Measure Theory

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2 A First Attempt at Measure Theory

We want to define a function which measures the size of a set. First let us work over $\mathbb{R}$. Then our measure is a map from subsets of the real line to nonegative reals or infinity if our set is infinite in length.

Definition: The domain of a mesure will be in the set,

$$\hat{\mathbb{R}}^{+} = \{x \in \mathbb{R} \mid x \geq 0\} \cup \{\infty\}$$

which has the topology of a closed interval.

Definition: A *measure* is a function $\mu : \mathcal{P}(\mathbb{R}) \to \hat{\mathbb{R}}^{+}$ satisfying,

1. $\mu(\varnothing) = 0$
2. For any countible collection of pairwise disjoint sets $\{E_i\}_{i=1}^{\infty}$ for $E_i \subset \mathbb{R}$ we have additivity,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Lemma 2.1. *Let μ be a measure. If $A \subset B$ then $\mu(A) \leq \mu(B)$.*

Proof. We can write $B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \varnothing$. Then, applying the second property of a measure,

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$$

because $\mu(B \setminus A) \geq 0$ for any set. □

Example 2.1. The following are well-defined measures on all subsets of $\mathbb{R}$:

1. The counting measure is defined by $\mu(()\, S) = \#(S)$ when S is finite and $\mu(S) = \infty$ when S is infinite.
2. The dirac measure δ_a for $a \in \mathbb{R}$ is given by,

$$\delta_a(S) = \mathbb{1}_S(a) = \begin{cases} 1 & a \in S \\ 0 & a \notin S \end{cases}$$

   where $\mathbb{1}_S$ is the indicator function given by,

$$\mathbb{1}_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

3. Let $\{q_i\}$ be a fixed enummeration of the rational numbers $\mathbb{Q}$. Define $\mu_{\mathbb{Q}} : \mathcal{P}(\mathbb{R}) \to \hat{\mathbb{R}}^+$ by,

$$\mu_{\mathbb{Q}}(S) = \sum_{i=1}^{\infty} \frac{\mathbb{1}_S(q_i)}{2^i}$$

Since the sum,

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

converges, the measure $\mu_{\mathbb{Q}}(S) \leq 1$ so it is never infinite. This function is indeed a measure because the measure of a disjoint union gives the sum over all rationals in each piece with is exactly the sum of the measures.

Definition: We say a measure $\mu : \mathcal{P}(\mathbb{R}) \to \hat{\mathbb{R}}^+$ is *translation-invariant* if $\mu(S + x) = \mu(S)$ for any $S \subset \mathbb{R}$ and $x \in R$ where,

$$S + x = \{s + x \mid s \in S\}$$

Example 2.2.

The counting measure is translation-invariant since $S + x$ has the same number of elements as S.

The dirac measure is not translation-invariant since $\delta_a(\{a\}) = 1$ but if $x \neq 0$ then $\delta_a(\{a\} + a) = \delta_a(\{a + x\}) = 0$.

$\mu_{\mathbb{Q}}$ is not translation-invariant because different rational numbers will appear in a shifted interval.

Definition: We say a measure $\mu : \mathcal{P}(\mathbb{R}) \to \hat{\mathbb{R}}^+$ is *interval-length-compatible* if for any real numbers $a < b$ we have $\mu([a, b]) = b - a$. The weaker notion of being *nontrivial on intervals* holds if $\mu([a, b]) \neq 0, \infty$ for all such intervals.

Example 2.3.

The counting measure is trivial on intervals because $\mu([a, b]) = \infty$.

The dirac measure δ_a is trivial on all intervals which do not conatain a.

$\mu_{\mathbb{Q}}$ is nontrivial on intervals since every interval contains a rational number $q_i \in [a, b]$ so $2^{-1} \leq \mu_{\mathbb{Q}}([a, b]) < \infty$.

Remark 2.0.1. None of the examples discussed are both translation-invariant and nontrivial on all intervals. This is not an accident as we will now demonstrate.

Theorem 2.2 (Vitali)**.** *There does not exist a translation-invariant measure on $\mathbb{R}$ which is nontrivial on intervals.*

Proof. We will define an equivalence relation $\sim$ on $\mathbb{R}$ by,

$$x \sim y \iff \exists q \in \mathbb{Q} : x + q = y$$

This equivalence relation measures the "irrational part" of a number. Consider the set of equivalence classes,

$$\mathbb{R}/\mathbb{Q} = \{[x] \mid x \in \mathbb{R}\} \quad \text{where} \quad [x] = \{t \in \mathbb{R} \mid x \sim y\}$$

This is actually a quotient of groups since $[x] = x + \mathbb{Q}$ so we can also write,

$$\mathbb{R}/\mathbb{Q} = \{x + \mathbb{Q} \mid x \in \mathbb{R}\}$$

Now we create a set V by choosing a single element of each equivalence class such that this element lies in $[0, 1]$. That is, if $x \in V$ then $V \cap [x] = \{x\}$ so no element equivalent to x (i.e. differing by a rational from x) can lie in V. Given any choice of a representitive for $[x]$ we can shif by rationals until we land in $[0, 1]$. Constructing V formally requires the axiom of choice but more on this latter.

Now, for $q \in \mathbb{Q} \cap [-1, 1] = \mathbb{Q}_1$ consider the sets $V + q$. Given any $x \in [-1, 1]$ we know that there exists some $y \in [x] \cap V$ with $y \in [0, 1]$. Thus, $x - y \in \mathbb{Q}$ since $x \sim y$ and $x - y \in [-1, 1]$ since $x, y \in [0, 1]$. Thus, $x = y + q$ for some $q \in \mathbb{Q} \cap [-1, 1]$. However, $y \in V$ so $x \in V + q$. But furthermore, if $x \in V$ then $x \in [0, 1]$ so $x + q \in [-1, 2]$ for $q \in \mathbb{Q} \cap [-1, 1]$. Therefore,

$$[0, 1] \subset \bigcup_{q \in \mathbb{Q}_1} V + q \subset [-1, 2]$$

Finally, let $\mu : \mathcal{P}(\mathbb{R}) \to \hat{\mathbb{R}}^+$ be a translation-invation measure on $\mathbb{R}$ which is nontrivial on intervals. Applying this measure,

$$\mu([0, 1]) \leq \mu\left(\bigcup_{q \in \mathbb{Q}_1} V + q\right) \leq \mu([-1, 2])$$

However, if $q \neq q'$ then $V + q$ and $V + q'$ are disjoint because if $x \in V + q$ and $x \in V + q'$ then we would have $x - q, x - q' \in V$ but $(x - q) + (q - q') = x - q'$ so these must lie in the same equivalence class and thus $x - q = x - q'$ so $q = q'$ since there is exactly one element from each equivalence class in V. Furthermore, since $\mathbb{Q}$ is countible $\mathbb{Q}_1 = \mathbb{Q} \cap [-1, 1]$ is also a countible index set. Therefore, since μ is a measure, it is additive over countible collections of disjoint set so we have,

$$\mu\left(\bigcup_{q \in \mathbb{Q}_1} V + q\right) = \sum_{q \in \mathbb{Q}_1} \mu(V + q)$$

Furthermore, μ is translation invariant so,

$$\mu(V + q) = \mu(V)$$

Therefore,

$$\mu\left(\bigcup_{q\in\mathbb{Q}_1} V+q\right) = \sum_{q\in\mathbb{Q}_1}\mu\left(V\right)$$

Plugging into the innequality,

$$\mu\left([0,1]\right) \le \sum_{q\in\mathbb{Q}_1}\mu\left(V\right) \le \mu\left([-1,2]\right)$$

Finally, because μ is nontrivial on intervals we know that $\mu\left([0,1]\right)$ and $\mu\left([-1,2]\right)$ are positive real numbers (not ∞). This is the desired contradiction because,

$$\sum_{q\in\mathbb{Q}_1}\mu\left(V\right) = \mu\left(V\right)\sum_{q\in\mathbb{Q}_1} 1 = \begin{cases}\infty & \mu\left(V\right)\neq 0\\ 0 & \mu\left(V\right) = 0\end{cases}$$

so this value cannot possibly fit in the innequality between two positive real numbers. □

Remark 2.0.2. The axiom of choice is a somewhat controversial axiom of set theory which states that given any collection of nonempty sets there exists a set which contains exactly one element from each set in the collection. Applying this axiom to $\mathbb{R}/\mathbb{Q}$ gives us a Vitali set V. We can write this axiom in formal logic as,

$$\forall X[\varnothing \notin X \implies \exists f : X \to \bigcup X \quad \forall A \in X : f(A) \in A]$$

which states that there exists a choice function taking a set A and choosing some element $f(A) \in A$.

Remark 2.0.3. This is a devestating result. We certainally wanted any candidate length function to be a translation-invariant measure which respects the lengths of intervals. Vitali showed that this is impossible. We will discuss how the modern theory circumvents this difficulty in the following section.

3 Sigma Algebras and Measure Spaces

Definition: An *outer-measure* is a function $\mu^* : \mathcal{P}\left(X\right) \to \hat{\mathbb{R}}^+$ satisfying,

1. $\mu^*\left(\varnothing\right) = 0$
2. For any subsets $A, B \subset X$ we have,

$$A \subset B \implies \mu^*\left(A\right) \le \mu^*\left(B\right)$$

3. For any countible collection of pairwise disjoint sets $\{E_i\}_{i=1}^{\infty}$ for $E_i \subset X$ we have subadditivity,

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty}\mu^*\left(E_i\right)$$

Definition: The *Lebesgue outer-measure* $\mu^* : \mathcal{P}(\mathbb{R}) \to \hat{\mathbb{R}}^+$ is defined as follows. Let I denote an open interval of the form $I = (a, b)$ and $\ell(I) = b - a$ its canonical length. Then for $S \subset \mathbb{R}$ we set,

$$\mu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; \{I_k\}_{k\in\mathbb{N}} \text{ is a cover of } E \text{ by open intervals i.e. } E \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

Proposition. The Lebesgue outer-measure defined above satisfies the outer-measure axioms.

Proof. □

Remark 3.0.1. The concept of an outer-measure will allow us to define the space of measureable sets. We first need to know what kind of space this will be.

Definition: A *σ-algebra* on X is a collection $\Sigma \subset X$ of subsets of X satisfying,

1. $X \in \Sigma$ and $\varnothing \in \Sigma$
2. If $E \in \Sigma$ then $E^c = X \setminus E \in \Sigma$.
3. or any countible collection of pairwise disjoint sets $\{E_i\}_{i=1}^{\infty}$ for $E_i \in \Sigma$ then,

$$\bigcup_{i=1}^{\infty} E_i \in \Sigma$$

By taking the compliment of the union of the compliments we also get coutible intersections i.e.

$$\bigcap_{i=1}^{\infty} E_i \in \Sigma$$

We call the pair (X, Σ) a measureable space.

Definition: Let (X, Σ_X) and (Y, Σ_Y) be measureable spaces. A function $f : X \to Y$ is called *measureable* if for any Y-measurable set $E \in \Sigma_Y$ its pre-image is X-measureable i.e. $f^{-1}(E) \in \Sigma_X$.

Remark 3.0.2. We now have the tools to give a correct modern definition of a measure.

Definition: Let (X, Σ) be a measureable space i.e. Σ is a σ-algebra on X. Then a *measure* on (X, Σ) is a function $\mu : \Sigma \to \hat{\mathbb{R}}^+$ satisfying,

1. $\mu(\varnothing) = 0$
2. For any countible collection of pairwise disjoint sets $\{E_i\}_{i=1}^{\infty}$ for $E_i \in \Sigma$ we have additivity,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

We call the triple (X, Σ, μ) a *measure* space.

Definition: A measure space (X, Σ, μ) is *complete* if for any $E \in \Sigma$ such that $\mu(E) = 0$ and any $S \subset E$ we have $S \in \Sigma$.

Definition: Let $\mu^* : \mathcal{P}(X) \to \hat{\mathbb{R}}^+$ be an outer-measure. We say that $E \subset X$ is *measureable* if for any $A \subset X$ we have,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Lemma 3.1. *If $E_1, E_2 \subset X$ are μ^*-measurable then $E_1 \cup E_2$ is also μ^*-measurable.*

Proof. If $E_1, E_2 \in \Sigma$ then,

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$$

for any A. Furthermore, taking $A \cap E_1^c$ as the arbitrary subset and applying the measurability of of E_2,

$$\mu^*(A \cap E_1^c) = \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c)$$

Furthermore, we can split the set $A \cap (E_1 \cup E_2)$ as the union of $A \cap E_1$ and $A \cap E_1^c \cap E_2$. By subadditivity,

$$\mu^*(A \cap (E_1 \cup E_2)) \leq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2)$$

Combining these results,

$$\begin{aligned}\mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1^c \cap E_2^c)) &\leq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap (E_1^c \cap E_2^c)) \\ &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) = \mu^*(A)\end{aligned}$$

However, A can be decomposed as the disjoint union of $A \cap (E_1 \cup E_2)$ and $A \cap (E_1^c \cap E_2^c)$ so by subadditivity,

$$\mu^*(A) \leq \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1^c \cap E_1^c))$$

Therefore,

$$\mu^*(A) = \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1^c \cap E_1^c))$$

for any set A. Thus, $E_1 \cup E_2 \in \Sigma$ is measureable. ∎

Lemma 3.2. *If $\{E_i\}_{i=1}^{\infty}$ is a countible increasing collection of μ^*-measureable sets then, for any set $A \subset X$,*

$$\mu^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu^*(A \cap E_n)$$

Proof. Define,

$$E = \bigcup_{i=1}^{\infty} E_i$$

By monotonicity,

$$\mu^*(A \cap E_n) \le \mu^*(A \cap E) \implies \lim_{n\to\infty} \mu^*(A \cap E_n) \le \mu^*(A \cap E)$$

We can write,

$$A \cap E = \bigcup_{i=1}^{\infty} A \cap E_i = \bigcup_{i=0}^{\infty} A \cap E_{i+1} \cap E_i^c$$

since $E_{i+1} \supset E_i$ this is a disjoint union since if $i < j$ then $E_{j+1} \cap E_j^c$ is disjoint from $E_j \supset E_i$. Applying subadditivity,

$$\mu^*(A \cap E) \le \sum_{i=0}^{\infty} \mu^*(A \cap E_{i+1} \cap E_i^c)$$

Since E_i is μ^*-measureable then taking $A \cap E_{i+1}$,

$$\mu^*(A \cap E_{i+1}) = \mu^*(A \cap E_{i+1} \cap E_i) + \mu^*(A \cap E_{i+1} \cap E_i^c)$$

with $E_0 = \varnothing$. Thus,

$$\begin{aligned}\mu^*(A \cap E) &\le \sum_{i=0}^{\infty} \mu^*(A \cap E_{i+1} \cap E_i^c) = \sum_{i=0}^{\infty} \left[\mu^*(A \cap E_{i+1}) - \mu^*(A \cap E_{i+1} \cap E_i)\right] \\ &= \sum_{i=0}^{\infty} \left[\mu^*(A \cap E_{i+1}) - \mu^*(A \cap E_i)\right] = \lim_{n\to\infty} \mu^*(A \cap E_n) - \mu^*(A \cap E_0) = \lim_{n\to\infty} \mu^*(A \cap E_n)\end{aligned}$$

because,

$$\mu^*(A \cap E_0) = \mu^*(A \cap \varnothing) = 0$$

Therefore,

$$\mu^*(A \cap E) = \lim_{n\to\infty} \mu^*(A \cap E_n)$$

□

Theorem 3.3. *The collection of μ^*-measureable sets Σ_μ is a σ-algebra on X and μ, the restiction of μ^* to Σ_μ, makes (X, Σ_μ, μ) a complete measure space.*

Proof. If $E = X$ or $E = \varnothing$ then clearly,

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A) + \mu^*(\varnothing) = \mu^*(A)$$

so $X, \varnothing \in \Sigma_\mu$. Furthermore $E \in \Sigma_\mu$ if and only if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for each $A \subset X$. So clearly $E \in \Sigma_\mu \iff E^c \in \Sigma$. We have shown that Σ_μ contains finite unions. Taking $A = E_1$ with disjoint $E_1, E_2 \in \Sigma_\mu$ gives,

$$\mu^*(E_1 \cup E_2) = \mu^*((E_1 \cup E_2) \cap E_1) + \mu^*((E_1 \cup E_2) \cap E_1^c) = \mu^*(E_1) + \mu^*(E_2)$$

so we have finite additivity on Σ_μ. If we have a countible collection of pairwise disjoint sets $\{E_i\}_{i=1}^\infty$ for $E_i \in \Sigma_\mu$. We have shown that the unions,

$$T_n = \bigcup_{i=1}^n E_n \in \Sigma_\mu$$

are measureable. Then,

$$\mu^*(A) = \mu^*(A \cap T_n) + \mu^*(A \cap T_n^c)$$

Furthermore, define,

$$E = \bigcup_{i=1}^\infty E_i$$

and then,

$$A \cap E^c \subset A \cap T_n^c$$

so we have,

$$\mu^*(A \cap E^c) \le \mu^*(A \cap T_n^c)$$

Thus,

$$\mu^*(A) \ge \mu^*(A \cap T_n) + \mu^*(A \cap E^c)$$

which implies, via Lemma 3.2, that

$$\mu^*(A) \ge \lim_{n\to\infty} \mu^*(A \cap T_n) + \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Finally, by subadditivty,

$$\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

and therefore,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

So $E \in \Sigma_\mu$. Therefore Σ_μ is a σ-algebra. Furthermore, if $E \in \Sigma_\mu$ with $\mu^*(E) = 0$ and take $S \subset E$ then for any $A \subset X$ using monotonicity we have,

$$\mu^*(A \cap S^c) \le \mu^*(A)$$

and also,

$$\mu^*(A \cap S) \le \mu^*(A \cap E) \le \mu^*(E) = 0$$

Thus,

$$\mu^*(A \cap S^c) + \mu^*(A \cap S) \le \mu^*(A)$$

and also, by subadditivity,

$$\mu^*(A) \leq \mu^*(A \cap S) + \mu^*(A \cap S^c)$$

Thus,

$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c)$$

so $S \in \Sigma_\mu$. Finally, we have,

$$\mu^*(T_n) = \sum_{i=1}^{n} \mu^*(E_i)$$

but finite additivity. Thus,

$$\mu^*(E) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu^*(E_i) = \sum_{i=1}^{\infty} E_i$$

Therefore, (X, Σ_μ, μ^*) is a complete measure space. □

Definition: A σ-algeba Σ on a topological space X is called *Borel* if Σ contains every open set of X. If Σ is Borel then we say that the measureable space (X, Σ) is a Borel space and any measure on (X, Σ) is a Borel measure. Furthermore, *the Borel algebra* $\mathfrak{B}(X)$ is the intersection of all Borel σ-algebras on X so $\mathfrak{B}(X)$ is the minimal σ-algebra containing all open and thus all closed sets of X.

Theorem 3.4. *The σ-algebra of Lebesgue-measurable sets $\Sigma_{\mathcal{L}}$ is Borel over $\mathbb{R}$.*

Proof. □

Theorem 3.5. *The Lebesgue measure on $(X, \Sigma_{\mathcal{L}})$ is a translation-invariant measure which is nontrivial on intervals.*

Proof. □

Remark 3.0.3. We can generalize the Lebesgue measure to $\mathbb{R}^n$ for arbitrary dimensions by,

$$\mu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; \{I_k\}_{k\in\mathbb{N}} \text{ is a cover of } E \text{ by open intervals i.e. } E \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

where I_k is a primitive open set $[a_1, b_1] \times \cdots \times [a_n, b_n]$ and

$$\ell(I_k) = (b_1 - a_1) \cdots (b_n - a_n)$$

is the canonical volume.

4 Haar Measures

5 Probability Theory

6 Lebesgue Integration

7 Hausdorff Measures

8 Banach-Tarski