

Artin's Axioms and Formal Deformation Theory

Stanford-Berkeley Number Theory Learning Seminar

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1 Definitions

Definition 1.0.1. An *algebraic space* is a functor $X : (\mathbf{Sch}_S)^{\text{op}}_{\text{fppf}} \rightarrow \text{Set}$ such that,

- (a) F is a sheaf in the fppf topology
- (b) the diagonal $\Delta_{X/S} : X \rightarrow X \times_S X$ is representable by schemes
- (c) there is a scheme U and an étale surjection $U \rightarrow X$.

Definition 1.0.2. An *algebraic stack* is a category fibered in groupoids $\mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$ such that,

- (a) \mathcal{X} is a stack in the fppf topology
- (b) $\Delta_{\mathcal{X}/S} : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces
- (c) there is an algebraic space U and an étale surjection $U \rightarrow \mathcal{X}$.

Remark. The map $U \rightarrow \mathcal{X}$ is only necessarily representable by algebraic spaces so to express the property of being an étale surjection consider any map from a scheme $T \rightarrow \mathcal{X}$ and an étale cover from a scheme $V \rightarrow U \times_{\mathcal{X}} T$ in the diagram,

$$\begin{array}{ccccc} & & \text{ét surj} & & \\ & \searrow & & \nearrow & \\ V & \longrightarrow & U \times_{\mathcal{X}} T & \longrightarrow & T \\ & & \downarrow & & \downarrow \\ & & U & \longrightarrow & \mathcal{X} \end{array}$$

This property is independent of the choice of étale cover $V \rightarrow U \times_{\mathcal{X}} T$ by étale descent for étale surjective morphisms.

Remark. Why do we only require that \mathcal{X} be smooth locally an algebraic space and its diagonal be representable by only algebraic spaces? The diagonal is closely related to the automorphism groups of objects \mathcal{X} parametrizes. When $\pi : X \rightarrow S$ a proper finitely presented map of schemes, $\text{Hilb}_{X/S}$ is representable by an algebraic space but not generally by a scheme unless π is projective. This shows that $\text{Isom}_S(X, Y)$ between two proper S -schemes is usually only representable by an algebraic space. Therefore, we want to allow for Δ to be representable by algebraic spaces not just schemes to capture moduli of proper non-projective objects.

Definition 1.0.3. Consider $f : \mathcal{X} \rightarrow \mathcal{Z}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ morphisms of categories fibered in groupoids. Then the 2-fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is defined as the category fibered in groupoids,

- (a) objects are (x, y, γ) with $p(x) = p(y)$ and $\gamma : f(x) \rightarrow g(y)$ a morphism over id
- (b) morphisms are $\varphi : (x, y, \gamma) \rightarrow (x', y', \gamma')$ are given by pairs $(\varphi_x : x \rightarrow x', \varphi_y : y \rightarrow y')$ such that the diagram,

$$\begin{array}{ccc} f(x) & \xrightarrow{\gamma} & g(y) \\ \downarrow \varphi_x & & \downarrow \varphi_y \\ f(x') & \xrightarrow{\gamma'} & g(y') \end{array}$$

commutes.

Proposition 1.0.4. For any objects $x, y \in \mathcal{X}(U)$. There is a 2-fiber product diagram,

$$\begin{array}{ccc} \text{Isom}(x, y) & \longrightarrow & U \\ \downarrow & & \downarrow y \\ U & \xrightarrow{x} & \mathcal{X} \end{array}$$

Definition 1.0.5. The *inertia stack* of \mathcal{X} is the category fibered in groupoids $\mathcal{I}_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$.

Proposition 1.0.6. For any $x \in \mathcal{X}(U)$ there is a 2-fiber product diagram,

$$\begin{array}{ccc} \text{Isom}(x, x) & \longrightarrow & \mathcal{I}_X \\ \downarrow & \lrcorner & \downarrow \\ U & \xrightarrow{x} & \mathcal{X} \end{array}$$

2 Presentations

Proposition 2.0.1. Let X be an algebraic space over S and $f : U \twoheadrightarrow X$ an étale surjection from a scheme U . Set $R = U \times_X U$ in the pullback diagram,

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

then we have,

- (a) $j : R \rightarrow U \times_S U$ is a monomorphism and $R(T) \subset U(T) \times U(T)$ is an equivalence relation for all $T \rightarrow S$
- (b) the projections $s, t : R \rightarrow U$ are étale
- (c) the diagram,

$$R \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} U \longrightarrow X$$

is a coequalizer in $\text{Sh}((\mathbf{Sch}_S)_{\text{fppf}})$.

Proof. The first two are immediate. The last holds in any category of sheaves given that $U \rightarrow X$ is surjective. \square

Definition 2.0.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S . The quotient stack,

$$p : [U/R] \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$$

is the stackification of the category fibered in groupoids,

$$(T \rightarrow S) \mapsto (U(T), R(T), s, t, c)$$

Proposition 2.0.3 (04T5). Given an algebraic stack \mathcal{X} there is a smooth morphism $U \rightarrow \mathcal{X}$ from a scheme. We recover the groupoid presentation by taking the 2-fiber product,

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

and R is an algebraic space because we assumed that $\Delta_{\mathcal{X}}$ is representable by algebraic spaces. Then there is a natural equivalence $[U/R] \xrightarrow{\sim} \mathcal{X}$.

3 Infinitesimal Deformation Theory

Remark. First we recall how to apply infinitesimal deformation theory in the relative setting. In the basic case, we want to probe properties of a morphisms of schemes $f : X \rightarrow S$ near a finite type point $x : \text{Spec}(k) \rightarrow S$. There is some affine open $\text{Spec}(\Lambda) \subset X$ containing x . Then we need to consider Artinian local rings A and diagrams,

$$\begin{array}{ccccccc} & & & & & & X \\ & & & & & & \downarrow f \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(A) & \longrightarrow & \text{Spec}(\Lambda) & \hookrightarrow & S \end{array}$$

and consider the set of dashed arrows. This means our base category should be the category of local Artinian Λ -algebras with residue field k .

Definition 3.0.1. Let Λ be a noetherian ring and $\Lambda \rightarrow k$ a finite ring map with k a field. Let \mathcal{C}_{Λ} be the category of,

- (a) (A, φ) where A is an Artinian local Λ -algebra and $\varphi : A/\mathfrak{m}_A \rightarrow k$ a Λ -algebra isomorphism
- (b) morphisms $f : (B, \psi) \rightarrow (A, \varphi)$ are local Λ -algebra maps such that $\varphi \circ (f \bmod \mathfrak{m}_A) = \psi$

Remark. As in the absolute case (which corresponds to $\Lambda = k$) we can factor any extension $B \twoheadrightarrow A$ into *small* extensions $\varphi : B' \twoheadrightarrow A$ where $\ker \varphi$ is principal and annihilated by \mathfrak{m}_B .

Definition 3.0.2. Let Λ be a Noetherian ring and let $\Lambda \rightarrow k$ be a finite ring map where k is a field. Define the category $\widehat{\mathcal{C}}_{\Lambda}$ of,

- (a) pairs (R, φ) where R is a Noetherian complete local Λ -algebra and $\varphi : R/\mathfrak{m}_R \rightarrow k$ is a Λ -algebra isomorphism,
- (b) morphisms $f : (S, \psi) \rightarrow (R, \varphi)$ are local Λ -algebra map such that $\varphi \circ (f \bmod \mathfrak{m}_S) = \psi$.

Remark. Then $\mathcal{C}_{\Lambda} \subset \widehat{\mathcal{C}}_{\Lambda}$ is naturally a full subcategory.

3.1 Deformation Functors

Definition 3.1.1. A *predeformation functor* is a functor $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ such that $F(k) = \{*\}$.

Remark. The condition $F(k) = \{*\}$ corresponds to choosing a fixed base object for the deformations.

Definition 3.1.2. Given a predeformation functor $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ we extend it to $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \rightarrow \text{Set}$ via,

$$\widehat{F}(R) = \varprojlim_n F(R/\mathfrak{m}_R^n)$$

A functor F is *pro-representable* if \widehat{F} is representable.

Definition 3.1.3. We say a morphism $\varphi : F \rightarrow G$ of functors on \mathcal{C}_Λ is *smooth* the map,

$$F(B) \rightarrow F(A) \times_{G(A)} G(B)$$

induced by an extension $B \twoheadrightarrow A$ is surjective.

Definition 3.1.4. Let $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ be a predeformation functor. The *tangent space* of F is the set $TF = F(k[\epsilon])$. We will see under some assumptions this set is naturally a k -vectorspace.

Definition 3.1.5. Let $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ be a predeformation functor. A *hull*¹ for F is a pair (R, η) where $R \in \widehat{\mathcal{C}}_\Lambda$ and $\eta \in \widehat{F}(R)$ such that $h_R \rightarrow F$ is formally smooth and bijective on tangent spaces.

Remark. Let $k[\epsilon]$ be the ring $k[\epsilon]/(\epsilon^2)$ with the trivial Λ -algebra structure.

Definition 3.1.6. Let $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ be a predeformation functor. If $A' \rightarrow A$ and $A'' \rightarrow A$ are morphisms in \mathcal{C}_Λ there is a natural map,

$$(*) \quad F(A' \times_A A'') \rightarrow F(A') \times_{F(A)} F(A'')$$

Then Schlessinger's conditions are as follows,

(H1) if $A'' \twoheadrightarrow A$ is a small thickening then $(*)$ is surjective

(H2) if $A = k$ and $A'' = k[\epsilon]$ then $(*)$ is bijective

(H3) $TF = F(k[\epsilon])$ is finite-dimensional

(H4) if $A'' = A'$ and $A' \twoheadrightarrow A$ is a small thickening, then $(*)$ is bijective.

Remark. $TF = F(k[\epsilon])$ has a canonical vectorspace structure when F satisfies (H2) since we get,

$$F(k[\epsilon]) \times F(k[\epsilon]) \xrightarrow{\sim} F(k[\epsilon_1, \epsilon_2]) \rightarrow F(k[\epsilon])$$

using the map $k[\epsilon_1, \epsilon_2] \rightarrow k[\epsilon]$ via $\epsilon_1 \mapsto \epsilon$ and $\epsilon_2 \mapsto \epsilon$. The scalar multiplication is defined by $F(k[\epsilon]) \rightarrow F(k[\epsilon])$ induced by the map $\epsilon \mapsto c\epsilon$.

We cannot give TF a vectorspace structure without (H2) so it is more correct to group the Schlessinger conditions into pairs (H1) + (H2) and (H3) + (H4) as we do in the sequel.

¹Some authors use the terminology *miniversal* formal object. However, in the deformation category setting, a minimal versal object may not induce an isomorphism of the tangent space so we reserve the term *miniversal* for a minimal versal object see [Tag 06T0](#).

Definition 3.1.7. A pre-deformation functor $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is a *deformation functor* if it satisfies (H1) and (H2).

Theorem 3.1.8 (Schlessinger). Let $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ be a deformation functor. Then,

- (a) F admits a hull if and only if it satisfies (H3)
- (b) F is pro-representable if and only if it satisfies (H3) and (H4).

Example 3.1.9. Let X be a k -scheme, the functor $\text{Def}_X : \mathcal{C}_k \rightarrow \text{Set}$ defined by,

$$\text{Def}_X : A \mapsto \{(X', \psi) \mid X' \text{ flat } A\text{-scheme with } \psi : X' \otimes_A k' \xrightarrow{\sim} X\} / \cong$$

is a deformation functor.

Example 3.1.10. Let $X = \text{Spec}(k[x, y]/(xy))$ and $F = \text{Def}_X$. If A is a finite type k -algebra and $P \rightarrow A$ is a presentation from a polynomial ring with kernel K then [H, Ex. 9.8] shows that,

$$\text{Hom}_A(\Omega_{P/k} \otimes_k A, A) \longrightarrow \text{Hom}_A(J/J^2, A) \longrightarrow T\text{Def}_A \longrightarrow 0$$

arising from the conormal exact sequence,

$$J/J^2 \longrightarrow \Omega_{P/k} \otimes_P A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

In our case, let $P = k[x, y]$ and $J = (xy)$. Then we have,

$$A\partial_x \oplus A\partial_y \longrightarrow A \longrightarrow T\text{Def}_A \longrightarrow 0$$

and therefore $T\text{Def}_A = A/(x, y) = k$. Thus Def_X satisfies (H3) so it should have a hull. Indeed,

$$(k[[t]], \text{Spf}(k[[t]][x, y]/(xy - t)))$$

is a hull (note the formal object is effective). Let's first understand why this hull is not a pro-representing object. For any map, $\varphi : k[[t]] \rightarrow A$ the induced object,

$$\varphi_*(\text{Spf}(k[[t]][x, y]/(xy - t))) = \text{Spec}(A[x, y]/(xy - \varphi(t)))$$

is unchanged (in isomorphism class) if we replace φ by $\varphi' = u\varphi$ for any unit $u \in A$ since then we can scale x or y to remove u . However, recall that a deformation X' is equipped with a distinguished isomorphism $\varphi : X' \otimes_A k \xrightarrow{\sim} X$ with which isomorphisms of deformations must be compatible. Therefore, $\varphi' = u\varphi$ and φ define the same deformation if $u \in A^\times$ is a unit and $u \equiv 1 \pmod{\mathfrak{m}_A}$. Therefore, the map, $h_R \rightarrow \text{Def}_X$ is not injective for general A but is injective for $A = k[\epsilon]$ (since $(1 + a\epsilon) \cdot \epsilon = \epsilon$ so multiplication by such a does nothing) as must be true for a hull.

However Def_X is not pro-representable since it does not satisfy (H4). Indeed, consider $A = k[\epsilon]/(\epsilon^3)$ and consider,

$$\text{Def}_X(A \times_k A) \rightarrow \text{Def}_X(A) \times \text{Def}_X(A)$$

I claim this is not injective. Indeed, $t = \epsilon_1 + \epsilon_2$ and $t = \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$ map to the same pair of deformations but I claim they are not related by such a unit. Write,

$$u = 1 + a\epsilon_1 + b\epsilon_2 + O(\epsilon^2)$$

then,

$$u(\epsilon_1 + \epsilon_2) = \epsilon_1 + \epsilon_2 + a\epsilon_1^2 + (a + b)\epsilon_1\epsilon_2 + b\epsilon_2^2 + O(\epsilon^3)$$

and we cannot have $a = b = 0$ but $a + b = 1$.

Remark. The above illustrates why it is necessary to define deformations of a scheme as equipped with a distinguished isomorphism $\varphi : X' \otimes_A k \xrightarrow{\sim} X$ otherwise Def_X will not be a deformation functor. Indeed, let Def'_X be the pre-deformation functor,

$$\text{Def}'_X : A \mapsto \{X' \mid X' \text{ flat } A\text{-schemes such that } X' \otimes_A k' \cong X\} / \cong$$

but forgetting the isomorphism. Then for $X = \text{Spec}(k[x, y]/(xy))$

$$\text{Def}'_X(k[\epsilon_1, \epsilon_2]) \rightarrow \text{Def}'_X(k[\epsilon]) \times \text{Def}'_X(k[\epsilon])$$

is not injective. Indeed,

$$\text{Spec}(k[\epsilon_1, \epsilon_2][x, y]/(xy + \epsilon_1 + \epsilon_2)) \quad \text{and} \quad \text{Spec}(k[\epsilon_1, \epsilon_2][x, y]/(xy + \epsilon_1 + 2\epsilon_2))$$

have the same image but are not isomorphic.

3.2 Deformation Categories

Definition 3.2.1. A *predeformation category* is a category cofibered in groupoids $\mathcal{F} \rightarrow \mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to the trivial category.

Remark. Let \mathcal{F} be a predeformation category and $x_0 \in \mathcal{F}(k)$. Then for any $x \in \mathcal{F}$ over A let $q : A \rightarrow k$ then there is a pushforward $x \rightarrow q_*x$ and $q_*x \in \mathcal{F}(k)$ so there is a unique isomorphism $q_*x \xrightarrow{\sim} x_0$ and hence there is a canonical morphism $x \rightarrow x_0$ in \mathcal{F} .

Remark. If $F : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is a predeformation functor then the associated cofibered set $\mathcal{F}_F \rightarrow \mathcal{C}_\Lambda$ is a predeformation category. Likewise, if $\mathcal{F} \rightarrow \mathcal{C}_\Lambda$ is a predeformation category then the functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \rightarrow \text{Set}$ is a predeformation functor.

Definition 3.2.2. Let $\mathcal{F} \rightarrow \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The *category of formal objects of $\widehat{\mathcal{F}}$* is the category of,

- (a) formal objects (R, ξ_n, f_n) consists of an object $R \in \widehat{\mathcal{C}}_\Lambda$, and objects $\xi_n \in \mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \rightarrow \xi_n$ over the projection $R/\mathfrak{m}_R^{n+1} \rightarrow R/\mathfrak{m}_R^n$
- (b) morphisms $a : (R, \xi_n, f_n) \rightarrow (S, \eta_n, g_n)$ consists of a map $a_0 : R \rightarrow S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \rightarrow \eta_n$ of morphisms in \mathcal{F} lying over $R/\mathfrak{m}_R^n \rightarrow S/\mathfrak{m}_S^n$ such that the diagrams,

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow{f_n} & \xi_n \\ \downarrow a_{n+1} & & \downarrow a_n \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commute for each $n \in \mathbb{N}$.

Proposition 3.2.3 (06H4). The formal objects forms a category cofibered in groupoids $\widehat{p} : \widehat{\mathcal{F}} \rightarrow \widehat{\mathcal{C}}_\Lambda$.

Definition 3.2.4. Let $p : \mathcal{F} \rightarrow \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We say that \mathcal{F} satisfies the *Rim-Schlessinger (RS) condition* if for all $A_1 \rightarrow A$ and $A_2 \rightarrow A$ in \mathcal{C}_Λ with $A_2 \twoheadrightarrow A$ surjective,

$$\mathcal{F}(A_1 \times_A A_2) \rightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence. A *deformation category* is a predeformation category \mathcal{F} satisfying (RS).

Lemma 3.2.5 ([06J5](#)). Condition (RS) is equivalent to: for every diagram in \mathcal{F} ,

$$\begin{array}{ccc} & x_2 & \\ & \downarrow & \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & A_2 & \\ & \downarrow & \\ A_1 & \longrightarrow & A \end{array}$$

in \mathcal{C}_Λ with $A_2 \rightarrow A$ surjective, there exists a fiber product $x_1 \times_x x_2$ in \mathcal{F} such that the diagram,

$$\begin{array}{ccc} x_1 \times_x x_2 & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

Lemma 3.2.6 ([07WQ](#)). If $\mathcal{X} \rightarrow S$ is an algebraic stack then for any $\text{Spec}(k) \rightarrow S$ and $x_0 \in \mathcal{X}(k)$ the deformation category $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (RS).

Remark. By Schlessinger's theorem, this is telling us that a deformation functor $F = D_{X,x_0}$ represented by some pointed finite-type quasi-separated² algebraic space $x_0 \in X$ over a noetherian scheme S is pro-representable. So even though X does not have a canonical local ring it does have a formal local ring $\widehat{\mathcal{O}_{X,x_0}}$. We can calculate it from the formal local ring of any étale cover $U \rightarrow X$. This is well-defined because for two étale covers $U_1 \rightarrow X$ and $U_2 \rightarrow X$ we have $U_1 \times_X U_2$ is an étale cover of both and these maps identify the formal local rings. There is a subtlety here about the residue field of the preimage of x_0 in these étale covers meaning that the complete local rings will not be isomorphic until after a field extension. The technical assumptions ensure that X is decent and then the discussion of [Tag 0EMV](#) applies.

3.3 Versality

Remark. A versal object is a universal object without the “uni” i.e. without the uniqueness.

Definition 3.3.1. A morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of categories cofibered in groupoids over \mathcal{C}_Λ is *smooth* if for every extension $B \rightarrow A$ in \mathcal{C}_Λ the map,

$$\mathcal{F}(B) \rightarrow \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{F}(B)$$

is essentially surjective.

Remark. This is basically the formal lifting criterion for formal smoothness. Indeed, if these deformation categories are induced by the representable functors for a morphism of schemes $f : X \rightarrow Y$ then we get that,

$$X(B) \rightarrow X(A) \times_{Y(A)} Y(B)$$

is surjective which is equivalent to there existing a dashed arrow in each lifting diagram,

$$\begin{array}{ccc} \text{Spec}(A) & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow f \\ \text{Spec}(B) & \longrightarrow & Y \end{array}$$

²I don't know if these are the *right* conditions but they make the discussion work.

Lemma 3.3.2. Smoothness of $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is equivalent to the following explicit condition. For every surjection $B \twoheadrightarrow A$ in \mathcal{C}_Λ and $y \in \mathcal{G}(B)$ and $x \in \mathcal{F}(A)$ equipped with a map $y \rightarrow \varphi(x)$ over $B \twoheadrightarrow A$ there is $x' \in \mathcal{F}(B)$ and a morphism $x' \rightarrow x$ over $B \twoheadrightarrow A$ and a morphism $\varphi(x') \rightarrow y$ over $\text{id} : B \rightarrow V$ such that,

$$\begin{array}{ccc} \varphi(x') & \longrightarrow & y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

Definition 3.3.3. Let $R \in \widehat{\mathcal{C}}_\Lambda$. We say $\xi \in \widehat{\mathcal{F}}(R)$ is *versal* if the morphism $\xi : \underline{R}|_{\mathcal{C}_\Lambda} \rightarrow \mathcal{F}$ defined by ξ is smooth.

Remark. The morphism is defined as follows. For any $A \in \mathcal{C}_\Lambda$ and map $\varphi : R \rightarrow A$ it will factor as $\varphi_n : R/\mathfrak{m}^n \rightarrow A$ we send $(A, \varphi) \mapsto (\varphi_n)_* \xi_n$. The compatibility isomorphisms of the formal object ξ make this well-defined.

Remark. Let ξ be a formal object of \mathcal{F} . Versality of ξ is equivalent to: the existence of a dashed arrow for any diagram,

$$\begin{array}{ccc} & & y \\ & \nearrow \text{dashed} & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

in $\widehat{\mathcal{F}}$ such that $y \rightarrow x$ lies over a surjective map $B \twoheadrightarrow A$ of Artinian rings.

Theorem 3.3.4 (Rim-Schlessinger). A deformation category \mathcal{F} such that $T\mathcal{F} = \overline{\mathcal{F}}(k[\epsilon])$ is finite dimensional admits a versal formal object.

Example 3.3.5. Let X be a k -scheme. The cofibered category of deformations $\mathcal{D}ef_X \rightarrow \mathcal{C}_k$ is a deformation category. If X is finite type and either proper or affine then $T\mathcal{D}ef_X = T\text{Def}_X$ is finite dimensional so X admits a versal formal deformation $\mathcal{X} \rightarrow \text{Spf}(R)$.

Definition 3.3.6. Given a category fibered in groupoids,

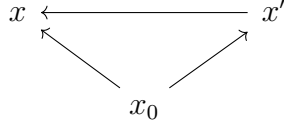
$$p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$$

and a finite type point $\text{Spec}(k) \rightarrow S$ and $x_0 \in \mathcal{X}(k)$. First factor $\text{Spec}(k) \rightarrow \text{Spec}(\Lambda) \hookrightarrow S$ through some affine open such that $\Lambda \rightarrow k$ is finite. The category \mathcal{C}_Λ , up to canonical equivalence, does not depend of the choice of affine open $\text{Spec}(\Lambda) \subset S$. Note that \mathcal{C}_Λ is equivalent to the opposite category of factorizations,

$$\text{Spec}(k) \rightarrow \text{Spec}(A) \rightarrow S$$

such that A is Artin local and $A \rightarrow k$ identifies k with the residue field. Now let $\mathcal{F}_{\mathcal{X}, k, x_0}$ be the category of,

- (a) morphisms $x_0 \rightarrow x$ of \mathcal{X} over $\text{Spec}(k) \rightarrow \text{Spec}(A)$ as S -map in \mathcal{C}_Λ ,
- (b) morphisms $(x_0 \rightarrow x) \rightarrow (x_0 \rightarrow x')$ are diagrams,



in \mathcal{X} (notice the reversal of arrows).

Then $p : \mathcal{F}_{\mathcal{X},k,x_0} \rightarrow \mathcal{C}_\Lambda$ is a predeformation category. We say that a formal object $\xi = (R, \xi_n, f_n)$ of \mathcal{X} is *versal* if ξ is versal as a formal object of $\mathcal{F}_{\mathcal{X},k,x_0}$ with $k = R/\mathfrak{m}_R$ and $x_0 = \xi_1$. We say that $x \in \mathcal{X}(U)$ is versal at a finite type point $u_0 \in U$ if $\hat{x} \in \widehat{\mathcal{F}}_{\mathcal{X},\kappa(u_0),x_0}$ is versal where $x_0 : \text{Spec}(k) \rightarrow U \rightarrow \mathcal{X}$ is the image.

Definition 3.3.7. Let S be a locally noetherian scheme and $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$ a category fibered in groupoids. We say \mathcal{X} satisfies *openness of versality* if given a scheme U locally of finite type over S , an object $x \in \mathcal{X}(U)$, and a finite type point $u_0 \in U$ such that x is versal at u_0 then there exists an open neighborhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U' .

3.4 Effectivity

Definition 3.4.1. A formal object $\xi = (R, \xi_n, f_n) \in \widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$ is *effective* if it arises from $\tilde{\xi} \in \mathcal{X}(R)$.

Lemma 3.4.2 (07X3). If $\mathcal{X} \rightarrow S$ is an algebraic stack over a locally noetherian scheme S then every formal object is effective.

Proof. First, if X is a scheme then for all local rings R factoring $\text{Spec}(k) \rightarrow X$ the map corresponds to $\text{Spec}(R) \rightarrow \text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ so if R is complete,

$$X(R) = \text{Hom}_{\text{loc}}(\mathcal{O}_{X,x}, R) = \varprojlim_n \text{Hom}_{\text{loc}}(\mathcal{O}_{X,x}, R/\mathfrak{m}_R^n) = \varprojlim_n X(R/\mathfrak{m}_R^n)$$

The general case follows from an intricate descent argument. □

4 Artin's Axioms

Theorem 4.0.1 (Artin Approximation). Let S be a locally noetherian scheme and a category fibered in groupoids $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$. Let R be a Noetherian complete local ring with residue field k with $\text{Spec}(R) \rightarrow S$ finite type and $x \in \mathcal{X}(R)$. Let $s \in S$ be the image of $\text{Spec}(k) \rightarrow \text{Spec}(R) \rightarrow S$. Assume that,

- (a) $\mathcal{O}_{S,s}$ is a G -ring
- (b) p is limit-preserving on objects.

Then for every $N \geq 1$ there exist,

- (a) a finite type S -algebra A
- (b) a maximal ideal $\mathfrak{m}_A \subset A$
- (c) an object $x_A \in \mathcal{X}(A)$
- (d) an S -isomorphism $R/\mathfrak{m}_R^N \xrightarrow{\sim} A/\mathfrak{m}_A^N$

- (e) an isomorphism $x|_{R/\mathfrak{m}_R^N} \xrightarrow{\sim} x_A|_{A/\mathfrak{m}_A^N}$ over the previous map
- (f) an isomorphism $\mathbf{gr}_{\mathfrak{m}_R}(R) \xrightarrow{\sim} \mathbf{gr}_{\mathfrak{m}_A}(A)$ of graded k -algebras.

Lemma 4.0.2. Let S be a locally noetherian scheme and $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$ a category fibered in groupoids. Let ξ be a formal object of \mathcal{X} with $x_0 = \xi_1$ lying over $\text{Spec}(k) \rightarrow S$ with image $s \in S$ such that,

- (a) ξ is versal
- (b) ξ is effective
- (c) $\mathcal{O}_{S,s}$ is a G -ring
- (d) $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$ is limit-preserving

then there exists a finite type morphism $U \rightarrow S$, a finite type point $u_0 \in U$ with residue field k and $x \in \mathcal{X}(U)$ such that $x : U \rightarrow \mathcal{X}$ is versal at u_0 and $x|_{\text{Spec}(\mathcal{O}_{U,u_0})}$ induces ξ .

Proof. Choose an object $x_R \in \mathcal{X}(R)$ whose completion is ξ . Apply Artin approximation with $N = 2$ to obtain $A, \mathfrak{m}_A, x_A \in \mathcal{X}(A)$ approximating ξ . Let η be the formal object completing $x_A|_{\text{Spec}(\hat{A})}$ (the completion of A at \mathfrak{m}_A). Then a lift for the diagram in $\widehat{\mathcal{F}}_{\mathcal{X},k,x_0}$,

$$\begin{array}{ccc}
 & \eta & \\
 \nearrow & \downarrow & \\
 \xi & \longrightarrow & \xi_2 = \eta_2
 \end{array}
 \quad \text{lying over} \quad
 \begin{array}{ccc}
 & \hat{A} & \\
 \nearrow & \downarrow & \\
 R & \longrightarrow & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2
 \end{array}$$

exists because ξ is versal. Since the map $R \rightarrow \hat{A}$ induces an isomorphism on tangent spaces and by construction $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ we conclude that $R \rightarrow \hat{A}$ is an isomorphism. Hence $\eta \cong \xi$ is versal so the map $x_A : \text{Spec}(A) \rightarrow \mathcal{X}$ is versal at $x_A|_{\widehat{\text{Spec}(\hat{A})}} = \eta$. \square

Theorem 4.0.3. Let S be a locally Noetherian base scheme and consider a category fibered in groupoids $p : \mathcal{X} \rightarrow (\mathbf{Sch}_S)_{\text{fppf}}$. For each finite type morphism $\text{Spec}(k) \rightarrow S$ with k a field and $x_0 \in \mathcal{X}(\text{Spec}(k))$ assume that,

- (a) \mathcal{X} is a stack for the étale topology
- (b) $\Delta_{\mathcal{X}/S} : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces
- (c) \mathcal{X} is limit preserving (preserves filtered colimits)
- (d) \mathcal{X} satisfies the Rim-Schlessinger condition (RS)
- (e) $T\mathcal{F}_{\mathcal{X},k,x_0}$ is finite dimensional for all k and all $x_0 \in \mathcal{F}(k)$
- (f) every formal object of \mathcal{X} is effective
- (g) \mathcal{X} satisfies openness of versality
- (h) $\mathcal{O}_{S,s}$ is a G -ring for all finite type points $s \in S$
- (i) a set theoretic condition

then \mathcal{X} is an algebraic stack.

Proof. It suffices to show that for each finite type $\mathrm{Spec}(k) \rightarrow S$ and $x_0 \in \mathcal{X}(k)$ there is a finite type morphism $U \rightarrow S$ and a smooth map $U \rightarrow \mathcal{X}$ such that there is a finite type point $u_0 : \mathrm{Spec}(k) \rightarrow U$ such that $x|_{u_0} \cong x_0$.

By Rim-Schlessinger $\mathcal{F}_{\mathcal{X},k,x_0}$ admits a versal formal object ξ which is then effective. Artin approximation allows us to approximate an effective formal object by a finite type object $U \rightarrow \mathcal{X}$ which is versal at $u_0 \in U$. By openness of versality, we can shrink U such that $U \rightarrow \mathcal{X}$ is versal at every finite type point.

Finally, prove that a representable morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of limit preserving categories fibered in groupoids which is smooth on deformation categories is smooth (Tag [07XX](#)). Indeed, for $T \rightarrow \mathcal{Y}$ the condition says that $f : \mathcal{X}_T \rightarrow T$ is a formally smooth map of algebraic spaces³ and the limit-preserving condition gives finitely presented. \square

Remark. Usually most difficult to prove openness of versality. There a number of deformation-theoretic techniques for proving this but require effectivity of formal objects over more general formal schemes. There are also tangent-obstruction theory methods for proving openness of versality.

³There is a subtlety here with changing fields that requires the full strength of (RS) where as proving that a versal object exists only requires (S1) and (S2) and finite-dimensionality of tangent spaces