

# Artin's Axioms and Formal Deformation Theory

Stanford-Berkeley Number Theory Learning Seminar

November 16, 2022

## 1 Definitions

**Definition 1.0.1.** An *algebraic space* is a functor  $X : (\mathbf{Sch}_S)^{\text{op}}_{\text{fppf}} \rightarrow \text{Set}$  such that,

- (a)  $F$  is a sheaf in the fppf topology
- (b) the diagonal  $\Delta_{X/S} : X \rightarrow X \times_S X$  is representable by schemes
- (c) there is a scheme  $U$  and an étale surjection  $U \twoheadrightarrow X$ .

**Definition 1.0.2.** An *algebraic stack* is a category fibered in groupoids  $\mathcal{X} \rightarrow (\mathbf{Sch}_S)^{\text{op}}_{\text{fppf}}$  such that,

- (a)  $\mathcal{X}$  is a stack in the fppf topology
- (b)  $\Delta_{\mathcal{X}/S} : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$  is representable by algebraic spaces
- (c) there is an algebraic space  $U$  and an étale surjection  $U \twoheadrightarrow \mathcal{X}$ .

*Remark.* The map  $U \rightarrow \mathcal{X}$  is only necessarily representable by algebraic spaces so to express the property of being an étale surjection consider any map from a scheme  $T \rightarrow \mathcal{X}$  and an étale cover from a scheme  $V \rightarrow U \times_{\mathcal{X}} T$  in the diagram,

$$\begin{array}{ccccc} & & \text{ét surj} & & \\ & \swarrow & & \searrow & \\ V & \longrightarrow & U \times_{\mathcal{X}} T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} & & \end{array}$$

This property is independent of the choice of étale cover  $V \rightarrow U \times_{\mathcal{X}} T$  by étale descent for étale surjective morphisms.

*Remark.* Why do we only require that  $\mathcal{X}$  be smooth locally an algebraic space and its diagonal be representable by only algebraic spaces? The diagonal is closely related to the automorphism groups of objects  $\mathcal{X}$  parametrizes. When  $\pi : X \rightarrow S$  a proper finitely presented map of schemes,  $\text{Hilb}_{X/S}$  is representable by an algebraic space but not generally by a scheme unless  $\pi$  is projective. This shows that  $\text{Isom}_S(X, Y)$  between two proper  $S$ -schemes is usually only representable by an algebraic space. Therefore, we want to allow for  $\Delta$  to be representable by algebraic spaces not just schemes to capture moduli of proper non-projective objects.

**Definition 1.0.3.** Consider  $f : \mathcal{X} \rightarrow \mathcal{Z}$  and  $g : \mathcal{Y} \rightarrow \mathcal{Z}$  morphisms of categories fibered in groupoids. Then the 2-fiber product  $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$  is defined as the category fibered in groupoids,

- (a) objects are  $(x, y, \gamma)$  with  $p(x) = p(y)$  and  $\gamma : f(x) \rightarrow g(y)$  a morphism over  $\text{id}$
- (b) morphisms are  $\varphi : (x, y, \gamma) \rightarrow (x', y', \gamma')$  are given by pairs  $(\varphi_x : x \rightarrow x', \varphi_y : y \rightarrow y')$  such that the diagram,

$$\begin{array}{ccc} f(x) & \xrightarrow{\gamma} & g(x) \\ \downarrow \varphi_x & & \downarrow \varphi_y \\ f(x') & \xrightarrow{\gamma'} & g(x') \end{array}$$

commutes.

**Proposition 1.0.4.** For any objects  $x, y \in \mathcal{X}(U)$ . There is a 2-fiber product diagram,

$$\begin{array}{ccc} \text{Isom}(x, y) & \longrightarrow & U \\ \downarrow & & \downarrow y \\ U & \xrightarrow{x} & \mathcal{X} \end{array}$$

**Definition 1.0.5.** The *inertia stack* of  $\mathcal{X}$  is the category fibered in groupoids  $\mathcal{I}_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ .

**Proposition 1.0.6.** For any  $x \in \mathcal{X}(U)$  there is a 2-fiber product diagram,

$$\begin{array}{ccc} \text{Isom}(x, x) & \longrightarrow & \mathcal{I}_{\mathcal{X}} \\ \downarrow & \lrcorner & \downarrow \\ U & \xrightarrow{x} & \mathcal{X} \end{array}$$

## 2 Presentations

**Proposition 2.0.1.** Let  $X$  be an algebraic space over  $S$  and  $f : U \rightarrow X$  an étale surjection from a scheme  $U$ . Set  $R = U \times_X U$  in the pullback diagram,

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

then we have,

- (a)  $j : R \rightarrow U \times_S U$  is a monomorphism and  $R(T) \subset U(T) \times U(T)$  is an equivalence relation for all  $T \rightarrow S$
- (b) the projections  $s, t : R \rightarrow U$  are étale
- (c) the diagram,

$$R \xrightarrow[s]{t} U \longrightarrow X$$

is a coequalizer in  $\text{Sh}((\mathbf{Sch}_S)_{\text{fppf}})$ .

*Proof.* The first two are immediate. The last holds in any category of sheaves given that  $U \rightarrow X$  is surjective.  $\square$

**Definition 2.0.2.** Let  $(U, R, s, t, c)$  be a groupoid in algebraic spaces over  $S$ . The quotient stack,

$$p : [U/R] \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$$

is the stackification of the category fibered in groupoids,

$$(T \rightarrow S) \mapsto (U(T), R(T), s, t, c)$$

**Proposition 2.0.3 (04T5).** Given an algebraic stack  $\mathcal{X}$  there is a smooth morphism  $U \rightarrow \mathcal{X}$  from a scheme. We recover the groupoid presentation by taking the 2-fiber product,

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

and  $R$  is an algebraic space because we assumed that  $\Delta_{\mathcal{X}}$  is representable by algebraic spaces. Then there is a natural equivalence  $[U/R] \xrightarrow{\sim} \mathcal{X}$ .

### 3 Infinitesimal Deformation Theory

*Remark.* First we recall how to apply infinitesimal deformation theory in the relative setting. In the basic case, we want to probe properties of a morphisms of schemes  $f : X \rightarrow S$  near a finite type point  $x : \text{Spec}(k) \rightarrow S$ . There is some affine open  $\text{Spec}(\Lambda) \subset X$  containing  $x$ . Then we need to consider Artinian local rings  $A$  and diagrams,

$$\begin{array}{ccccc} & & & \nearrow X & \\ & & & \downarrow f & \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(A) & \xrightarrow{\quad} & \text{Spec}(\Lambda) \xrightarrow{\quad} S \end{array}$$

and consider the set of dashed arrows. This means our base category should be the category of local Artinian  $\Lambda$ -algebras with residue field  $k$ .

**Definition 3.0.1.** Let  $\Lambda$  be a noetherian ring and  $\Lambda \rightarrow k$  a finite ring map with  $k$  a field. Let  $\mathcal{C}_{\Lambda}$  be the category of,

- (a)  $(A, \varphi)$  where  $A$  is an Artinian local  $\Lambda$ -algebra and  $\varphi : A/\mathfrak{m}_A \rightarrow k$  a  $\Lambda$ -algebra isomorphism
- (b) morphisms  $f : (B, \psi) \rightarrow (A, \varphi)$  are local  $\Lambda$ -algebra maps such that  $\varphi \circ (f \bmod \mathfrak{m}_A) = \psi$

*Remark.* As in the absolute case (which corresponds to  $\Lambda = k$ ) we can factor any extension  $B \twoheadrightarrow A$  into *small* extensions  $\varphi : B' \twoheadrightarrow A$  where  $\ker \varphi$  is principal and annihilated by  $\mathfrak{m}_B$ .

**Definition 3.0.2.** Let  $\Lambda$  be a Noetherian ring and let  $\Lambda \rightarrow k$  be a finite ring map where  $k$  is a field. Define the category  $\widehat{\mathcal{C}}_{\Lambda}$  of,

- (a) pairs  $(R, \varphi)$  where  $R$  is a Noetherian complete local  $\Lambda$ -algebra and  $\varphi : R/\mathfrak{m}_R \rightarrow k$  is a  $\Lambda$ -algebra isomorphism,
- (b) morphisms  $f : (S, \psi) \rightarrow (R, \varphi)$  are local  $\Lambda$ -algebra map such that  $\varphi \circ (f \bmod \mathfrak{m}_S) = \psi$ .

*Remark.* Then  $\mathcal{C}_{\Lambda} \subset \widehat{\mathcal{C}}_{\Lambda}$  is naturally a full subcategory.

### 3.1 Deformation Functors

**Definition 3.1.1.** A *predeformation functor* is a functor  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  such that  $F(k) = \{*\}$ .

*Remark.* The condition  $F(k) = \{*\}$  corresponds to choosing a fixed base object for the deformations.

**Definition 3.1.2.** Given a predeformation functor  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  we extend it to  $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \rightarrow \text{Set}$  via,

$$\widehat{F}(R) = \varprojlim_n F(R/\mathfrak{m}_R^n)$$

A functor  $F$  is *pro-representable* if  $\widehat{F}$  is representable.

**Definition 3.1.3.** We say a morphism  $\varphi : F \rightarrow G$  of functors on  $\mathcal{C}_\Lambda$  is *smooth* the map,

$$F(B) \rightarrow F(A) \times_{G(A)} G(B)$$

induced by an extension  $B \twoheadrightarrow A$  is surjective.

**Definition 3.1.4.** Let  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  be a predeformation functor. The *tangent space* of  $F$  is the set  $TF = F(k[\epsilon])$ . We will see under some assumptions this set is naturally a  $k$ -vectorspace.

**Definition 3.1.5.** Let  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  be a predeformation functor. A *hull*<sup>1</sup> for  $F$  is a pair  $(R, \eta)$  where  $R \in \widehat{\mathcal{C}}_\Lambda$  and  $\eta \in \widehat{F}(R)$  such that  $h_R \rightarrow F$  is formally smooth and bijective on tangent spaces.

*Remark.* Let  $k[\epsilon]$  be the ring  $k[\epsilon]/(\epsilon^2)$  with the trivial  $\Lambda$ -algebra structure.

**Definition 3.1.6.** Let  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  be a predeformation functor. If  $A' \rightarrow A$  and  $A'' \rightarrow A$  are morphisms in  $\mathcal{C}_\Lambda$  there is a natural map,

$$(*) \quad F(A' \times_A A'') \rightarrow F(A') \times_{F(A)} F(A'')$$

Then Schlessinger's conditions are as follows,

- (H1) if  $A'' \twoheadrightarrow A$  is a small thickening then  $(*)$  is surjective
- (H2) if  $A = k$  and  $A'' = k[\epsilon]$  then  $(*)$  is bijective
- (H3)  $TF = F(k[\epsilon])$  is finite-dimensional
- (H4) if  $A'' = A'$  and  $A' \twoheadrightarrow A$  is a small thickening, then  $(*)$  is bijective.

*Remark.*  $TF = F(k[\epsilon])$  has a canonical vectorspace structure when  $F$  satisfies (H2) since we get,

$$F(k[\epsilon]) \times F(k[\epsilon]) \xrightarrow{\sim} F(k[\epsilon_1, \epsilon_2]) \rightarrow F(k[\epsilon])$$

using the map  $k[\epsilon_1, \epsilon_2] \rightarrow k[\epsilon]$  via  $\epsilon_1 \mapsto \epsilon$  and  $\epsilon_2 \mapsto \epsilon$ . The scalar multiplication is defined by  $F(k[\epsilon]) \rightarrow F(k[\epsilon])$  induced by the map  $\epsilon \mapsto c\epsilon$ .

We cannot give  $TF$  a vectorspace structure without (H2) so it is more correct to group the Schlessinger conditions into pairs (H1) + (H2) and (H3) + (H4) as we do in the sequel.

---

<sup>1</sup>Some authors use the terminology *miniversal* formal object. However, in the deformation category setting, a minimal versal object may not induce an isomorphism of the tangent space so we reserve the term *miniversal* for a minimal versal object see [Tag 06T0](#).

**Definition 3.1.7.** A pre-deformation functor  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  is a *deformation functor* if it satisfies (H1) and (H2).

**Theorem 3.1.8** (Schlessinger). Let  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  be a deformation functor. Then,

- (a)  $F$  admits a hull if and only if it satisfies (H3)
- (b)  $F$  is pro-representable if and only if it satisfies (H3) and (H4).

**Example 3.1.9.** Let  $X$  be a  $k$ -scheme, the functor  $\text{Def}_X : \mathcal{C}_k \rightarrow \text{Set}$  defined by,

$$\text{Def}_X : A \mapsto \{(X', \psi) \mid X' \text{ flat } A\text{-scheme with } \psi : X' \otimes_A k' \xrightarrow{\sim} X\} / \cong$$

is a deformation functor.

**Example 3.1.10.** Let  $X = \text{Spec}(k[x, y]/(xy))$  and  $F = \text{Def}_X$ . If  $A$  is a finite type  $k$ -algebra and  $P \twoheadrightarrow A$  is a presentation from a polynomial ring with kernel  $K$  then [H, Ex. 9.8] shows that,

$$\text{Hom}_A(\Omega_{P/k} \otimes_k A, A) \longrightarrow \text{Hom}_A(J/J^2, A) \longrightarrow T\text{Def}_A \longrightarrow 0$$

arising from the conormal exact sequence,

$$J/J^2 \longrightarrow \Omega_{P/k} \otimes_P A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

In our case, let  $P = k[x, y]$  and  $J = (xy)$ . Then we have,

$$A\partial_x \oplus A\partial_y \longrightarrow A \longrightarrow T\text{Def}_A \longrightarrow 0$$

and therefore  $T\text{Def}_A = A/(x, y) = k$ . Thus  $\text{Def}_X$  satisfies (H3) so it should have a hull. Indeed,

$$(k[[t]], \text{Spf}(k[[t]][x, y]/(xy - t)))$$

is a hull (note the formal object is effective). Let's first understand why this hull is not a pro-representing object. For any map,  $\varphi : k[[t]] \rightarrow A$  the induced object,

$$\varphi_*(\text{Spf}(k[[t]][x, y]/(xy - t))) = \text{Spec}(A[x, y]/(xy - \varphi(t)))$$

is unchanged (in isomorphism class) if we replace  $\varphi$  by  $\varphi' = u\varphi$  for any unit  $u \in A$  since then we can scale  $x$  or  $y$  to remove  $u$ . However, recall that a deformation  $X'$  is equipped with a distinguished isomorphism  $\varphi : X' \otimes_A k \xrightarrow{\sim} X$  with which isomorphisms of deformations must be compatible. Therefore,  $\varphi' = u\varphi$  and  $\varphi$  define the same deformation if  $u \in A^\times$  is a unit and  $u \equiv 1 \pmod{\mathfrak{m}_A}$ . Therefore, the map,  $h_R \rightarrow \text{Def}_X$  is not injective for general  $A$  but is injective for  $A = k[\epsilon]$  (since  $(1 + a\epsilon) \cdot \epsilon = \epsilon$  so multiplication by such  $a$  does nothing) as must be true for a hull.

However  $\text{Def}_X$  is not pro-representable since it does not satisfy (H4). Indeed, consider  $A = k[\epsilon]/(\epsilon^3)$  and consider,

$$\text{Def}_X(A \times_k A) \rightarrow \text{Def}_X(A) \times \text{Def}_X(A)$$

I claim this is not injective. Indeed,  $t = \epsilon_1 + \epsilon_2$  and  $t = \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$  map to the same pair of deformations but I claim they are not related by such a unit. Write,

$$u = 1 + a\epsilon_1 + b\epsilon_2 + O(\epsilon^2)$$

then,

$$u(\epsilon_1 + \epsilon_2) = \epsilon_1 + \epsilon_2 + a\epsilon_1^2 + (a + b)\epsilon_1\epsilon_2 + b\epsilon_2^2 + O(\epsilon^3)$$

and we cannot have  $a = b = 0$  but  $a + b = 1$ .

*Remark.* The above illustrates why it is necessary to define deformations of a scheme as equipped with a distinguished isomorphism  $\varphi : X' \otimes_A k \xrightarrow{\sim} X$  otherwise  $\text{Def}_X$  will not be a deformation functor. Indeed, let  $\text{Def}'_X$  be the pre-deformation functor,

$$\text{Def}'_X : A \mapsto \{X' \mid X' \text{ flat } A\text{-schemes such that } X' \otimes_A k' \cong X\} / \cong$$

but forgetting the isomorphism. Then for  $X = \text{Spec}(k[x, y]/(xy))$

$$\text{Def}'_X(k[\epsilon_1, \epsilon_2]) \rightarrow \text{Def}'_X(k[\epsilon]) \times \text{Def}'_X(k[\epsilon])$$

is not injective. Indeed,

$$\text{Spec}(k[\epsilon_1, \epsilon_2][x, y]/(xy + \epsilon_1 + \epsilon_2)) \quad \text{and} \quad \text{Spec}(k[\epsilon_1, \epsilon_2][x, y]/(xy + \epsilon_1 + 2\epsilon_2))$$

have the same image but are not isomorphic.

## 3.2 Deformation Categories

**Definition 3.2.1.** A *predeformation category* is a category cofibered in groupoids  $\mathcal{F} \rightarrow \mathcal{C}_\Lambda$  such that  $\mathcal{F}(k)$  is equivalent to the trivial category.

*Remark.* Let  $\mathcal{F}$  be a predeformation category and  $x_0 \in \mathcal{F}(k)$ . Then for any  $x \in \mathcal{F}$  over  $A$  let  $q : A \rightarrow k$  then there is a pushforward  $x \rightarrow q_*x$  and  $q_*x \in \mathcal{F}(k)$  so there is a unique isomorphism  $q_*x \xrightarrow{\sim} x_0$  and hence there is a canonical morphism  $x \rightarrow x_0$  in  $\mathcal{F}$ .

*Remark.* If  $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$  is a predeformation functor then the associated cofibered set  $\mathcal{F}_F \rightarrow \mathcal{C}_\Lambda$  is a predeformation category. Likewise, if  $\mathcal{F} \rightarrow \mathcal{C}_\Lambda$  is a predeformation category then the functor of isomorphism classes  $\widehat{\mathcal{F}} : \mathcal{C}_\Lambda \rightarrow \text{Set}$  is a predeformation functor.

**Definition 3.2.2.** Let  $\mathcal{F} \rightarrow \mathcal{C}_\Lambda$  be a category cofibered in groupoids. The *category of formal objects of  $\widehat{\mathcal{F}}$*  is the category of,

- (a) formal objects  $(R, \xi_n, f_n)$  consists of an object  $R \in \widehat{\mathcal{C}}_\Lambda$ , and objects  $\xi_n \in \mathcal{F}(R/\mathfrak{m}_R^n)$  and morphisms  $f_n : \xi_{n+1} \rightarrow \xi_n$  over the projection  $R/\mathfrak{m}_R^{n+1} \rightarrow R/\mathfrak{m}_R^n$
- (b) morphisms  $a : (R, \xi_n, f_n) \rightarrow (S, \eta_n, g_n)$  consists of a map  $a_0 : R \rightarrow S$  in  $\widehat{\mathcal{C}}_\Lambda$  and a collection  $a_n : \xi_n \rightarrow \eta_n$  of morphisms in  $\mathcal{F}$  lying over  $R/\mathfrak{m}_R^n \rightarrow S/\mathfrak{m}_S^n$  such that the diagrams,

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow{f_n} & \xi_n \\ \downarrow a_{n+1} & & \downarrow a_n \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commute for each  $n \in \mathbb{N}$ .

**Proposition 3.2.3 (06H4).** The formal objects forms a category cofibered in groupoids  $\hat{p} : \widehat{\mathcal{F}} \rightarrow \widehat{\mathcal{C}}_\Lambda$ .

**Definition 3.2.4.** Let  $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$  be a category cofibered in groupoids. We say that  $\mathcal{F}$  satisfies the *Rim-Schlessinger (RS) condition* if for all  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  in  $\mathcal{C}_\Lambda$  with  $A_2 \twoheadrightarrow A$  surjective,

$$\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence. A *deformation category* is a predeformation category  $\mathcal{F}$  satisfying (RS).

**Lemma 3.2.5 (06J5).** Condition (RS) is equivalent to: for every diagram in  $\mathcal{F}$ ,

$$\begin{array}{ccc} x_2 & & A_2 \\ \downarrow & \text{lying over} & \downarrow \\ x_1 \longrightarrow x & & A_1 \longrightarrow A \end{array}$$

in  $\mathcal{C}_\Lambda$  with  $A_2 \rightarrow A$  surjective, there exists a fiber product  $x_1 \times_x x_2$  in  $\mathcal{F}$  such that the diagram,

$$\begin{array}{ccc} x_1 \times_x x_2 \longrightarrow x_2 & & A_1 \times_A A_2 \longrightarrow A_2 \\ \downarrow & \text{lying over} & \downarrow \\ x_1 \longrightarrow x & & A_1 \longrightarrow A \end{array}$$

**Lemma 3.2.6 (07WQ).** If  $\mathcal{X} \rightarrow S$  is an algebraic stack then for any  $\text{Spec}(k) \rightarrow S$  and  $x_0 \in \mathcal{X}(k)$  the deformation category  $\mathcal{F}_{\mathcal{X},k,x_0}$  satisfies (RS).

*Remark.* By Schlessinger's theorem, this is telling us that a deformation functor  $F = D_{X,x_0}$  represented by some pointed finite-type quasi-separated<sup>2</sup> algebraic space  $x_0 \in X$  over a noetherian scheme  $S$  is pro-representable. So even though  $X$  does not have a canonical local ring it does have a formal local ring  $\widehat{\mathcal{O}_{X,x_0}}$ . We can calculate it from the formal local ring of any étale cover  $U \rightarrow X$ . This is well-defined because for two étale covers  $U_1 \rightarrow X$  and  $U_2 \rightarrow X$  we have  $U_1 \times_X U_2$  is an étale cover of both and these maps identify the formal local rings. There is a subtlety here about the residue field of the preimage of  $x_0$  in these étale covers meaning that the complete local rings will not be isomorphic until after a field extension. The technical assumptions ensure that  $X$  is decent and then the discussion of [Tag 0EMV](#) applies.

### 3.3 Versality

*Remark.* A versal object is a universal object without the “uni” i.e. without the uniqueness.

**Definition 3.3.1.** A morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  of categories cofibered in groupoids over  $\mathcal{C}_\Lambda$  is *smooth* if for every extension  $B \twoheadrightarrow A$  in  $\mathcal{C}_\Lambda$  the map,

$$\mathcal{F}(B) \rightarrow \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{F}(B)$$

is essentially surjective.

*Remark.* This is basically the formal lifting criterion for formal smoothness. Indeed, if these deformation categories are induced by the representable functors for a morphism of schemes  $f : X \rightarrow Y$  then we get that,

$$X(B) \rightarrow X(A) \times_{Y(A)} Y(B)$$

is surjective which is equivalent to there existing a dashed arrow in each lifting diagram,

$$\begin{array}{ccc} \text{Spec}(A) & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow f \\ \text{Spec}(B) & \longrightarrow & Y \end{array}$$

---

<sup>2</sup>I don't know if these are the *right* conditions but they make the discussion work.

**Lemma 3.3.2.** Smoothness of  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is equivalent to the following explicit condition. For every surjection  $B \twoheadrightarrow A$  in  $\mathcal{C}_\Lambda$  and  $y \in \mathcal{G}(B)$  and  $x \in \mathcal{F}(A)$  equipped with a map  $y \rightarrow \varphi(x)$  over  $B \twoheadrightarrow A$  there is  $x' \in \mathcal{F}(B)$  and a morphism  $x' \rightarrow x$  over  $B \twoheadrightarrow A$  and a morphism  $\varphi(x') \rightarrow y$  over  $\text{id} : B \rightarrow V$  such that,

$$\begin{array}{ccc} \varphi(x') & \longrightarrow & y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

**Definition 3.3.3.** Let  $R \in \widehat{\mathcal{C}}_\Lambda$ . We say  $\xi \in \widehat{\mathcal{F}}(R)$  is *versal* if the morphism  $\xi : \underline{R}|_{e_\Lambda} \rightarrow \mathcal{F}$  defined by  $\xi$  is smooth.

*Remark.* The morphism is defined as follows. For any  $A \in \mathcal{C}_\Lambda$  and map  $\varphi : R \rightarrow A$  it will factor as  $\varphi_n : R/\mathfrak{m}^n \rightarrow A$  we send  $(A, \varphi) \mapsto (\varphi_n)_* \xi_n$ . The compatibility isomorphisms of the formal object  $\xi$  make this well-defined.

*Remark.* Let  $\xi$  be a formal object of  $\mathcal{F}$ . Versality of  $\xi$  is equivalent to: the existence of a dashed arrow for any diagram,

$$\begin{array}{ccc} & y & \\ & \nearrow & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

in  $\widehat{\mathcal{F}}$  such that  $y \rightarrow x$  lies over a surjective map  $B \twoheadrightarrow A$  of Artinian rings.

**Theorem 3.3.4** (Rim-Schlessinger). A deformation category  $\mathcal{F}$  such that  $T\mathcal{F} = \overline{\mathcal{F}}(k[\epsilon])$  is finite dimensional admits a versal formal object.

**Example 3.3.5.** Let  $X$  be a  $k$ -scheme. The cofibered category of deformations  $\mathcal{D}\text{ef}_X \rightarrow \mathcal{C}_k$  is a deformation category. If  $X$  is finite type and either proper or affine then  $T\mathcal{D}\text{ef}_X = T\text{Def}_X$  is finite dimensional so  $X$  admits a versal formal deformation  $\mathcal{X} \rightarrow \text{Spf}(R)$ .

**Definition 3.3.6.** Given a category fibered in groupoids,

$$p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$$

and a finite type point  $\text{Spec}(k) \rightarrow S$  and  $x_0 \in \mathcal{X}(k)$ . First factor  $\text{Spec}(k) \rightarrow \text{Spec}(\Lambda) \hookrightarrow S$  through some affine open such that  $\Lambda \rightarrow k$  is finite. The category  $\mathcal{C}_\Lambda$ , up to canonical equivalence, does not depend of the choice of affine open  $\text{Spec}(\Lambda) \subset S$ . Note that  $\mathcal{C}_\Lambda$  is equivalent to the opposite category of factorizations,

$$\text{Spec}(k) \rightarrow \text{Spec}(A) \rightarrow S$$

such that  $A$  is Artin local and  $A \rightarrow k$  identifies  $k$  with the residue field. Now let  $\mathcal{F}_{\mathcal{X}, k, x_0}$  be the category of,

- (a) morphisms  $x_0 \rightarrow x$  of  $\mathcal{X}$  over  $\text{Spec}(k) \rightarrow \text{Spec}(A)$  as  $S$ -map in  $\mathcal{C}_\Lambda$ ,
- (b) morphisms  $(x_0 \rightarrow x) \rightarrow (x_0 \rightarrow x')$  are diagrams,

$$\begin{array}{ccc}
& x & x' \\
& \swarrow & \searrow \\
x_0 & &
\end{array}$$

in  $\mathcal{X}$  (notice the reversal of arrows).

Then  $p : \mathcal{F}_{\mathcal{X}, k, x_0} \rightarrow \mathcal{C}_\Lambda$  is a predeformation category. We say that a formal object  $\xi = (R, \xi_n, f_n)$  of  $\mathcal{X}$  is *versal* if  $\xi$  is versal as a formal object of  $\mathcal{F}_{\mathcal{X}, k, x_0}$  with  $k = R/\mathfrak{m}_R$  and  $x_0 = \xi_1$ . We say that  $x \in \mathcal{X}(U)$  is versal at a finite type point  $u_0 \in U$  if  $\hat{x} \in \widehat{\mathcal{F}}_{\mathcal{X}, k(u_0), x_0}$  is versal where  $x_0 : \text{Spec}(k) \rightarrow U \rightarrow \mathcal{X}$  is the image.

**Definition 3.3.7.** Let  $S$  be a locally noetherian scheme and  $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$  a category fibered in groupoids. We say  $\mathcal{X}$  satisfies *openness of versality* if given a scheme  $U$  locally of finite type over  $S$ , an object  $x \in \mathcal{X}(U)$ , and a finite type point  $u_0 \in U$  such that  $x$  is versal at  $u_0$  then there exists an open neighborhood  $U' \subset U$  such that  $x$  is versal at every finite type point of  $U'$ .

### 3.4 Effectivity

**Definition 3.4.1.** A formal object  $\xi = (R, \xi_n, f_n) \in \widehat{\mathcal{F}}_{\mathcal{X}, k, x_0}$  is *effective* if it arises from  $\tilde{\xi} \in \mathcal{X}(R)$ .

**Lemma 3.4.2 (07X3).** If  $\mathcal{X} \rightarrow S$  is an algebraic stack over a locally noetherian scheme  $S$  then every formal object is effective.

*Proof.* First, if  $X$  is a scheme then for all local rings  $R$  factoring  $\text{Spec}(k) \rightarrow X$  the map corresponds to  $\text{Spec}(R) \rightarrow \text{Spec}(\mathcal{O}_{X, x}) \rightarrow X$  so if  $R$  is complete,

$$X(R) = \text{Hom}_{\text{loc}}(\mathcal{O}_{X, x}, R) = \varprojlim_n \text{Hom}_{\text{loc}}(\mathcal{O}_{X, x}, R/\mathfrak{m}_R^n) = \varprojlim_n X(R/\mathfrak{m}_R^n)$$

The general case follows from an intricate descent argument.  $\square$

## 4 Artin's Axioms

**Theorem 4.0.1** (Artin Approximation). Let  $S$  be a locally noetherian scheme and a category fibered in groupoids  $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$ . Let  $R$  be a Noetherian complete local ring with residue field  $k$  with  $\text{Spec}(R) \rightarrow S$  finite type and  $x \in \mathcal{X}(R)$ . Let  $s \in S$  be the image of  $\text{Spec}(k) \rightarrow \text{Spec}(R) \rightarrow S$ . Assume that,

- (a)  $\mathcal{O}_{S, s}$  is a  $G$ -ring
- (b)  $p$  is limit-preserving on objects.

Then for every  $N \geq 1$  there exist,

- (a) a finite type  $S$ -algebra  $A$
- (b) a maximal ideal  $\mathfrak{m}_A \subset A$
- (c) an object  $x_A \in \mathcal{X}(A)$
- (d) an  $S$ -isomorphism  $R/\mathfrak{m}_R^N \xrightarrow{\sim} A/\mathfrak{m}_A^N$

- (e) an isomorphism  $x|_{R/\mathfrak{m}_R^N} \xrightarrow{\sim} x_A|_{A/\mathfrak{m}_A^N}$  over the previous map
- (f) an isomorphism  $\mathbf{gr}_{\mathfrak{m}_R}(R) \xrightarrow{\sim} \mathbf{gr}_{\mathfrak{m}_A}(A)$  of graded  $k$ -algebras.

**Lemma 4.0.2.** Let  $S$  be a locally noetherian scheme and  $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$  a category fibered in groupoids. Let  $\xi$  be a formal object of  $\mathcal{X}$  with  $x_0 = \xi_1$  lying over  $\text{Spec}(k) \rightarrow S$  with image  $s \in S$  such that,

- (a)  $\xi$  is versal
- (b)  $\xi$  is effective
- (c)  $\mathcal{O}_{S,s}$  is a  $G$ -ring
- (d)  $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$  is limit-preserving

then there exists a finite type morphism  $U \rightarrow S$ , a finite type point  $u_0 \in U$  with residue field  $k$  and  $x \in \mathcal{X}(U)$  such that  $x : U \rightarrow \mathcal{X}$  is versal at  $u_0$  and  $x|_{\text{Spec}(\mathcal{O}_{U,u_0})}$  induces  $\xi$ .

*Proof.* Choose an object  $x_R \in \mathcal{X}(R)$  whose completion is  $\xi$ . Apply Artin approximation with  $N = 2$  to obtain  $A, \mathfrak{m}_A, x_A \in \mathcal{X}(A)$  approximating  $\xi$ . Let  $\eta$  be the formal object completing  $x_A|_{\text{Spec}(\hat{A})}$  (the completion of  $A$  at  $\mathfrak{m}_A$ ). Then a lift for the diagram in  $\widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$ ,

$$\begin{array}{ccc} \xi & \xrightarrow{\eta} & \hat{A} \\ \downarrow & \text{lying over} & \downarrow \\ \xi_2 = \eta_2 & \longrightarrow & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{array}$$

exists because  $\xi$  is versal. Since the map  $R \rightarrow \hat{A}$  induces an isomorphism on tangent spaces and by construction  $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$  we conclude that  $R \rightarrow \hat{A}$  is an isomorphism. Hence  $\eta \cong \xi$  is versal so the map  $x_A : \text{Spec}(A) \rightarrow \mathcal{X}$  is versal at  $x_A|_{\text{Spec}(\hat{A})} = \eta$ .  $\square$

**Theorem 4.0.3.** Let  $S$  be a locally Noetherian base scheme and consider a category fibered in groupoids  $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$ . For each finite type morphism  $\text{Spec}(k) \rightarrow S$  with  $k$  a field and  $x_0 \in \mathcal{X}(\text{Spec}(k))$  assume that,

- (a)  $\mathcal{X}$  is a stack for the étale topology
- (b)  $\Delta_{\mathcal{X}/S} : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$  is representable by algebraic spaces
- (c)  $\mathcal{X}$  is limit preserving (preserves filtered colimits)
- (d)  $\mathcal{X}$  satisfies the Rim-Schlessinger condition (RS)
- (e)  $T\mathcal{F}_{\mathcal{X},k,x_0}$  is finite dimensional for all  $k$  and all  $x_0 \in \mathcal{F}(k)$
- (f) every formal object of  $\mathcal{X}$  is effective
- (g)  $\mathcal{X}$  satisfies openness of versality
- (h)  $\mathcal{O}_{S,s}$  is a  $G$ -ring for all finite type points  $s \in S$
- (i) a set theoretic condition

then  $\mathcal{X}$  is an algebraic stack.

*Proof.* It suffices to show that for each finite type  $\text{Spec}(k) \rightarrow S$  and  $x_0 \in \mathcal{X}(k)$  there is a finite type morphism  $U \rightarrow S$  and a smooth map  $U \rightarrow \mathcal{X}$  such that there is a finite type point  $u_0 : \text{Spec}(k) \rightarrow U$  such that  $x|_{u_0} \cong x_0$ .

By Rim-Schlessinger  $\mathcal{F}_{\mathcal{X}, k, x_0}$  admits a versal formal object  $\xi$  which is then effective. Artin approximation allows us to approximate an effective formal object by a finite type object  $U \rightarrow \mathcal{X}$  which is versal at  $u_0 \in U$ . By openness of versality, we can shrink  $U$  such that  $U \rightarrow \mathcal{X}$  is versal at every finite type point.

Finally, prove that a representable morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  of limit preserving categories fibered in groupoids which is smooth on deformation categories is smooth (Tag 07XX). Indeed, for  $T \rightarrow \mathcal{Y}$  the condition says that  $f : \mathcal{X}_T \rightarrow T$  is a formally smooth map of algebraic spaces<sup>3</sup> and the limit-preserving condition gives finitely presented.  $\square$

*Remark.* Usually most difficult to prove openness of versality. There a number of deformation-theoretic techniques for proving this but require effectivity of formal objects over more general formal schemes. There are also tangent-obstruction theory methods for proving openness of versality.

---

<sup>3</sup>There is a subtly here with changing fields that requires the full strength of (RS) where as proving that a versal object exists only requires (S1) and (S2) and finite-dimensionality of tangent spaces