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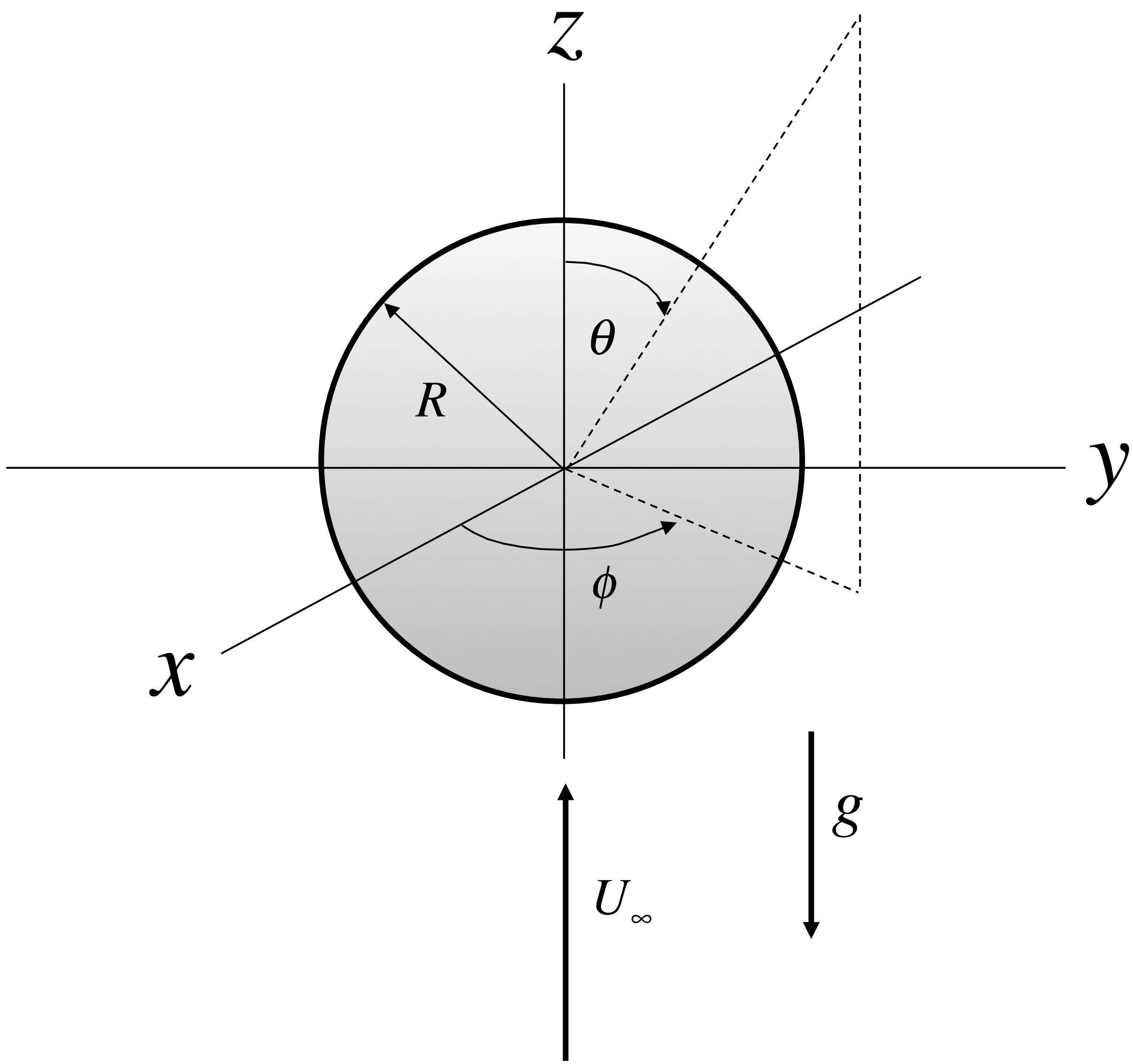
Viscous flow past a sphere at low Reynolds number



The Stokes stream function satisfies the biharmonic equation

$$\nabla^2\left(\nabla^2\Psi\right)=0$$

In spherical polar coordinates

$$\left(\frac{\partial^2}{\partial r^2}+\frac{Sin(\theta)}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{Sin(\theta)}\frac{\partial}{\partial\theta}\right)\right)^2\Psi=0$$

Velocities

$$U_r=-\frac{1}{r^2Sin(\theta)}\frac{\partial\Psi}{\partial\theta}$$

$$U_\theta=\frac{1}{rSin(\theta)}\frac{\partial\Psi}{\partial r}$$

Boundary conditions

No-slip condition

$$U_r(R,\theta)=0$$

$$U_\theta(R,\theta)=0$$

Uniform flow at infinity

$$\lim_{r\to\infty}\Psi\to-\frac{1}{2}U_\infty r^2Sin^2(\theta)$$

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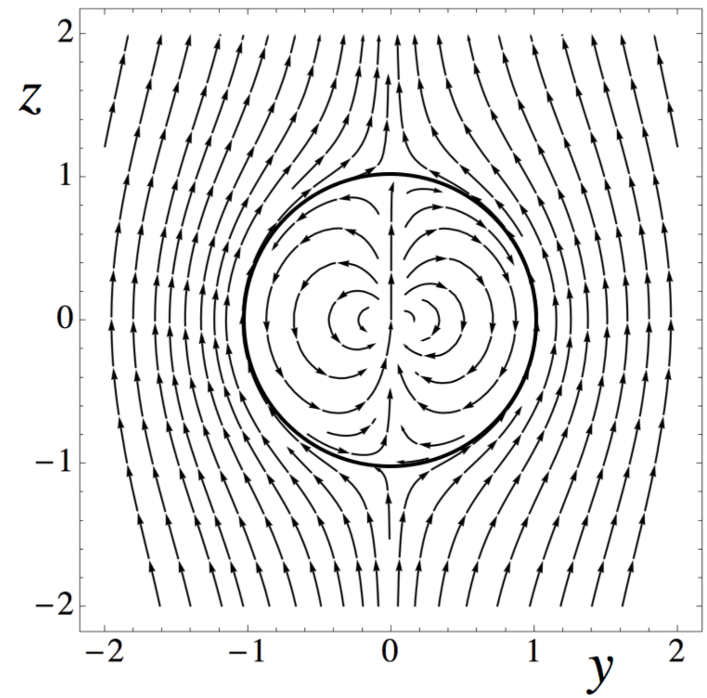
Assume $$\Psi = f(r)\,Sin^2(\theta)$$

$$f(r) = \frac{a}{r} + br + cr^2 + dr^4$$

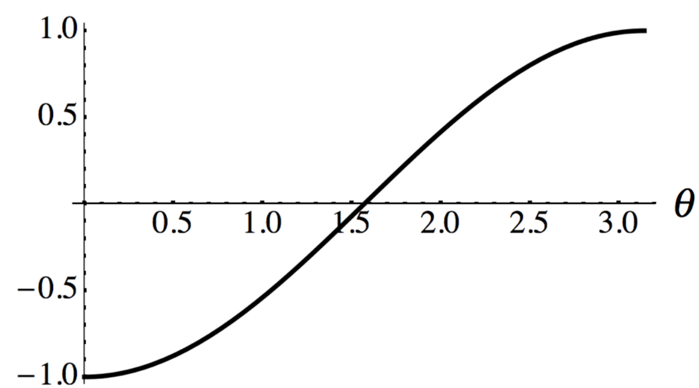


Solution
$$\frac{\Psi}{U_\infty R^2} = \left(-\frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{3}{4}\left(\frac{r}{R}\right) - \frac{1}{4}\left(\frac{r}{R}\right)^{-1}\right) Sin^2(\theta)$$

$$\frac{U_r}{U_\infty} = \left(1 - \frac{3}{2}\left(\frac{r}{R}\right)^{-1} + \frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right) Cos(\theta)$$

$$\frac{U_\theta}{U_\infty} = \left(-1 + \frac{3}{4}\left(\frac{r}{R}\right)^{-1} + \frac{1}{4}\left(\frac{r}{R}\right)^{-3}\right) Sin(\theta)$$

Viscous stress
$$\frac{\tau_{r\theta} R}{\mu U_\infty} = \frac{3}{2}\left(\frac{r}{R}\right)^{-4} Sin(\theta)$$



Pressure
$$\frac{(P - P_\infty)R}{\mu U_\infty} = -\frac{\rho g R^2}{\mu U_\infty}\left(\frac{z}{R}\right) - \frac{3}{2}\left(\frac{r}{R}\right)^2 Cos(\theta)$$

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Drag components

$$F_{z_{Pressure}} = -\int_0^{2\pi}\int_0^{\pi}\left(P\left(R,\theta\right)Cos\left(\theta\right)\right)R^2 Sin\left(\theta\right)d\theta d\phi = \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R U_\infty$$

$$F_{z_{Viscous}} = \int_0^{2\pi}\int_0^{\pi}\left(\tau_{r\theta}\left(R,\theta\right)Sin\left(\theta\right)\right)R^2 Sin\left(\theta\right)d\theta d\phi = 4\pi\mu R U_\infty$$

$$F_z = \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R U_\infty + 4\pi\mu R U_\infty$$

Buoyancy force

Pressure drag

Viscous drag

$$D_{Stokes} = 6\pi\mu R U_{\infty}$$

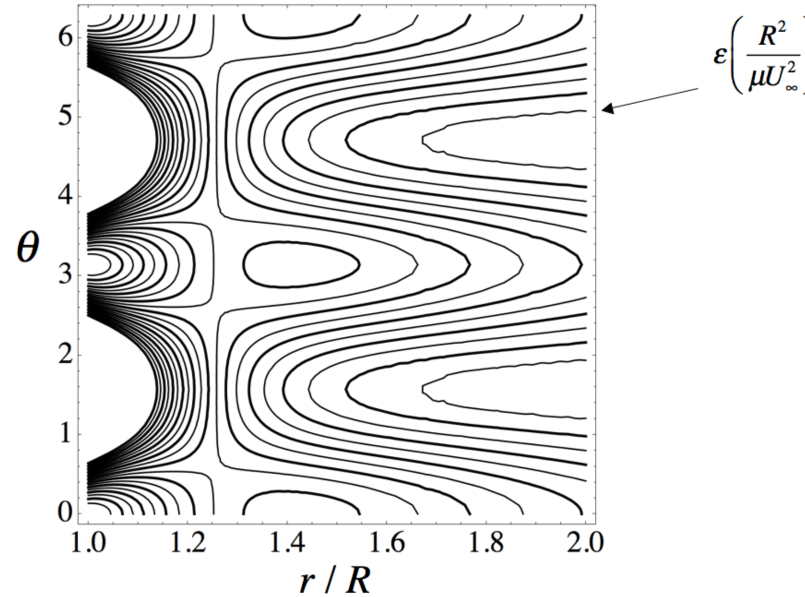
Reynolds number

$$R_e = \frac{\rho U_{\infty}(2R)}{\mu}$$

$$C_D = \frac{D_{Stokes}}{\frac{1}{2}\rho U_{\infty}^2(\pi R^2)} = \frac{12\pi\mu R U_{\infty}}{\rho U_{\infty}^2(\pi R^2)} = \frac{12\mu}{\rho U_{\infty} R} = \frac{24}{R_e}$$

Dissipation of kinetic energy

$$\varepsilon = \tau_{ij}\frac{\partial U_i}{\partial x_j}$$

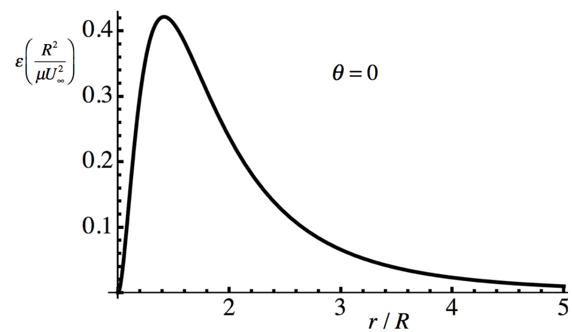


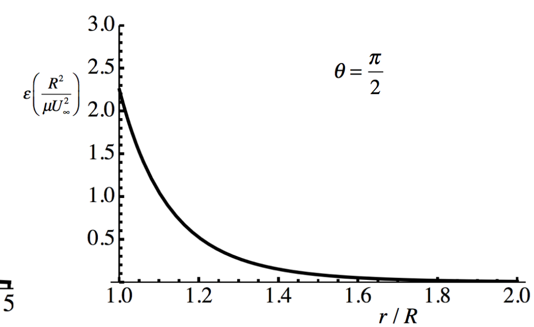
$$\frac{\varepsilon}{\mu} = 2\left(\frac{\partial U_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right)^2 +$$

$$2\left(\frac{U_r}{r} + \frac{U_\theta}{r}Cot(\theta)\right)^2 + \left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)^2$$

$$\frac{\varepsilon R^2}{\mu U_\infty^2} = \frac{9}{4}\left(\frac{r}{R}\right)^{-8}\left(3\left(\left(\frac{r}{R}\right)^2 - 1\right)^2 Cos^2(\theta) + Sin^2(\theta)\right)$$

Integrate the dissipation over the volume

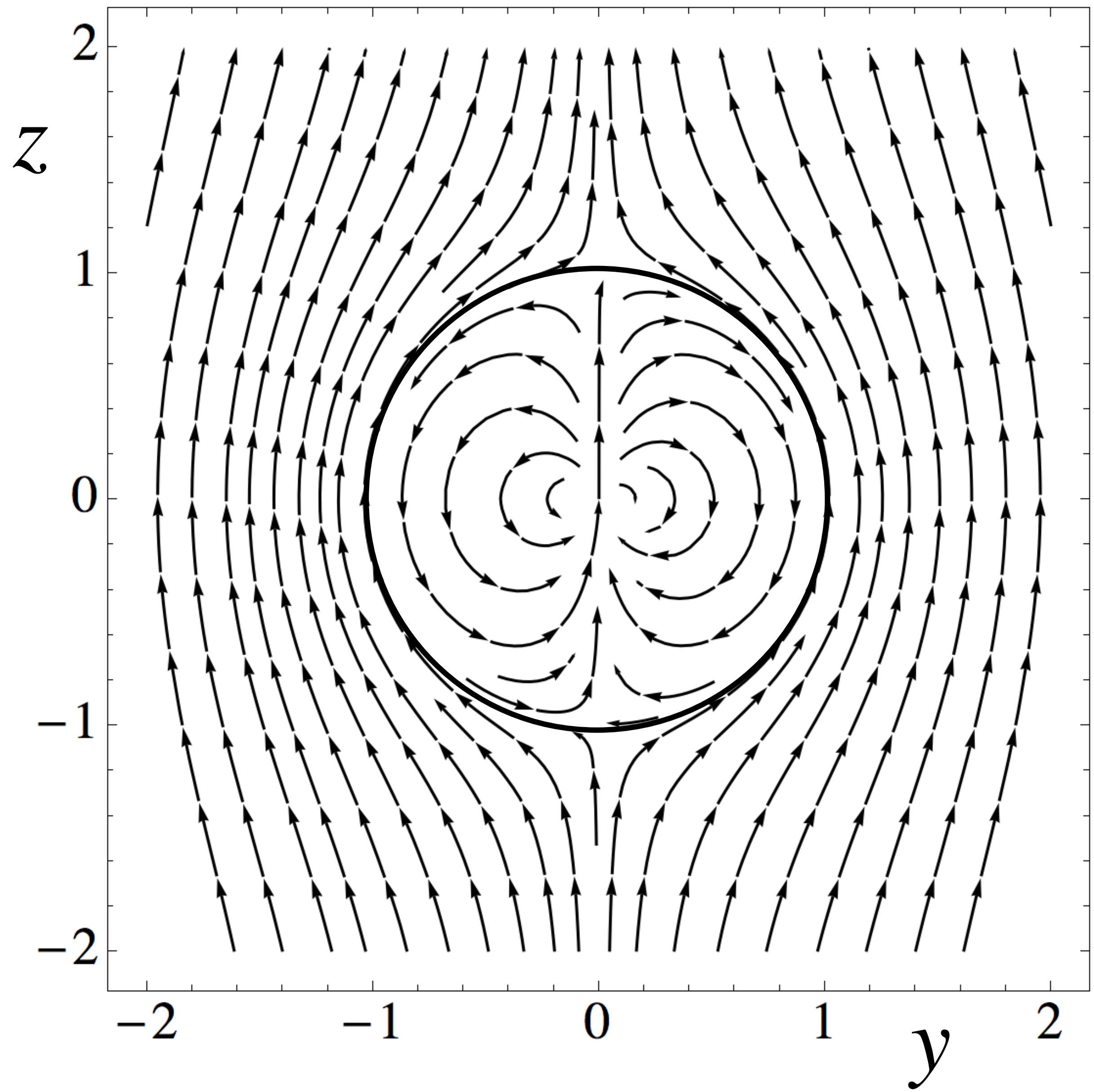


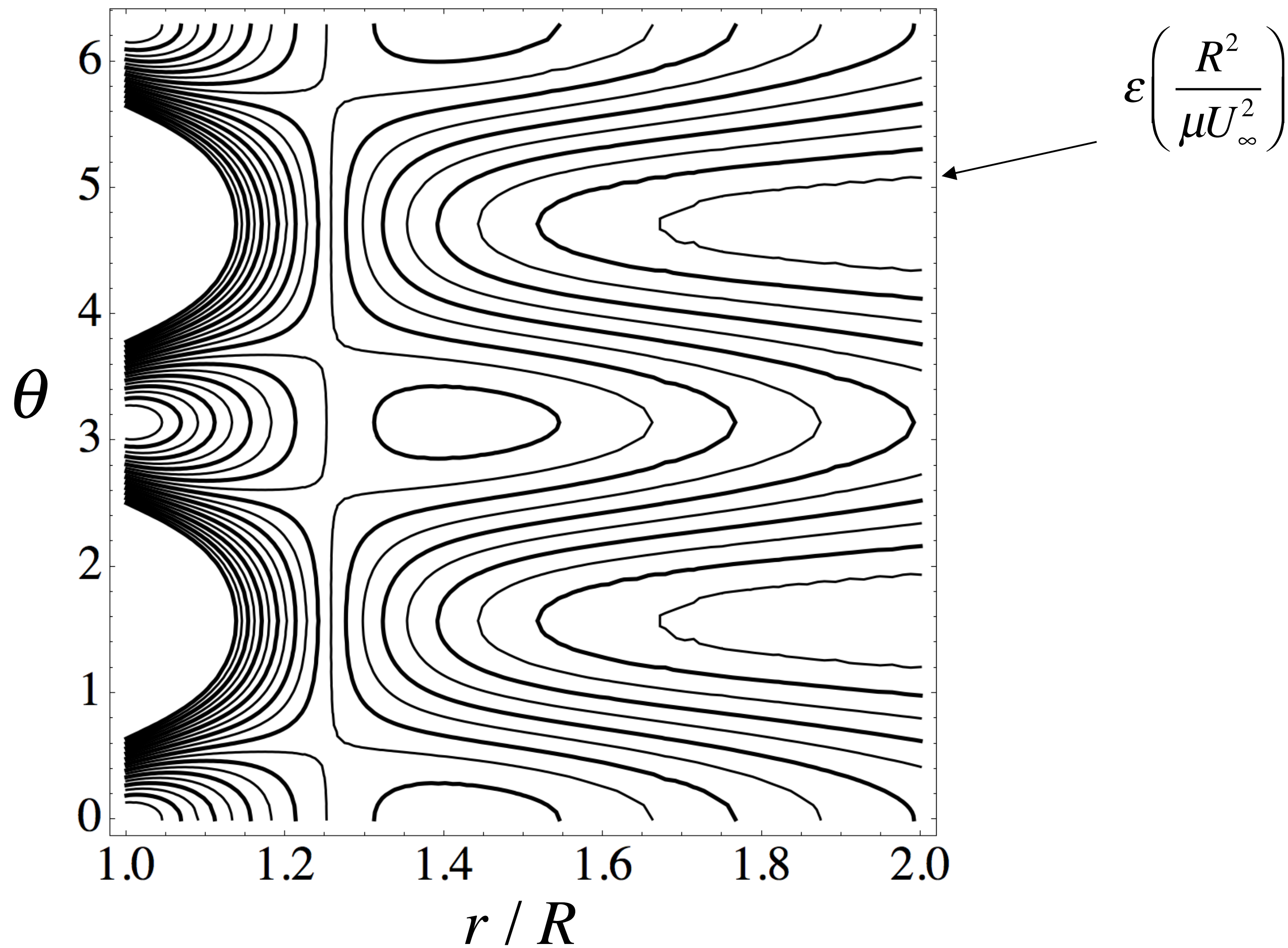


$$\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\varepsilon r^2 Sin(\theta)\,dr\,d\theta\,d\phi =$$

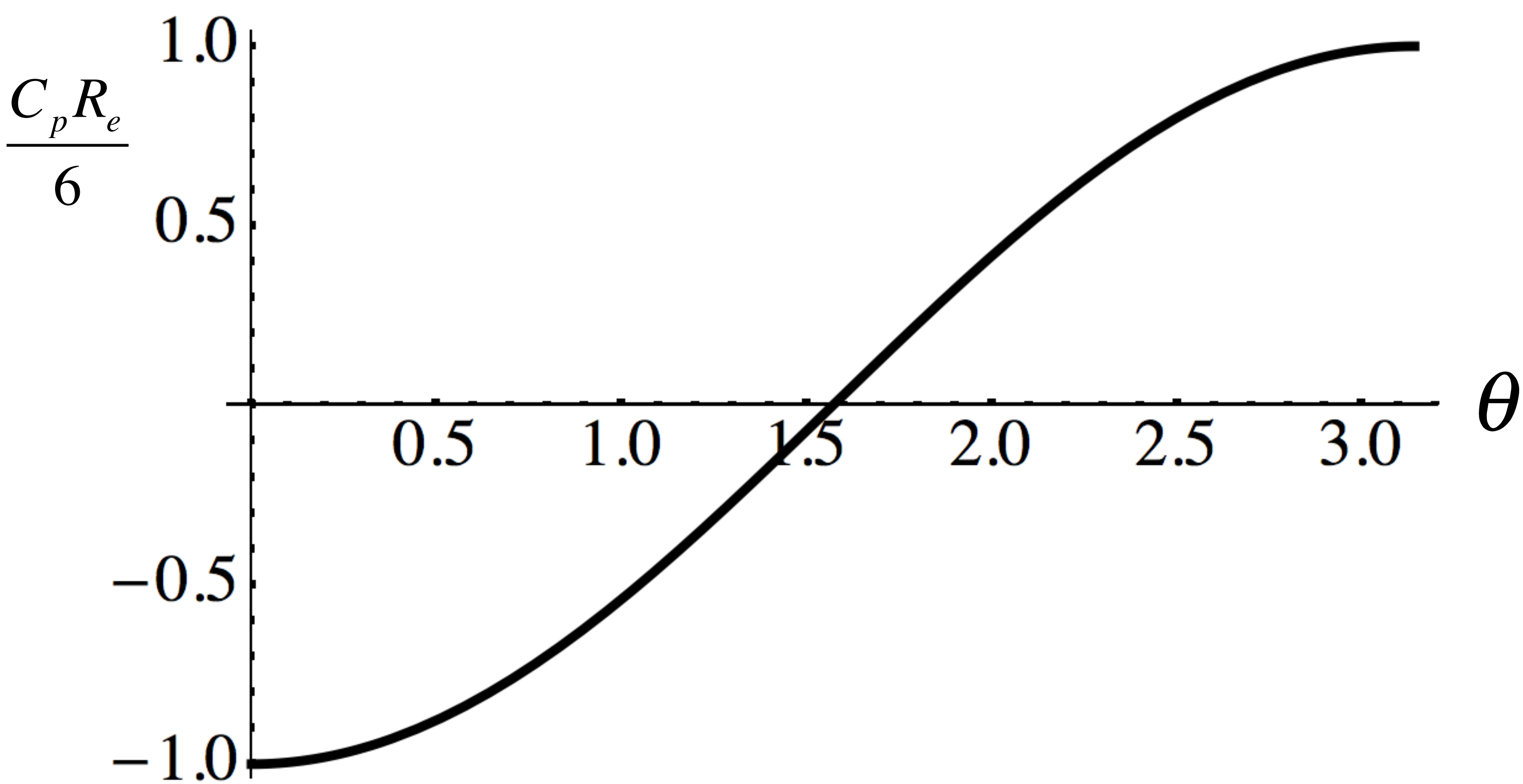
$$\frac{9}{4}\mu U_\infty^2 R\int_0^{2\pi}\int_0^{\pi}\int_1^{\infty}\left(\frac{r}{R}\right)^{-8}\left(3\left(\left(\frac{r}{R}\right)^2 - 1\right)^2 Cos^2(\theta) + Sin^2(\theta)\right)\left(\frac{r}{R}\right)^2 Sin(\theta)\,d\left(\frac{r}{R}\right)d\theta\,d\phi =$$

$$(6\pi\mu U_\infty R)U_\infty = D_{Stokes}U_\infty$$

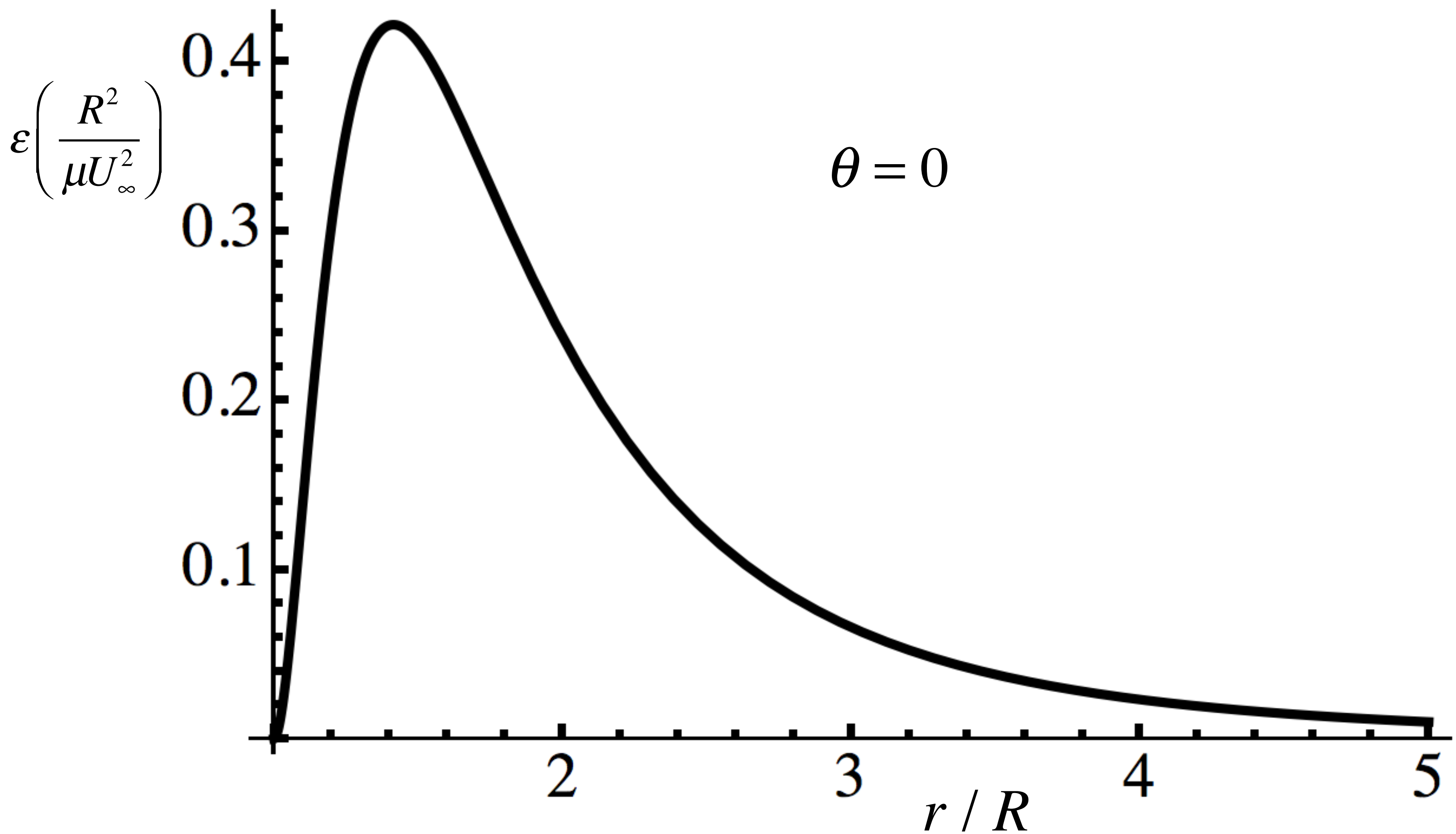

z
2
1
0
−1
−2
−2
−1
0
1
2
y



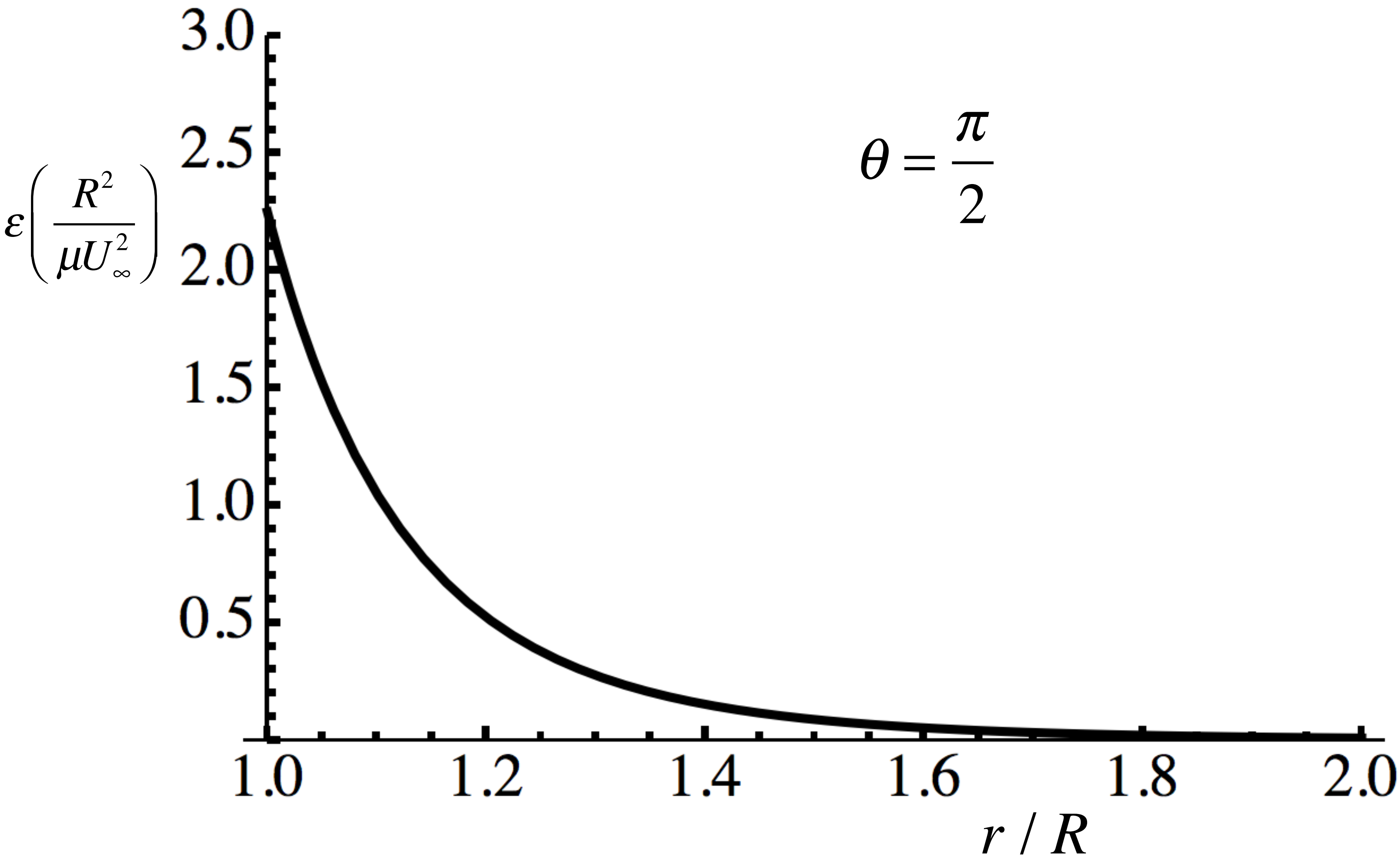
0
1
2
3
4
5
6
θ
1.0
1.2
1.4
1.6
1.8
2.0
r / R
$\varepsilon\left(\frac{R^2}{\mu U_\infty^2}\right)$



1.0
$\frac{C_p R_e}{6}$
0.5
−0.5
−1.0
0.5
1.0
1.5
2.0
2.5
3.0
θ



0.4
0.3
0.2
0.1
$\varepsilon\left(\frac{R^2}{\mu U_\infty^2}\right)$
$\theta = 0$
2
3
4
5
r / R


3.0
2.5
2.0
1.5
1.0
0.5
$\varepsilon\left(\frac{R^2}{\mu U_\infty^2}\right)$
$\theta = \frac{\pi}{2}$
1.0
1.2
1.4
1.6
1.8
2.0
$r\,/\,R$