 0 by definition. The remaining 11 roots in $\Phi^+ \setminus \Phi_P^+$ all have grade 1. □

The proofs of the cohomology lemmas is now immediate from Borel-Weil-Bott. The sheaf $\mathcal{O}_X(k)$ corresponds to the P-irrep of weight $k\omega_4$. The sheaf $\mathcal{T}_X(k)$ corresponds to the P-irrep of weight $\omega_2 + k\omega_4$. Magma handles computing the dominant representatives under the Weyl action. These scripts are available upon request.

4. Computing the Pushforward of the Relative Anti-Canonical

We are now going to consider the universal family of slices of X_{10}. From now on, unless otherwise specified we write $X = X_{10}$. Denote by

$$\mathcal{X} \subset X \times \mathrm{Gr}(12, 16)$$

the incidence correspondence of points on X that lie in a specified $\mathbb{P}^{11} \subset \mathbb{P}^{16}$. This corresponds to the case $r = 4$ above. Inside $X \times \mathrm{Gr}(12, 16)$ this is cut out by a section of the rank 4 bundle

$$\mathcal{E} = \mathcal{O}_X(1) \boxtimes \mathcal{Q}_4$$

where $\mathcal{Q}_4$ is the universal quotient bundle of rank 4 on the Grassmanian. We are interested in computing

$$R\pi_{2*}\omega^{-m}_{\mathcal{X}/\mathrm{Gr}}$$

Notice that

$$\omega_{\mathcal{X}/\mathrm{Gr}} = (\pi_1^*\omega_X)|_{\mathcal{X}} \otimes \det \mathcal{N}_{\mathcal{X}}$$

where $\mathcal{N}_{\mathcal{X}} = \mathcal{E}$. Therefore

$$\det \mathcal{N}_{\mathcal{X}} = \mathcal{O}_X(4) \boxtimes \det \mathcal{Q}_4$$
$$\omega_{\mathcal{X}/\mathrm{Gr}} = (\mathcal{O}_X(-4) \boxtimes \det \mathcal{Q}_4)|_{\mathcal{X}}$$

Now to compute the pushforward, we need to use the Kozul resolution of $\mathcal{O}_{\mathcal{X}}$

$$0 \to \mathcal{O}_X(-4) \boxtimes \det \mathcal{Q}_4^\vee \to \mathcal{O}_X(-3) \boxtimes \wedge^3\mathcal{Q}_4^\vee \to \cdots \to \mathcal{O}_X(-1) \boxtimes \mathcal{Q}_4^\vee \to \mathcal{O}_{X\times\mathrm{Gr}} \to \mathcal{O}_{\mathcal{X}} \to 0$$

Recall that for $k > -8$ there is no higher cohomology of $\mathcal{O}_X(k)$. Therefore, we get an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(4m-4)) \otimes \wedge^4\mathcal{Q}_4^\vee \to \cdots \to H^0(\mathcal{O}_X(4m-1)) \otimes \mathcal{Q}_4^\vee \to H^0(\mathcal{O}_X(4m)) \otimes \mathcal{O}_{\mathrm{Gr}} \to \mathcal{G}_m \to 0$$

with $\pi_{2*}\omega^{-m}_{\mathcal{X}/\mathrm{Gr}} = \mathcal{G}_m \otimes (\det \mathcal{Q}_4)^{-m}$ and there is no higher cohomology in $\mathbb{R}\pi_{2*}\omega^{-m}_{\mathcal{X}/\mathrm{Gr}}$.

4.1. **Chow-Mumford Degree.** Now we choose a generic pencil of 11-plane sections (codimension 4 linear slices) of X. This corresponds to a generic line $\mathbb{P}^1 \hookrightarrow \mathrm{Gr}(12, 16)$ with respect to the Plucker embedding. Under the map, the universal quotient bundle pulls back as

$$\mathcal{Q}_4 \mapsto \mathcal{O}^3_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

Therefore, pulling back the sequences computing $\mathbb{R}\pi_{2*}\omega^{-m}_{\mathcal{X}/\mathrm{Gr}}$ we find that

$$\operatorname{rank} \mathcal{G}_m = h^0(\mathcal{O}_X(4m)) - 4h^0(\mathcal{O}_X(4m-1)) + 6h^0(\mathcal{O}_X(4m-2)) - 4h^0(\mathcal{O}_X(4m-3)) + h^0(\mathcal{O}_X(4m-4))$$

and

$$\deg \mathcal{G}_m = h^0(\mathcal{O}_X(4m-1)) - 3h^0(\mathcal{O}_X(4m-2)) + 3h^0(\mathcal{O}_X(4m-3)) - h^0(\mathcal{O}_X(4m-4))$$

Hence we need to know the Hilbert polynomial of X

$$p(k) := \chi(X, \mathcal{O}_X(k))$$

To compute this, we can use Borel-Weil Bott and interpolation. Since p has degree 10, we need 11 points to determine it. The lemma gives exactly 11 points and proves that

$$p(k) = \frac{1}{10!}\left[12k^3 + 144k^2 + 12 + 720\right] k(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)$$

plugging in gives

$$\operatorname{rank} \mathcal{G}_m = \frac{1024m^6}{15} + \frac{1024m^5}{5} + 256m^4 + \frac{512m^3}{3} + \frac{956m^2}{15} + \frac{188m}{15} + 1$$

$$\deg \mathcal{G}_m = \frac{4096m^7}{105} + \frac{512m^6}{5} + \frac{1664m^5}{15} + 64m^4 + \frac{104m^3}{5} + \frac{18m^2}{5} + \frac{9m}{35}$$

$$
\begin{aligned}
\deg \pi_{2*}\omega_{\mathcal{X}/\mathrm{Gr}}^{-m} &= \deg \mathcal{G}_m - m \cdot \operatorname{rank} \mathcal{G}_m \\
&= -\frac{2m}{105}\left(39 + 469m + 2254m^2 + 5600m^3 + 7616m^4 + 5376m^5 + 1536m^6\right)
\end{aligned}
$$

and likewise

$$
\begin{aligned}
\deg \pi_{2*}\omega_{\mathcal{X}}{}^{-m} = \deg\left(\pi_{2*}\omega_{\mathcal{X}/\mathrm{Gr}}^{-m} \otimes \omega_{\mathbb{P}^1}^{-m}\right) &= \deg \mathcal{G}_m + m \cdot \operatorname{rank} \mathcal{G}_m \\
&= \frac{2m}{105}\left(66 + 847m + 4438m^2 + 12320m^3 + 19264m^4 + 16128m^5 + 5632m^6\right)
\end{aligned}
$$

The Chow-Mumford bundle controls the growth rate of $\det \pi_{2*}\omega_{\mathcal{X}/\mathrm{Gr}}^{-m}$ in the sense that

$$\lambda_{CM} = \mathcal{M}_7^{-1}$$

where the $\mathcal{M}_i$ are coefficients in the Mumford-Knudsen expansion of $\det \pi_{2*}\omega_{\mathcal{X}/\mathrm{Gr}}^{-m}$ (reviewed in section 5). These are computed by finite differencing this sequence of line bundles. Hence the top coefficient is given by 7! times the leading coefficient in degree:

$$\deg \lambda_{CM} = 7! \cdot \frac{2}{105} \cdot 5632 = 540672$$

If the family is K-stable and non-isotrivial then an important theorem in K-stability says that $\det \lambda_{CM} > 0$. Given that this holds in our example, it is natural to ask:

Question 4.1. Are the linear slices $Y = X_{10} \cap V$ which are smooth Fano varieties K-stable?

5. Review of the Chow-Mumford Bundle

Theorem 5.1. [KM76, Theorem 4] *Let $f : X \to S$ be a flat projective morphism of relative dimension r with a relatively very ample line bundle $\mathcal{L}$. Then there are unique line bundles $\mathcal{M}_i \in \operatorname{Pic} S$ such that*

$$\det Rf_*\mathcal{L}^{\otimes m} = \bigotimes_{k=0}^{r+1} \mathcal{M}_k^{\otimes\binom{m}{k}}$$

called the Mumford-Knudsen expansion.

Question: does the theorem require very ample?

Remark 5.2. These $\mathcal{M}_i$ are the line bundle analogs of the binomial coefficients of the Hilbert polynomial as a numerical polynomial. Note that

$$\operatorname{ch}(Rf_*\mathcal{L}^{\otimes m})_1 = c_1(\det Rf_*\mathcal{L}^{\otimes m}) = \sum_{i=0}^{r+1} \binom{m}{k} \cdot c_1(\mathcal{M}_k)$$

which by Grothendieck-Riemann-Roch equals the degree 1 part in $\mathrm{CH}^\bullet(S)$ of

$$f_*(e^{mc_1(\mathcal{L})}\mathrm{Td}_{X/S})$$

This is a numerical polynomial valued in $\mathrm{CH}^1(S)_{\mathbb{Q}}$ and the $c_1(\mathcal{M}_k)$ are related to its coefficients by expanding the binomial terms as polynomials in m. As a polynomial this has denominators, however the Mumford-Knudsen expansion shows there is a "numerical polynomial" valued in $\operatorname{Pic} S$ integrally (without tensoring in $\mathbb{Q}$) which agrees with a polynomial after tensoring with $\mathbb{Q}$. For example, rationally,

$$c_1(\mathcal{M}_{r+1}) = f_*c_1(\mathcal{L})^{r+1}$$

Question: does this equality hold integrally in the Chow ring?

Definition 5.3. A *numerical polynomial* of degree n valued in a ring R is a function $p : \mathbb{Z} \to R$ of the form

$$p(m) = \sum_{k=0}^{n} \binom{m}{k} a_k$$

for some coefficients $a_k \in R$.

Remark 5.4. Numerical polynomials are an abelian group in the obvious way. Furthermore,

$$\binom{m}{k}\binom{m}{k'} = \sum_{j=0}^{\min(k,k')} \binom{k+k'-j}{j, k-j, k'-j} \binom{m}{k+k'-j}$$

where the first coefficient is a multinomial coefficient. Since the multinomial coefficient is an integer depending only on k, k' and j this shows that numerical polynomials form a ring.

Recall that numerical polynomials of the form

$$p(n) = \sum_{k=0}^{d} \binom{n}{k} a_k$$

satisfy the important property that their finite differencing behaves formally like differentiation for Taylor polynomials, it just shifts the coefficients. Indeed,

$$p(n+1) - p(n) = \sum_{k=0}^{d} \binom{n}{k} a_{k+1}$$

because of the fundamental recursion

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Therefore, the function p determines the coefficients $a_k \in R$ integrally while p is only a polynomial valued in $R_{\mathbb{Q}}$ and this polynomial may forget torsion in R.

This discussion gives a differencing formula for the $\mathcal{M}_k$ in the Mumford-Knudsen decomposition.

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