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# AA200 Applied Aerodynamics

## Chapter 10 - Elements of potential flow and the concept of a vortex stick

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# 10.1 Incompressible flow

*Continuity*

$$\nabla \bullet \bar{U} = 0 \qquad (10.1)$$

*Momentum*

$$\frac{\partial \bar{U}}{\partial t} + \nabla \bullet \left( \bar{U}\bar{U} + \frac{P}{\rho}\bar{\bar{I}} \right) = \nu \nabla^2 \bar{U} \qquad (10.2)$$

The convective term can be rearranged using $\nabla \bullet \bar{U} = 0$ and the identity

$$\bar{U} \bullet \nabla \bar{U} = \left( \nabla \times \bar{U} \right) \times \bar{U} + \nabla \left( \frac{\bar{U} \bullet \bar{U}}{2} \right) \qquad (10.3)$$

The viscous term in (10.2) can be rearranged using the identity

$$\nabla \times \left( \nabla \times \bar{U} \right) = \nabla \left( \nabla \bullet \bar{U} \right) - \nabla^2 \bar{U} \qquad (10.4)$$

Using these results and $\nabla \bullet \bar{U} = 0$ the momentum equation can be written in terms of the vorticity.

$$\bar{\Omega} = \nabla \times \bar{U} \qquad (10.5)$$

in the form

$$\frac{\partial \bar{U}}{\partial t} + \bar{\Omega} \times \bar{U} + \nabla \left( \frac{P}{\rho} + \frac{\bar{U} \bullet \bar{U}}{2} \right) + \nu \nabla \times \bar{\Omega} = 0 \qquad (10.6)$$

If the flow is irrotational, $\bar{\Omega} = 0$, the velocity can be expressed in terms of a velocity potential.

$$\bar{U} = \nabla \Phi \tag{10.7}$$

The continuity equation becomes Laplace's equation

$$\nabla \bullet \bar{U} = \nabla \bullet \nabla \Phi = \nabla^2 \Phi = 0 \tag{10.8}$$

and the momentum equation is fully integrable.

$$\nabla \left( \frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{\bar{U} \bullet \bar{U}}{2} \right) = 0 \tag{10.9}$$

The quantity in parentheses is at most a function of time

$$\frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{\bar{U} \bullet \bar{U}}{2} = f(t) \tag{10.10}$$

The expression (10.10) is called the Bernoulli integral and can be used to determine the pressure throughout the flow once the velocity potential is known from a solution of Laplace's equation (10.7).

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# 10.2 Potentials

If $\Phi_1$ and $\Phi_2$ are solutions of Laplace's equation then so is

$$\Phi_3 = \Phi_1 + \Phi_2$$

The linearity of Laplace's equation allows solutions to be constructed from the superposition of simpler, elementary, solutions. This is the key feature of the equation that makes it a powerful tool for analyzing fluid flows. In this approach the requirement that the flow be divergence free and curl free everywhere is relaxed to permit isolated regions to exist within the flow where mass and vorticity can be created.

One can view an unsteady, incompressible flow as a field constructed from a scalar distribution of mass sources, $Q(\bar{x},t)$ and a vector distribution of vorticity sources, $\bar{\Omega}(\bar{x},t)$. In this approach the velocity field is generated from the linear superposition of two fields.

$$\bar{U} = \bar{U}_{sources} + \bar{U}_{vortices} \tag{10.13}$$

The velocity field generated by the mass sources is irrotational and that generated by the vorticity sources is divergence free. The continuity equation for such a flow now has a source term.

$$\nabla \bullet \bar{U} = \nabla \bullet \bar{U}_{sources} = Q(\bar{x},t) \tag{10.14}$$

The curl of the velocity is

$$\nabla \times \bar{U} = \nabla \times \bar{U}_{vortices} = \bar{\Omega}(\bar{x},t) \tag{10.15}$$

The velocity field is constructed from the superposition of the velocities generated by a scalar potential $\Phi$ generated by the mass sources and a vector potential $\bar{A}$ generated by the vorticity sources.

$$\bar{U} = \nabla\Phi + \nabla \times \bar{A} \tag{10.16}$$

The potentials satisfy a system of Poisson equations, a single equation for the scalar potential

$$\nabla \bullet \nabla\Phi = \nabla^2\Phi = Q(\bar{x}, t) \tag{10.17}$$

and three equations for the Cartesian components of the vector potential.

$$\nabla^2 \bar{A} = -\bar{\Omega}(\bar{x}, t) \tag{10.18}$$

# 10.4 Point source solution of Laplace's equation

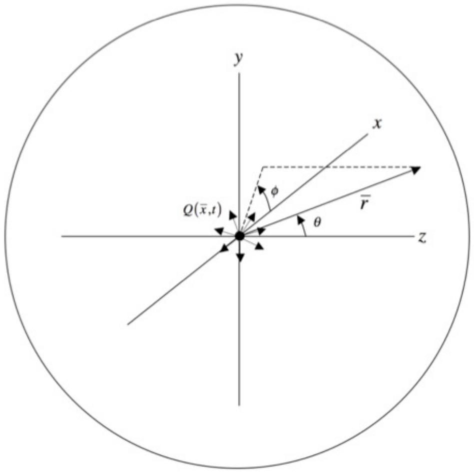


*Figure 10.2 Mass source at the origin.*

$$\Phi(r,t) = -\frac{Q(t)}{4\pi\rho r}$$

Radial velocity

$$U_r = \frac{\partial\Phi}{\partial r} = \frac{Q(t)}{4\pi\rho r^2}$$

Integrate over any closed surface surrounding the source

$$\int_0^{2\pi}\int_0^{\pi} U_r r^2 Sin(\theta)\, d\theta d\phi = \frac{Q(t)}{\rho}$$

Potential at $\bar{x}$ due to a source at $\bar{x}_s$

$$\Phi(\bar{x},\bar{x}_s,t) = -\frac{Q(t)}{4\pi\rho\left|\bar{x}-\bar{x}_s\right|} \qquad (10.34)$$

# Potential generated by a distribution of sources – vortex sticks

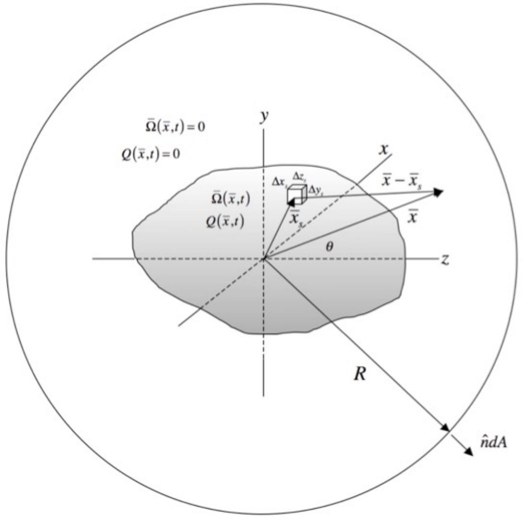


Figure 10.3 Smooth, finite distribution of mass and vorticity sources near the origin.

$$\nabla^2\Phi = Q(\bar{x},t)$$

$$\nabla^2\bar{A} = -\Omega(\bar{x},t)$$

Scalar potential

$$d\Phi = -\frac{Q(\bar{x}_s,t)}{4\pi\rho|\bar{x}-\bar{x}_s|}dx_s dy_s dz_s \qquad (10.37)$$

$$\Phi(\bar{x},t) = -\frac{1}{4\pi\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Q(\bar{x}_s,t)}{|\bar{x}-\bar{x}_s|}dx_s\,dy_s\,dz_s \qquad (10.38)$$

Vector potential

$$d\bar{A} = \frac{\bar{\Omega}(\bar{x}_s,t)dx_s dy_s dz_s}{4\pi|\bar{x}-\bar{x}_s|} \qquad (10.41)$$

$$\bar{A}(\bar{x},t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{\Omega}(\bar{x}_s,t)}{|\bar{x}-\bar{x}_s|}dx_s\,dy_s\,dz_s \qquad (10.42)$$

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# Example - Scalar potential generated by a line distribution of sources

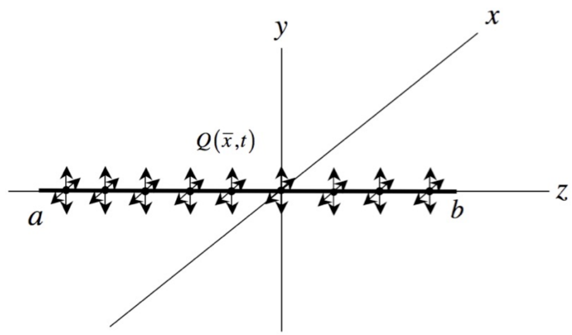


*Figure 10.4 Finite line distribution of mass sources*

Source distribution

$$\frac{Q(\bar{x},t)}{\rho} = \dot{S}(t)\delta(x)\delta(y)u(b-z)u(z-a) \tag{10.43}$$

Units of $\dot{S}$ = Area/Sec

Integrate

$$\Phi = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\dot{S}(t)\delta(x_s)\delta(y_s)u(b-z_s)u(z_s-a)}{\left((x-x_s)^2+(y-y_s)^2+(z-z_z)^2\right)^{1/2}}dx_s\,dy_s\,dz_s =$$

$$-\frac{1}{4\pi}\int_a^b \frac{\dot{S}(t)}{\left(x^2+y^2+(z-z_z)^2\right)^{1/2}}dz_s = \tag{10.44}$$

$$\frac{\dot{S}(t)}{4\pi}Ln\left(2\left(z-z_s+\sqrt{x^2+y^2+(z-z_z)^2}\right)\right)\Bigg|_a^b$$

Potential

$$\Phi(x,y,z,t;a,b) = \frac{\dot{S}(t)}{4\pi}Ln\left(\frac{z-b+\sqrt{x^2+y^2+(z-b)^2}}{z-a+\sqrt{x^2+y^2+(z-a)^2}}\right) \tag{10.45}$$

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$$\Phi(x,y,z,t;a,b) = \frac{\dot{S}(t)}{4\pi} Ln\left( \frac{z-b+\sqrt{x^2+y^2+(z-b)^2}}{z-a+\sqrt{x^2+y^2+(z-a)^2}} \right) \qquad (10.45)$$

Semi-infinite line of sources - Let $a \to -\infty$

$$\lim_{a\to-\infty} \Phi(x,y,z,t;a,b) =$$

$$\frac{\dot{S}(t)}{4\pi}\left( -Ln(2) + Ln\left(-\frac{1}{a}\right) + Ln\left(z-b+\sqrt{x^2+y^2+(z-b)^2}\right)\right) + \qquad (10.46)$$

$$\frac{\dot{S}(t)y}{4\pi a} - \frac{\dot{S}(t)}{16\pi a^2}\left(x^2+y^2-2z^2\right) + \frac{\dot{S}(t)}{24\pi a^3}\left(3x^2z+3y^2z-2z^3\right) + O\left(\frac{1}{a^4}\right)$$

$$\lim_{a\to-\infty} \Phi(x,y,z,t;a,b) = \frac{\dot{S}(t)}{4\pi} Ln\left(z-b+\sqrt{x^2+y^2+(z-b)^2}\right) + \frac{\dot{S}(t)}{4\pi}\left( Ln\left(-\frac{1}{2a}\right)\right) \qquad (10.47)$$

# Infinite line of sources

Let $b \to \infty$

$$\lim_{b\to\infty} \Phi(x,y,z,t;b) =$$
$$\frac{\dot{S}(t)}{4\pi}\left(-2Ln(2)+2Ln\left(\frac{1}{b}\right)+Ln\left(x^2+y^2\right)\right)-\frac{\dot{S}(t)}{8\pi b^2}\left(x^2+y^2-2z^2\right)+O\left(\frac{1}{b^4}\right) \quad (10.48)$$

$$\lim_{b\to\infty} \Phi(x,y,t;b) = \frac{\dot{S}(t)}{4\pi}Ln\left(x^2+y^2\right)+\frac{\dot{S}(t)}{2\pi}Ln\left(\frac{1}{2b}\right) \quad (10.49)$$

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## 2-D potential for a point source Q    Units of $Q(t)$ = Mass/Length-Sec

$$\Phi(x,y,t)=\frac{Q(t)}{2\pi\rho}Ln\left(x^2+y^2\right)^{1/2} \qquad (10.50)$$

2-D radial velocity

$$U_r=\frac{Q(t)}{2\pi\rho}\left(\frac{1}{r}\right) \qquad (10.51)$$

Integrate over any closed contour surrounding the source

$$\int_0^{2\pi}U_r r d\theta=\frac{Q(t)}{\rho} \qquad (10.52)$$

The fundamental source solution can be used to construct the Poisson solution for a distribution of sources in two dimensions

$$\Phi(x,y,t)=\frac{1}{2\pi\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q(\bar{x}_s,t)\,Ln\left(\left|\bar{x}-\bar{x}_s\right|^{1/2}\right)dx_s\,dy_s \qquad (10.53)$$

# Example – A vortex monopole - A finite length distribution of vortex monopoles will be used to generate what we will call a vortex stick. Lifting line theory will be developed using a superposition of vortex sticks.

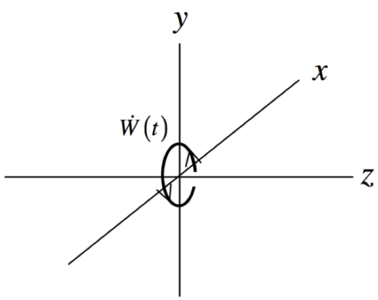


*Figure 10.5 A vortex monopole*

Vorticity point source

$$\bar{\Omega}(\bar{x},t) = \left\{ 0,0,\dot{W}(t)\delta(x)\delta(y)\delta(z) \right\} \quad (10.54)$$

Vector potential

$$A_z(\bar{x},t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\dot{W}(t)\delta(x_s)\delta(y_s)\delta(z_s)}{\left|\bar{x}-\bar{x}_s\right|}dx_s\,dy_s\,dz_s \quad (10.55)$$

$$\bar{A} = \left\{ 0,0,\frac{\dot{W}}{4\pi\left(x^2+y^2+z^2\right)^{1/2}} \right\} \quad (10.56)$$

Velocity field

$$\bar{U} = \left\{ -\frac{\dot{W}y}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},\frac{\dot{W}x}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},0 \right\} \quad (10.58)$$

Example - Scalar potential generated by a line distribution of vortices – the Vortex Stick, a finite length distribution of vortex monopoles

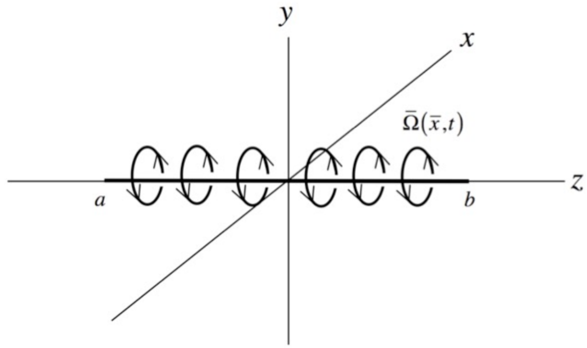


*Figure 10.6 A line distribution of vortex monopoles*

Vector source distribution

$$\bar{\Omega}(\bar{x},t) = \{0,0,\Gamma(t)\delta(x)\delta(y)u(b-z)u(z-a)\} \quad (10.60)$$

Vector potential has only one non-zero component

$$\bar{A} = (0,0,A_z)$$

$$A_z(x,y,z,t;a,b) =$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Gamma(t)\delta(x_s)\delta(y_s)u(b-z_s)u(z_s-a)}{\left((x-x_s)^2+(y-y_s)^2+(z-z_s)^2\right)^{1/2}}dx_s\,dy_s\,dz_s = \quad (10.61)$$

$$\frac{1}{4\pi}\int_a^b\frac{\Gamma(t)}{\left(x^2+y^2+(z-z_s)^2\right)^{1/2}}dz_s = \frac{-\Gamma(t)}{4\pi}Ln\left(\frac{z-b+\sqrt{x^2+y^2+(z-b)^2}}{z-a+\sqrt{x^2+y^2+(z-a)^2}}\right)$$

The velocity has two nonzero components

$$\bar{U} = \frac{-\Gamma}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+(z-b)^2}\left(z-b+\sqrt{x^2+y^2+(z-b)^2}\right)} - \frac{1}{\sqrt{x^2+y^2+(z-a)^2}\left(z-a+\sqrt{x^2+y^2+(z-a)^2}\right)}\right)\times\{y,-x,0\} \quad (10.62)$$

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# Infinite line of vortices

Let $a = -b$ and take the limit $b \to \infty$

$$\lim_{b\to\infty} A_z(x,y,z,t;b) = \frac{-\Gamma(t)}{4\pi} Ln\left(x^2+y^2\right) - \frac{\Gamma(t)}{2\pi} Ln\left(\frac{1}{2b}\right) \qquad (10.66)$$

2-D vector potential for a point vortex $\Gamma$ aka the stream function

$$\Psi(x,y,t) = \frac{-\Gamma(t)}{2\pi} Ln\left(\left(x^2+y^2\right)^{1/2}\right) \qquad (10.67)$$

The fundamental point vortex solution can be used to construct the Poisson solution for a distribution of vortices in two dimensions

$$\Psi(x,y,t) = \frac{-1}{2\pi\rho} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x_s,y_s,t) Ln\left(\left|\bar{x}-\bar{x}_s\right|^{1/2}\right) dx_s\, dy_s \qquad (10.68)$$

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# Example - Uniform Flow past a sphere

$$\Phi_{Dipole} = \frac{\kappa x}{\left(x^2 + y^2 + z^2\right)^{3/2}} \quad (10.69)$$

$$\Phi_{Sphere} = \Phi_{Uniform\ Flow} + \Phi_{Dipole} = U_\infty x + \frac{\kappa x}{\left(x^2 + y^2 + z^2\right)^{3/2}} \quad (10.70)$$

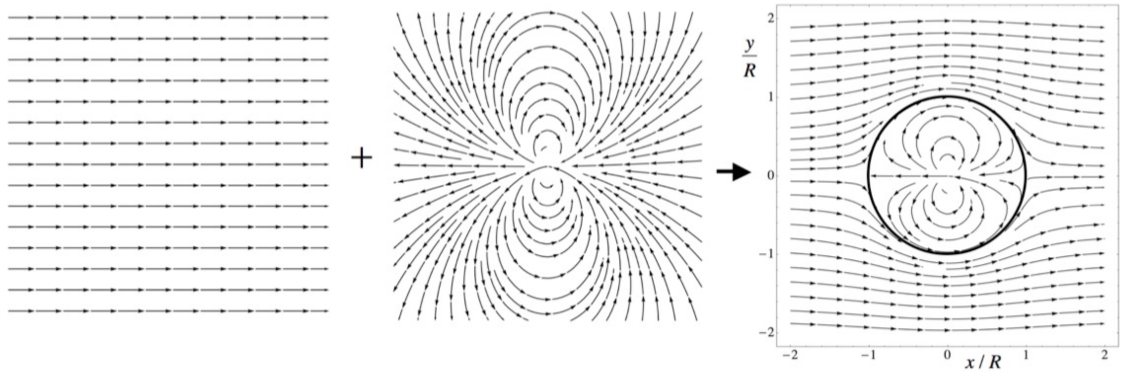


*Figure 10.7 Potential flow past a sphere.*

$$\begin{aligned} U_x(x,y,z) &= U_\infty - \frac{3\kappa x^2}{r^5} + \frac{\kappa}{r^3} \\ U_y(x,y,z) &= -\frac{3\kappa xy}{r^5} \\ U_z(x,y,z) &= -\frac{3\kappa xz}{r^5} \end{aligned} \quad (10.71)$$

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$$R_{Sphere} = \left(\frac{2\kappa}{U_\infty}\right)^{1/3} \tag{10.72}$$

$$\kappa = \frac{U_\infty}{2}\left(R_{Sphere}\right)^3 \tag{10.73}$$

$$\Phi_{Sphere} = U_\infty x\left(1 + \frac{\left(R_{Sphere}\right)^3}{2\left(x^2+y^2+z^2\right)^{3/2}}\right) \tag{10.74}$$

$$U_x(x,y,z) = U_\infty\left(1 - \frac{3\left(R_{Sphere}\right)^3 x^2}{2r^5} + \frac{\left(R_{Sphere}\right)^3}{2r^3}\right)$$

$$U_y(x,y,z) = -U_\infty \frac{3\left(R_{Sphere}\right)^3 xy}{2r^5} \tag{10.75}$$

$$U_z(x,y,z) = -U_\infty \frac{3\left(R_{Sphere}\right)^3 xz}{2r^5}$$

Disturbance velocity from a 3-D body decays like $1/r^3$

# 10.6 Elementary 2-D potential flows

2-D potential flows satisfy the Cauchy Riemann conditions

$$U = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}$$
$$V = \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \qquad (10.76)$$

$$z = x + iy \qquad (10.77)$$

Complex potential

$$W(z) = \Phi(x,y) + i\Psi(x,y) \qquad (10.78)$$

$$z = x + iy = r\left(Cos(\theta) + iSin(\theta)\right) = re^{i\theta} \qquad (10.79)$$

$$r = \left(x^2 + y^2\right)^{1/2} \qquad (10.80)$$

$$Tan(\theta) = \frac{y}{x} \qquad (10.81)$$

Both components of the complex potential satisfy Laplace's equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial^2\Psi}{\partial y\partial x}=0 \qquad (10.82)$$

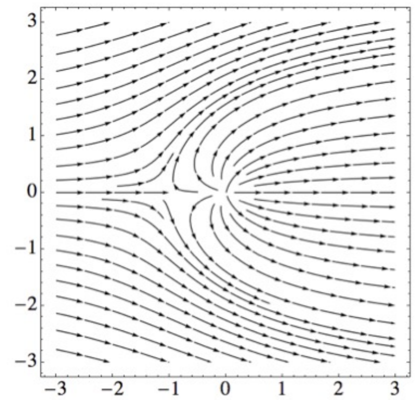
$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=-\frac{\partial^2\Phi}{\partial y\partial x}-\frac{\partial^2\Phi}{\partial x\partial y}=0 \qquad (10.83)$$

Complex velocity

$$\begin{aligned}\frac{dW}{dz}&=\frac{\partial\Phi}{\partial x}\frac{dx}{dz}+i\frac{\partial\Psi}{\partial x}\frac{dx}{dz}=U-iV\\ \frac{dW}{dz}&=\frac{\partial\Phi}{\partial y}\frac{dy}{dz}+i\frac{\partial\Psi}{\partial y}\frac{dy}{dz}=\frac{V}{i}+i\frac{U}{i}=U-iV\end{aligned} \qquad (10.84)$$

3) *Source at the origin plus uniform flow*

$$W = U_\infty z + \frac{Q}{2\pi} Ln(z) \qquad \Phi = U_\infty x + \frac{Q}{2\pi} Ln\left(x^2 + y^2\right)^{1/2} \qquad \Psi = U_\infty y + \frac{Q}{2\pi} ArcTan\left(\frac{y}{x}\right) \tag{10.90}$$

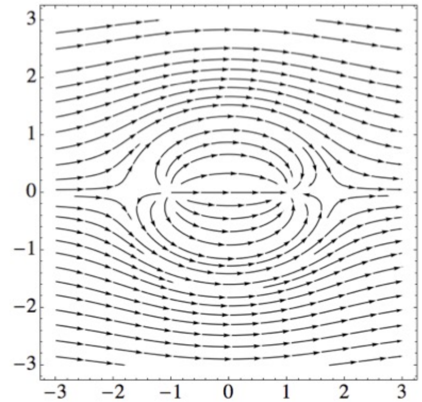


4) *Uniform flow plus a source at* $x = -a$ *and a sink of equal strength at* $x = a$

$$\Phi = U_\infty x + \frac{Q}{2\pi} Ln\left(\left(x+a\right)^2 + y^2\right)^{1/2} - \frac{Q}{2\pi} Ln\left(\left(x-a\right)^2 + y^2\right)^{1/2} \tag{10.91}$$

$$\Psi = U_\infty y + \frac{Q}{2\pi} ArcTan\left(\frac{y}{x+a}\right) - \frac{Q}{2\pi} ArcTan\left(\frac{y}{x-a}\right) \tag{10.92}$$

$$W = U_\infty z + \frac{Q}{2\pi} Ln(z-a) - \frac{Q}{2\pi} Ln(z+a)$$



*5) Point vortex*

Here we solve the Poisson equation for the stream function with a point source of circulation at the origin.

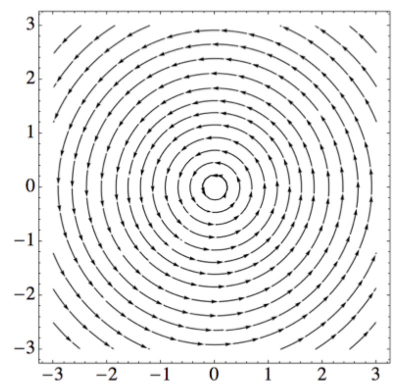
$$\nabla^2\Psi = -\Gamma\delta(\bar{x}) \tag{10.93}$$

where $\Gamma$ is the strength of the source. The Greens function solution is

$$\Psi(\bar{x}) = \frac{-1}{2\pi}\int_A \Gamma\delta(\bar{x}_s)Ln(|\bar{x}-\bar{x}_s|)dA = \frac{-1}{2\pi}\int_0^{2\pi}\int_0^r \Gamma\delta(r_s)Ln(|r-r_s|)drd\theta = \frac{-\Gamma}{2\pi}Ln(r) \tag{10.94}$$

This is the same solution we derived earlier through a limiting process of allowing a finite vortex line become infinite. The potentials for a point vortex are

$$W = -\frac{i\Gamma}{2\pi}Ln(z) \qquad \Phi = \frac{\Gamma}{2\pi}\theta \qquad \Psi = -\frac{\Gamma}{2\pi}Ln(r) \tag{10.95}$$



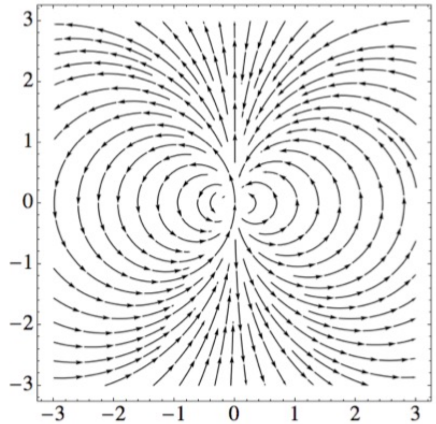

For any contour $C$ surrounding the origin

$$\int_A \Omega\, dA = \int_A \nabla\times\bar{U}\, dA = \oint_C \bar{U}\hat{c}dC = \int_0^{2\pi}\frac{\Gamma}{2\pi r}rd\theta = \Gamma \tag{10.96}$$

*6) Vortex doublet*

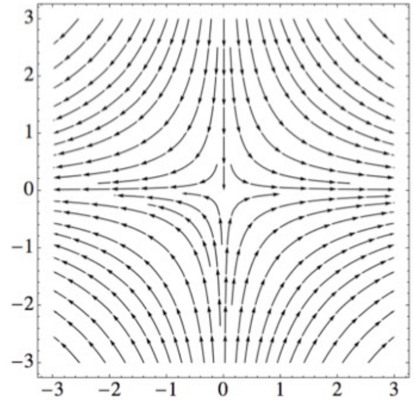
This is constructed from two point vortices of opposite circulation separated by the distance a. As they are brought together the strength $\lambda = a\Gamma$ is held constant.

$$W = \frac{\lambda}{2\pi}\left(\frac{i}{z}\right) = \frac{\lambda}{2\pi}\left(\frac{i}{r}\right)e^{-i\theta} \qquad \Phi = \frac{\lambda}{2\pi}\frac{Sin(\theta)}{r} \qquad \Psi = -\frac{\lambda}{2\pi}\frac{Cos(\theta)}{r} \qquad (10.97)$$




*7) Stagnation point flow*

$$W = Az^2 \qquad \Phi = A\left(x^2 - y^2\right) \qquad \Psi = 2Axy \qquad (10.98)$$

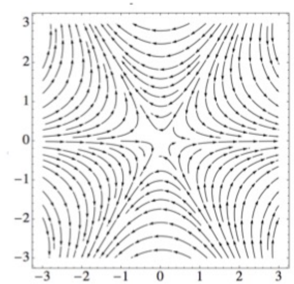


## 8) *Flow in a corner*

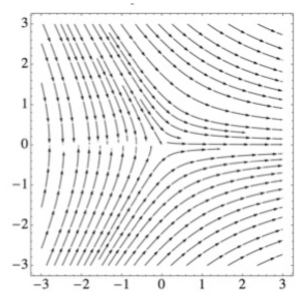
The potentials are ( $n=2$ is the stagnation point flow above).

$$W = Az^n = A\left(re^{i\theta}\right)^n \qquad \Phi = Ar^n Cos(n\theta) \qquad \Psi = Ar^n Sin(n\theta)$$

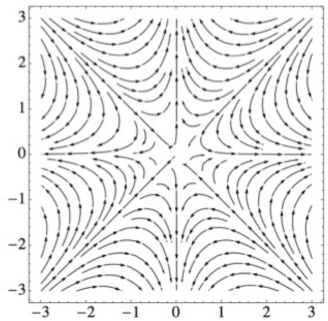
$n=3$




$n=3/2$




$n=4$



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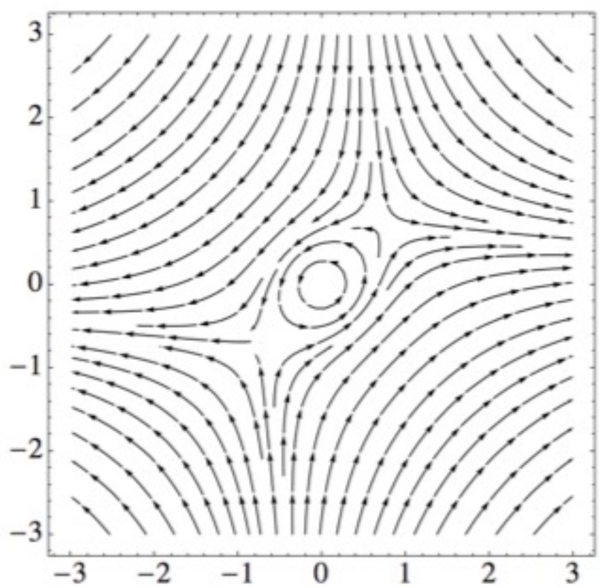
*9) Stagnation point flow plus vortex flow*

Add together the potentials for a stagnation point flow and a point vortex.

$$W = Az^2 - \frac{i\Gamma}{2\pi} Ln(z) \quad \Phi = A\left(x^2 - y^2\right) + \frac{\Gamma}{2\pi}\theta \quad \Psi = 2Axy - \frac{\Gamma}{2\pi} Ln(r)$$



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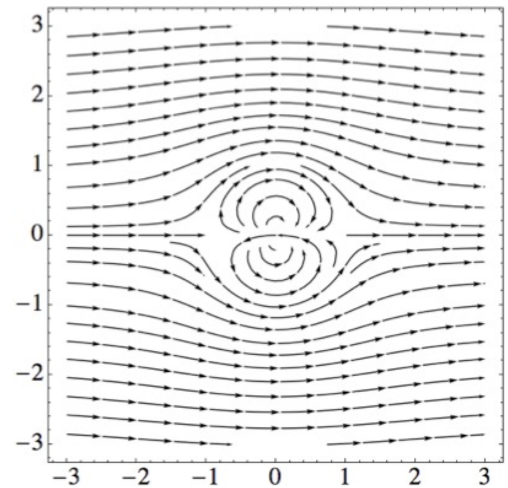
## *10) Flow past a circular cylinder*

Superpose a uniform flow with a dipole

$$W = U_\infty z + \frac{\kappa}{2\pi}\left(\frac{1}{z}\right) \quad \Phi = U_\infty x + \frac{\kappa}{2\pi}\left(\frac{x}{x^2+y^2}\right) \quad \Psi = U_\infty y + \frac{\kappa}{2\pi}\left(\frac{y}{x^2+y^2}\right) \tag{10.101}$$




The radius of the cylinder is

$$R = \left(\frac{\kappa}{2\pi U_\infty}\right)^{1/2} \tag{10.102}$$

and from the Bernoulli constant we get the pressure coefficient on the cylinder

$$\frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^2 = \left(\frac{P}{\rho} + \frac{1}{2}U^2\right)_{R=\left(\frac{\kappa}{2\pi U_\infty}\right)^{1/2}} \tag{10.103}$$

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$$
\begin{aligned}
&C_p = \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2} = \left(1 - \left(\frac{U}{U_\infty}\right)^2\right) = \\
&\left(1 - \left(\left(U_\infty + \frac{\kappa}{2\pi}\left(\frac{1}{x^2+y^2}\right) - \frac{\kappa}{2\pi}\left(\frac{2x^2}{\left(x^2+y^2\right)^2}\right)\right)^2 + \left(\frac{\kappa}{2\pi}\right)^2\left(\frac{2xy}{\left(x^2+y^2\right)^2}\right)^2\right)\right) = \\
&C_p = \left(1 - \frac{1}{U_\infty^2}\left(\left(2U_\infty - U_\infty\left(\frac{2x^2}{R^2}\right)\right)^2 + R^4U_\infty^2\left(\frac{2xy}{R^4}\right)^2\right)\right) = \\
&C_p = \left(1 - \left(4\left(1 - \left(\frac{x^2}{R^2}\right)\right)^2 + 4\left(\frac{xy}{R^2}\right)^2\right)\right) = \\
&x = RCos(\theta),\, y = RSin(\theta) \\
&C_p = 1 - 4Sin^2(\theta)
\end{aligned}
\tag{10.104}
$$

plotted below.

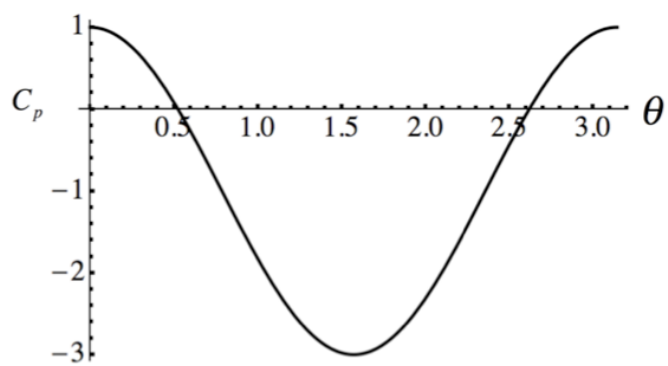



*Fig 10.8 Pressure coefficient for irrotational flow past a circle.*

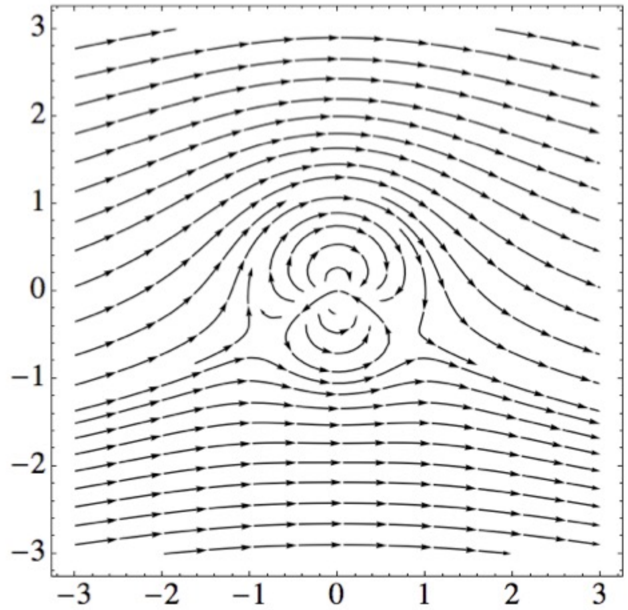
*11) Superpose a uniform flow with a dipole and a vortex.*

Take the circulation of the vortex to be in the clockwise direction.

$$W = U_\infty z + \frac{\kappa}{2\pi}\left(\frac{1}{z}\right) + \frac{i\Gamma}{2\pi} Ln(z)$$

$$\Phi = U_\infty x + \frac{\kappa}{2\pi}\left(\frac{x}{x^2+y^2}\right) - \frac{\Gamma}{2\pi}\theta \qquad (10.105)$$

$$\Psi = U_\infty y + \frac{\kappa}{2\pi}\left(\frac{y}{x^2+y^2}\right) + \frac{\Gamma}{2\pi} Ln(r)$$



# Force by a uniform flow on a 3-D rigid body in an inviscid fluid

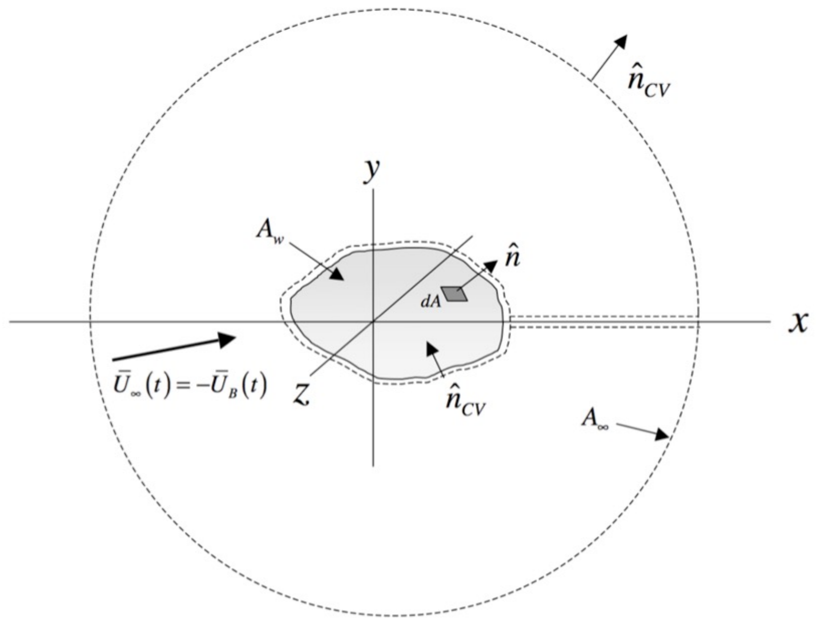



*Figure 10.10 Control volume surrounding a rigid body translating in an inviscid fluid.*

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi \hat{n} dA - \bar{U}_\infty(t) \times \int_{A_w} (\nabla\Phi \times \hat{n}) dA \qquad (10.145)$$

In steady flow the force is perpendicular to the velocity vector approaching the body

# Force on a 2-D rigid body

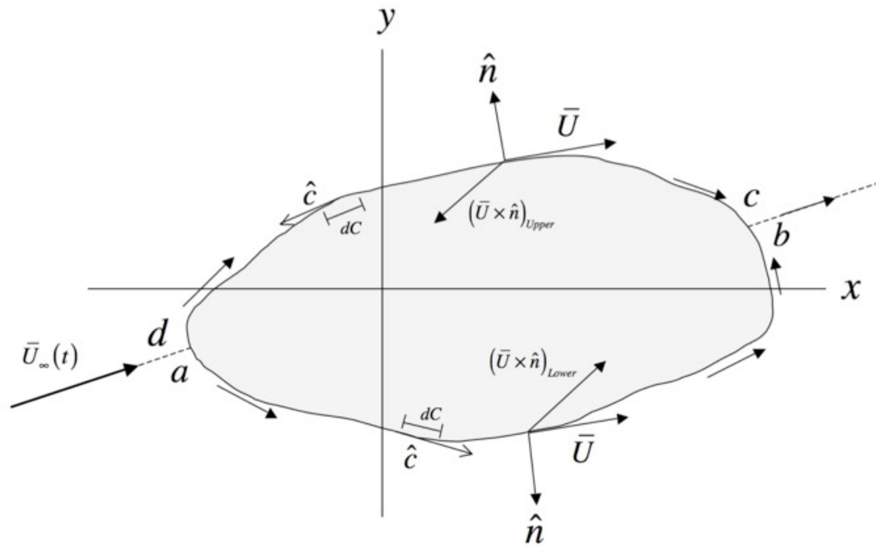



*Figure 10.11 Circulation about a two-dimensional rigid body translating in an inviscid fluid.*

$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{C_w} \Phi\hat{n}dC - \bar{U}_\infty(t)\times\oint_{C_w}(\nabla\Phi\times\hat{n})dC \qquad (10.146)$$

$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dl - \bar{U}_\infty(t)\times\left(\int_a^b(\bar{U}\times\hat{n})_{Lower}\,dl + \int_c^d(\bar{U}\times\hat{n})_{Upper}\,dl\right) \qquad (10.147)$$

$$(\bar{U}\times\hat{n})_{Upper} = |\bar{U}|\hat{k} \qquad (10.148)$$

$$(\bar{U}\times\hat{n})_{Lower} = -|\bar{U}|\hat{k} \qquad (10.149)$$

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$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dl - \bar{U}_{\infty}(t)\times\left(\int_a^b |\bar{U}|\,dl - \int_c^d |\bar{U}|\,dl\right)\hat{k} \qquad (10.150)$$

$$\int_a^b |\bar{U}|\,dl - \int_c^d |\bar{U}|\,dl = \oint_C \bar{U}\bullet\hat{c}\,dC$$
$$\Gamma(t) = \oint_C \bar{U}\bullet\hat{c}\,dC \qquad (10.151)$$

Force on a 2-D body in potential flow is

$$\frac{\bar{F}}{\rho(oneunitofspan)} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dl - \bar{U}_{\infty}(t)\times\Gamma(t)\hat{k} \qquad (10.152)$$

# 10.9 Virtual mass

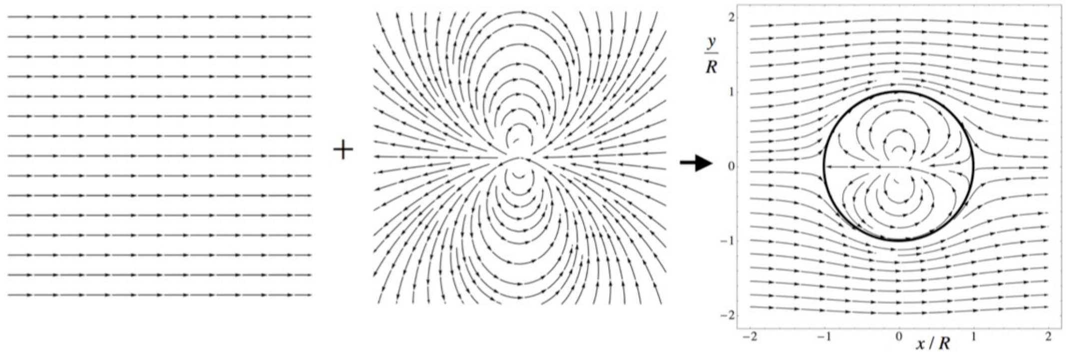



*Figure 10.12 Potential flow past a sphere.*

$$\frac{\tilde{F}_x}{\rho} = \frac{d}{dt}\int_{A_w}\left(U_\infty f(t)\left(\tilde{x}+U_\infty(\tilde{t})\right)\left(\frac{\left(R_{Sphere}\right)^3}{2\left(\left(\tilde{x}+U_\infty(\tilde{t})\right)^2+\tilde{y}^2+\tilde{z}^2\right)^{3/2}}\right)\right)\tilde{n}_x d\tilde{A} = \tag{10.159}$$

$$\frac{U_\infty}{2}\frac{df}{dt}\int_{A_w}\left(\tilde{x}+U_\infty(\tilde{t})\right)\tilde{n}_x d\tilde{A}$$

$$\tilde{n}_x = Cos\left(\tilde{\theta}\right) \text{ and } \tilde{x}+U_\infty\left(\tilde{t}\right) = R_{Sphere}Cos\left(\tilde{\theta}\right)$$

$$\frac{\tilde{F}_x}{\rho} = \frac{U_\infty\left(R_{Sphere}\right)^3}{2}\frac{df}{dt}\int_0^\pi\int_0^{2\pi}Cos^2\left(\tilde{\theta}\right)Sin\left(\tilde{\theta}\right)d\phi d\theta = \frac{2\pi}{3}\left(R_{Sphere}\right)^3 U_\infty\frac{df}{dt} \tag{10.160}$$

Virtual mass = 1/2 of the displaced mass

# Force on an accelerated sphere vs a sphere in an accelerated fluid

Fluid is inviscid

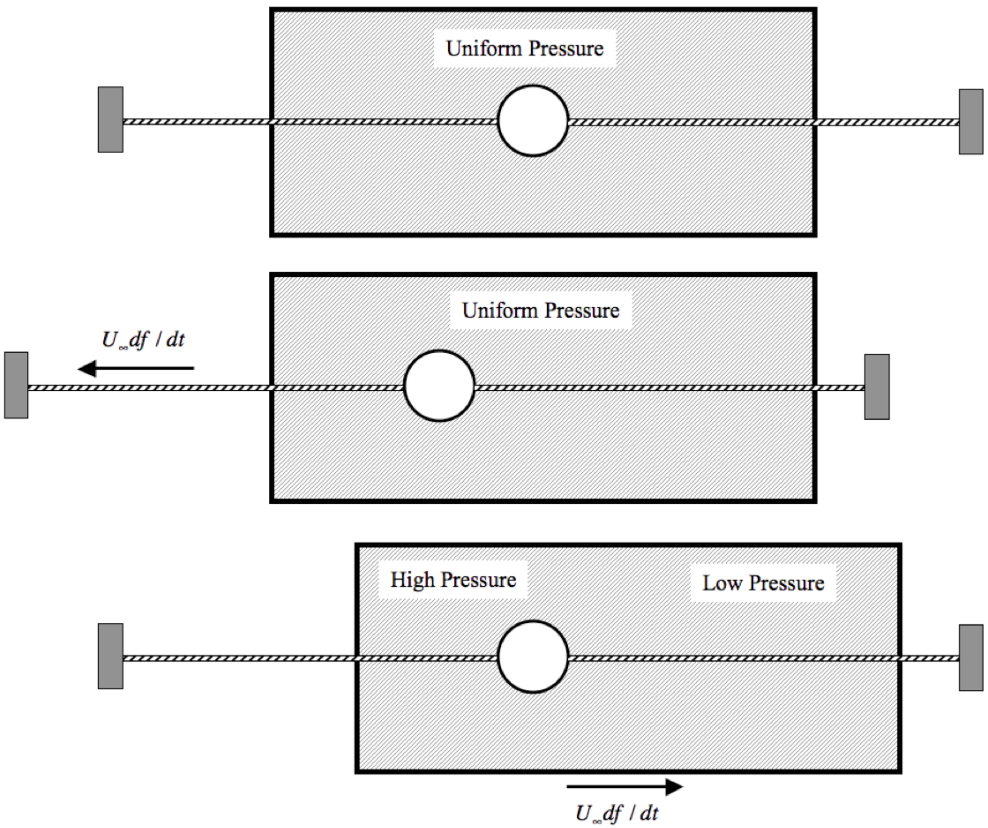



*Figure 10.13 Accelerated sphere versus accelerated fluid*

## Acceleration

$$U_\infty \frac{df(t)}{dt}$$

$f(t)$ is dimensionless

## Accelerate the sphere

$$F_x = \frac{2}{3}\pi\left(R_{sphere}\right)^3 \rho\left(U_\infty \frac{df}{dt}\right)$$

Virtual mass = 1/2 of the displaced mass

## Accelerate the fluid

$$F_x = 2\pi\left(R_{sphere}\right)^3 \rho\left(U_\infty \frac{df}{dt}\right) =$$

$$\frac{2}{3}\pi\left(R_{sphere}\right)^3 \rho\left(U_\infty \frac{df}{dt}\right) + \frac{4}{3}\pi\left(R_{sphere}\right)^3 \rho\left(U_\infty \frac{df}{dt}\right)$$

Virtual mass effect

Buoyancy effect

# Low Reynolds number flow

Take the curl of the incompressible momentum equation

$$\nabla \times \left( \frac{\partial \overline{U}}{\partial t} + \overline{U} \cdot \nabla \overline{U} + \frac{1}{\rho} \nabla P - \nu \nabla^2 \overline{U} \right) = 0$$

The result is the transport equation for the vorticity

$$\frac{\partial \overline{\Omega}}{\partial t} + \overline{U} \cdot \nabla \overline{\Omega} - \overline{\Omega} \cdot \nabla \overline{U} = \nu \nabla^2 \overline{\Omega}$$

If the flow is steady and the velocity is very small the equation reduces to

$$\nabla^2 \overline{\Omega} = 0$$

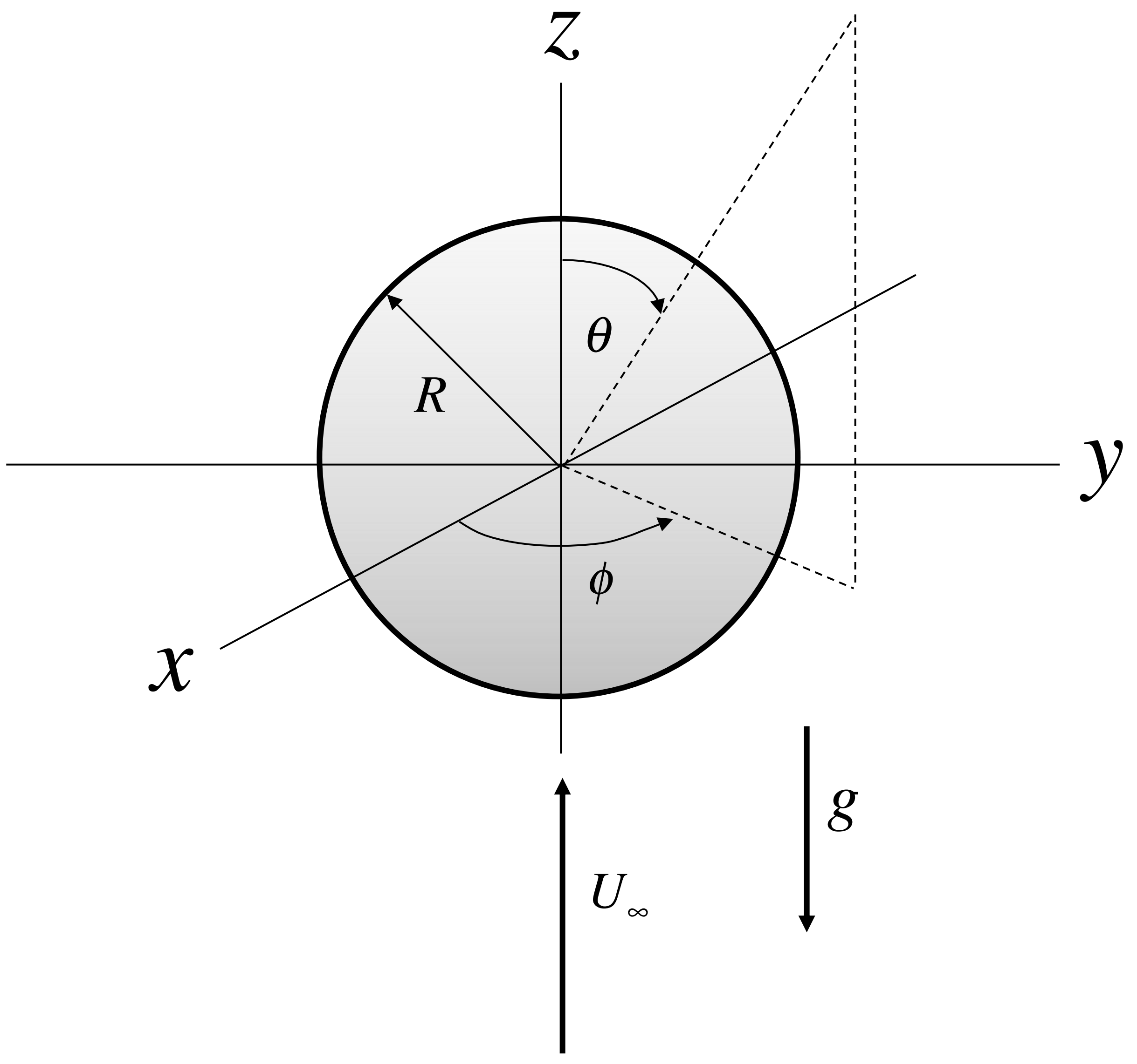
Recall the Poisson equation for the vector potential

$$\nabla^2 \overline{A} = -\overline{\Omega}$$

Low Reynolds number flow is governed by the biharmonic equation

$$\nabla^2 \left( \nabla^2 \overline{A} \right) = 0$$

# Viscous flow past a sphere at low Reynolds number



# The Stokes stream function

The flow is axisymmetric and best posed in spherical polar coordinates

$$\nabla^2\left(\nabla^2\overline{A}\right)=\left(\frac{\partial^2}{\partial r^2}+\frac{Sin(\theta)}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{Sin(\theta)}\frac{\partial}{\partial\theta}\right)\right)^2\Psi=0$$

Velocities

$$U_r=-\frac{1}{r^2 Sin(\theta)}\frac{\partial\Psi}{\partial\theta}$$

$$U_\theta=\frac{1}{rSin(\theta)}\frac{\partial\Psi}{\partial r}$$

Boundary conditions

No-slip condition

$$U_r(R,\theta)=0$$

$$U_\theta(R,\theta)=0$$

Uniform flow at infinity

$$\lim_{r\to\infty}\Psi\to-\frac{1}{2}U_\infty r^2 Sin^2(\theta)$$

Assume

$$\Psi = f(r)\,Sin^2(\theta)$$

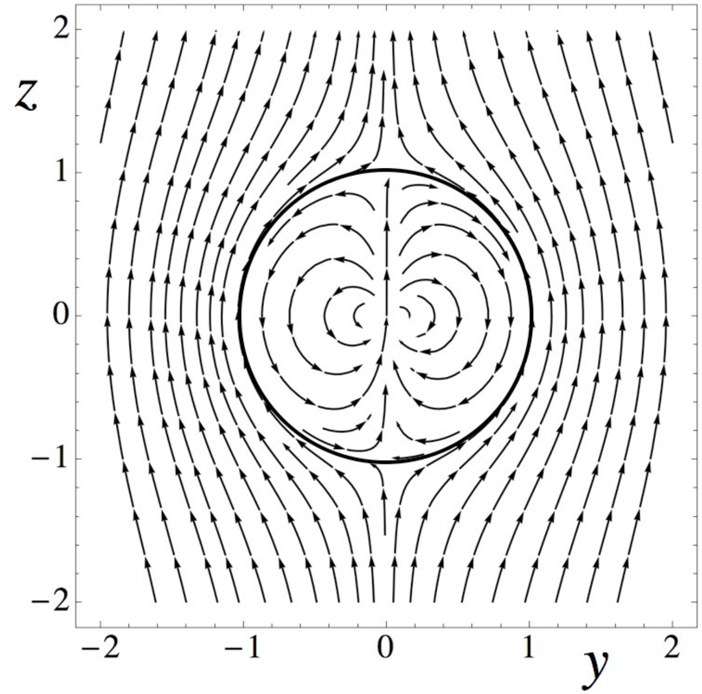
$$f(r) = \frac{a}{r} + br + cr^2 + dr^4$$

Solution

$$\frac{\Psi}{U_\infty R^2} = \left(-\frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{3}{4}\left(\frac{r}{R}\right) - \frac{1}{4}\left(\frac{r}{R}\right)^{-1}\right) Sin^2(\theta)$$

$$\frac{U_r}{U_\infty} = \left(1 - \frac{3}{2}\left(\frac{r}{R}\right)^{-1} + \frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right) Cos(\theta)$$

$$\frac{U_\theta}{U_\infty} = \left(-1 + \frac{3}{4}\left(\frac{r}{R}\right)^{-1} + \frac{1}{4}\left(\frac{r}{R}\right)^{-3}\right) Sin(\theta)$$

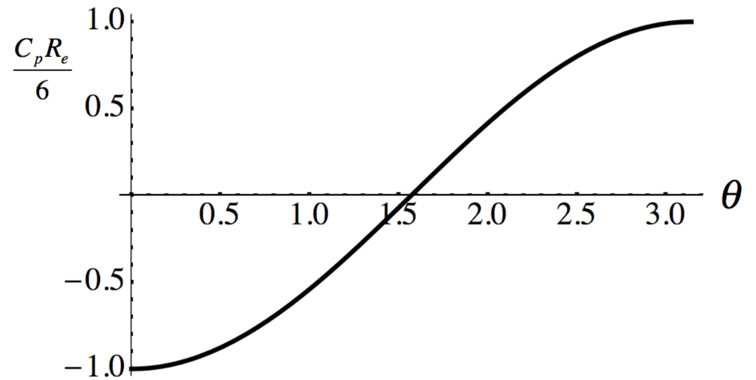



Viscous stress

$$\frac{\tau_{r\theta} R}{\mu U_\infty} = \frac{3}{2}\left(\frac{r}{R}\right)^{-4} Sin(\theta)$$

Pressure

$$\frac{(P - P_\infty)R}{\mu U_\infty} = -\frac{\rho g R^2}{\mu U_\infty}\left(\frac{z}{R}\right) - \frac{3}{2}\left(\frac{r}{R}\right)^2 Cos(\theta)$$



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# Drag components

$$F_{z_{Pressure}} = -\int_0^{2\pi}\int_0^{\pi}\left(P(R,\theta)Cos(\theta)\right)R^2 Sin(\theta)d\theta d\phi = \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R U_\infty$$

$$F_{z_{Viscous}} = \int_0^{2\pi}\int_0^{\pi}\left(\tau_{r\theta}(R,\theta)Sin(\theta)\right)R^2 Sin(\theta)d\theta d\phi = 4\pi\mu R U_\infty$$

$$F_z = \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R U_\infty + 4\pi\mu R U_\infty$$

Buoyancy force — Pressure drag — Viscous drag

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# Non buoyant pressure plus viscous drag

$$D_{Stokes} = 6\pi\mu R U_{\infty}$$

## Reynolds number

$$R_e = \frac{\rho U_{\infty}(2R)}{\mu}$$

$$C_D = \frac{D_{Stokes}}{\frac{1}{2}\rho U_{\infty}^2 (\pi R^2)} = \frac{12\pi\mu R U_{\infty}}{\rho U_{\infty}^2 (\pi R^2)} = \frac{12\mu}{\rho U_{\infty} R} = \frac{24}{R_e}$$

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# Dissipation of kinetic energy by viscous friction

$$\frac{\varepsilon}{\mu} = \frac{\tau_{ij}}{\mu}\frac{\partial U_i}{\partial x_j} = 2S_{ij}S_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

## Axisymmetric flow in spherical polar coordinates

$$\frac{\varepsilon}{\mu} = 2\left(\frac{\partial U_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right)^2 + 2\left(\frac{U_r}{r} + \frac{U_\theta}{r}Cot(\theta)\right)^2 + \left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} + \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)^2$$

## Low Reynolds number flow over a sphere

$$\frac{\varepsilon R^2}{\mu U_\infty^2} = \frac{9}{4}\left(\frac{r}{R}\right)^{-8}\left(3\left(\left(\frac{r}{R}\right)^2 - 1\right)^2 Cos^2(\theta) + Sin^2(\theta)\right)$$

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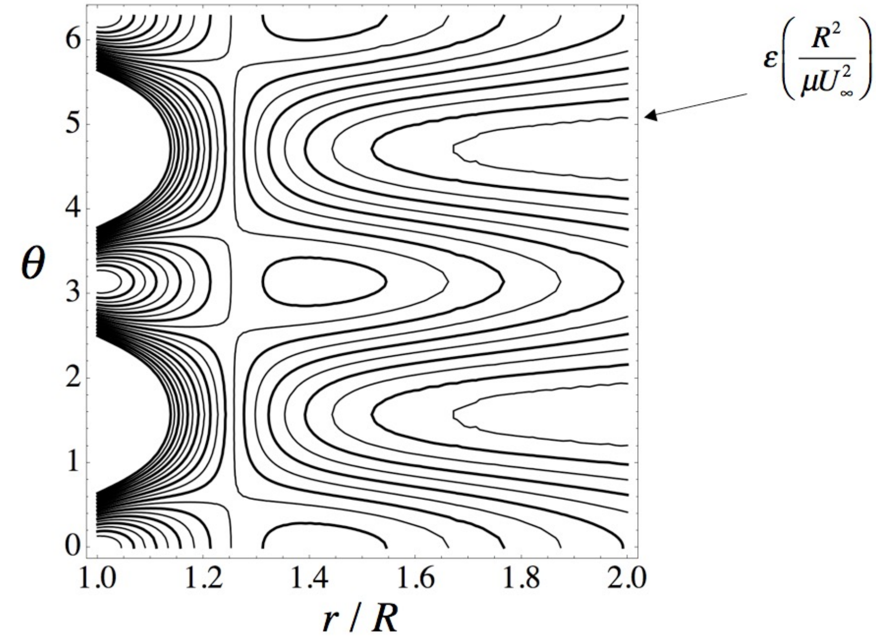
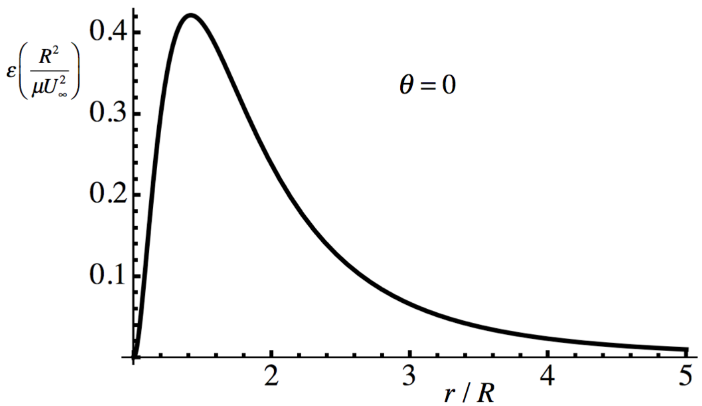
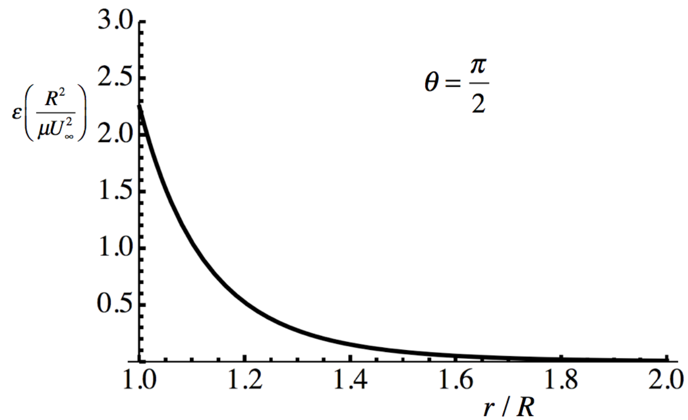
Integrate the kinetic energy dissipation over the flow volume out to infinity




$$\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} \varepsilon r^2 Sin(\theta)\, dr\, d\theta\, d\phi =$$

$$\frac{9}{4}\mu U^2{}_\infty R\int_0^{2\pi}\int_0^{\pi}\int_1^{\infty}\left(\frac{r}{R}\right)^{-8}\left(3\left(\left(\frac{r}{R}\right)^2-1\right)^2 Cos^2(\theta)+Sin^2(\theta)\right)\left(\frac{r}{R}\right)^2 Sin(\theta)\, d\left(\frac{r}{R}\right) d\theta\, d\phi = (6\pi\mu U_\infty R)U_\infty = D_{Stokes}U_\infty$$

# 10.7 Force on a rigid body translating in an <u>inviscid</u> fluid

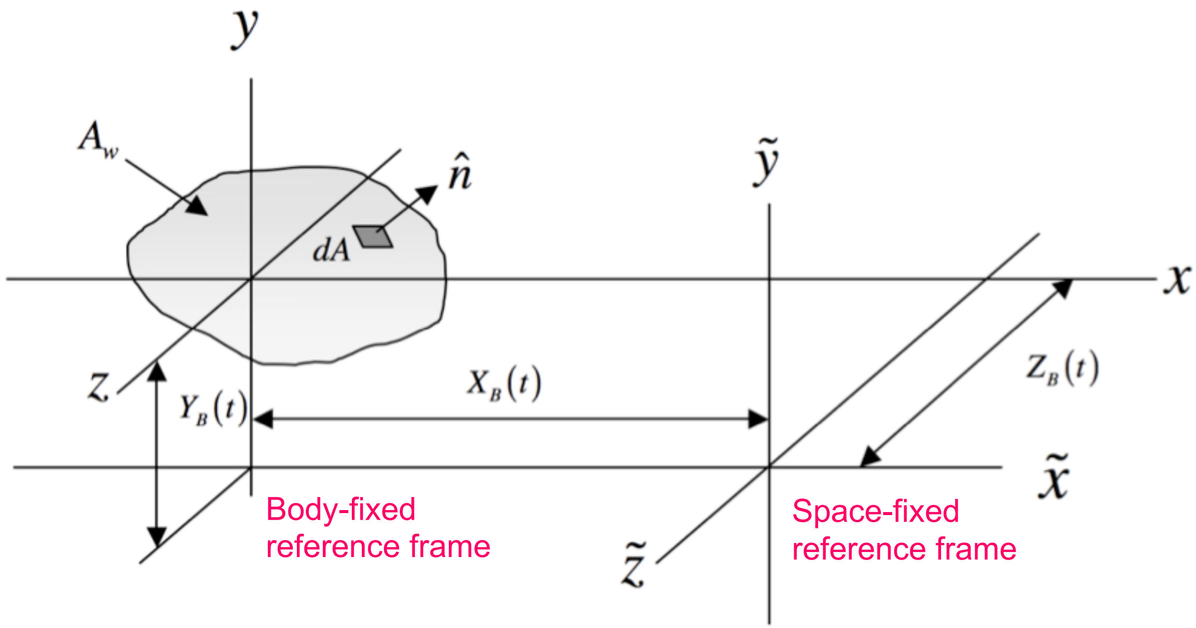



*Figure 10.9 Rigid body translating in an inviscid fluid*

$$
\begin{aligned}
\tilde{x} &= x + X_B(t) \\
\tilde{y} &= y + Y_B(t) \\
\tilde{z} &= z + Z_B(t) \\
\tilde{t} &= t \\
\tilde{U}_{\tilde{x}} &= U_x + \dot{X}_B(t) \\
\tilde{U}_{\tilde{y}} &= U_y + \dot{Y}_B(t) \\
\tilde{U}_z &= U_z + \dot{Z}_B(t) \\
\frac{\tilde{P}}{\rho} &= \frac{P}{\rho} - x\ddot{X}_B(t) - y\ddot{Y}_B(t) - z\ddot{Z}_B(t) \\
\tilde{\Phi} &= \Phi + x\dot{X}_B(t) + y\dot{Y}_B(t) + z\dot{Z}_B(t)
\end{aligned}
\qquad (10.106)
$$

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{x}} &= \frac{1}{\rho}\frac{\partial P}{\partial x} - \ddot{X}_B(t) \\
\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{y}} &= \frac{1}{\rho}\frac{\partial P}{\partial y} - \ddot{Y}_B(t) \\
\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{z}} &= \frac{1}{\rho}\frac{\partial P}{\partial z} - \ddot{Z}_B(t)
\end{aligned}
\qquad (10.107)
$$

# Transform the Bernoulli constant

$$\frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{\bar{U} \bullet \bar{U}}{2} = f(t) \qquad (10.108)$$

$$d\tilde{\Phi} - \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} d\tilde{x} - \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} d\tilde{y} - \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} d\tilde{z} - \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} d\tilde{t} = 0 \qquad (10.109)$$

$$\begin{aligned}
d\tilde{\Phi} &= \left( \frac{\partial \Phi}{\partial x} + \dot{X}_B(t) \right) dx + \left( \frac{\partial \Phi}{\partial y} + \dot{Y}_B(t) \right) dy + \left( \frac{\partial \Phi}{\partial z} + \dot{Z}_B(t) \right) dz + \\
&\left( \frac{\partial \Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) \right) dt \\
d\tilde{x} &= dx + \dot{X}_B(t) dt \\
d\tilde{y} &= dy + \dot{Y}_B(t) dt \\
d\tilde{z} &= dz + \dot{Z}_B(t) dt \\
d\tilde{t} &= dt
\end{aligned} \qquad (10.110)$$

$$\begin{aligned}
&d\tilde{\Phi} - \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} d\tilde{x} - \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} d\tilde{y} - \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} d\tilde{z} - \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} d\tilde{t} = \\
&\left( \frac{\partial \Phi}{\partial x} + \dot{X}_B(t) - \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} \right) dx + \left( \frac{\partial \Phi}{\partial y} + \dot{Y}_B(t) - \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} \right) dy + \left( \frac{\partial \Phi}{\partial z} + \dot{Z}_B(t) - \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} \right) dz + \\
&\left( \frac{\partial \Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) - \dot{X}_B(t) \frac{\partial \tilde{\Phi}}{\partial \tilde{x}} - \dot{Y}_B(t) \frac{\partial \tilde{\Phi}}{\partial \tilde{y}} - \dot{Z}_B(t) \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} - \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} \right) dt = 0
\end{aligned} \qquad (10.111)$$

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## Transformation of the time derivative of the potential

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{x}} = \frac{\partial\Phi}{\partial x} + \dot{X}_B(t)$$
$$\frac{\partial\tilde{\Phi}}{\partial\tilde{y}} = \frac{\partial\Phi}{\partial y} + \dot{Y}_B(t)$$
$$\frac{\partial\tilde{\Phi}}{\partial\tilde{z}} = \frac{\partial\Phi}{\partial z} + \dot{Z}_B(t) \qquad (10.112)$$
$$\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} = \frac{\partial\Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) - \dot{X}_B(t)\frac{\partial\tilde{\Phi}}{\partial\tilde{x}} - \dot{Y}_B(t)\frac{\partial\tilde{\Phi}}{\partial\tilde{y}} - \dot{Z}_B(t)\frac{\partial\tilde{\Phi}}{\partial\tilde{z}}$$

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} = \frac{\partial\Phi}{\partial t} + x\ddot{X}_B(t) + y\ddot{Y}_B(t) + z\ddot{Z}_B(t) - \dot{X}_B(t)\frac{\partial\Phi}{\partial x} - \dot{Y}_B(t)\frac{\partial\Phi}{\partial y} - \dot{Z}_B(t)\frac{\partial\Phi}{\partial z} - \left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right)$$

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} + \frac{\tilde{P}}{\rho} + \frac{1}{2}\left(\tilde{U}_{\tilde{x}}^2 + \tilde{U}_{\tilde{y}}^2 + \tilde{U}_{\tilde{z}}^2\right) =$$
$$\frac{\partial\Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\left(U_x^2 + U_y^2 + U_z^2\right) - \frac{1}{2}\left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right) \qquad (10.113)$$

$$\frac{\partial\tilde{\Phi}}{\partial\tilde{t}} + \frac{\tilde{P}}{\rho} + \frac{1}{2}\left(\tilde{U}_{\tilde{x}}^2 + \tilde{U}_{\tilde{y}}^2 + \tilde{U}_{\tilde{z}}^2\right) = \frac{P_\infty}{\rho} \qquad (10.114)$$

$$\frac{\partial\Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\left(U_x^2 + U_y^2 + U_z^2\right) = \frac{P_\infty}{\rho} + \frac{1}{2}\left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right) \qquad (10.115)$$

The force on the body in the body fixed frame is determined by

$$\frac{P}{\rho} - \frac{P_\infty}{\rho} = \frac{1}{2}\left(\dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2\right) - \frac{\partial\Phi}{\partial t} - \frac{1}{2}\bar{U}\bullet\bar{U} \qquad (10.116)$$

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## 10.7.3 Relation between the force acting on the body and the potential

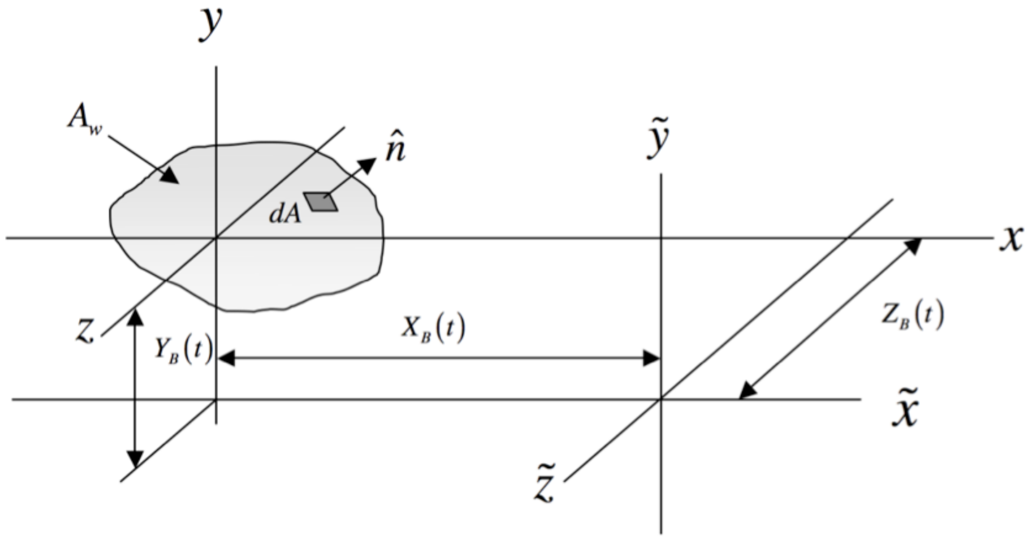


*Figure 10.9 Rigid body translating in an inviscid fluid*

$$\frac{\bar{F}}{\rho} = -\int_{A_w} \left( \frac{P}{\rho} - \frac{P_\infty}{\rho} \right) \hat{n}\, dA = \int_{A_w} \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \bar{U} \bullet \bar{U} - \frac{1}{2} \left( \dot{X}_B(t)^2 + \dot{Y}_B(t)^2 + \dot{Z}_B(t)^2 \right) \right) \hat{n}\, dA \quad (10.124)$$

$$\int_{A_w} \frac{\partial \Phi}{\partial t} \hat{n} dA = \frac{d}{dt} \int_{A_w} \Phi \hat{n} dA \quad (10.125)$$

$$\int_{A_w} \hat{n}\, dA = 0 \quad (10.126)$$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi \hat{n} dA + \int_{A_w} \left( \frac{\bar{U} \bullet \bar{U}}{2} \right) \hat{n}\, dA \quad (10.127)$$

$$\tilde{\bar{U}} = \bar{U} + \bar{U}_B \quad (10.117)$$

$$\bar{U}_B = \left( \dot{X}_B(t), \dot{Y}_B(t), \dot{Z}_B(t) \right)$$

Let $\tilde{\bar{U}} = \bar{u}$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi \hat{n} dA + \int_{A_w} \left( \frac{\bar{u} \bullet \bar{u}}{2} \right) \hat{n}\, dA - \int_{A_w} \left( \bar{U}_B(t) \bullet \bar{u} \right) \hat{n}\, dA \quad (10.129)$$

Use the vector identity

$$\left( \bar{U}_B \bullet \bar{u} \right) \hat{n} = \left( \bar{U}_B \bullet \hat{n} \right) \bar{u} + \bar{U}_B \times \left( \hat{n} \times \bar{u} \right) \quad (10.130)$$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi \hat{n} dA + \int_{A_w} \left( \frac{\bar{u} \bullet \bar{u}}{2} \right) \hat{n}\, dA - \int_{A_w} \left( \bar{U}_B(t) \bullet \hat{n} \right) \bar{u}\, dA - \bar{U}_B(t) \times \int_{A_w} \left( \hat{n} \times \bar{u} \right) dA \quad (10.131)$$

These two terms cancel

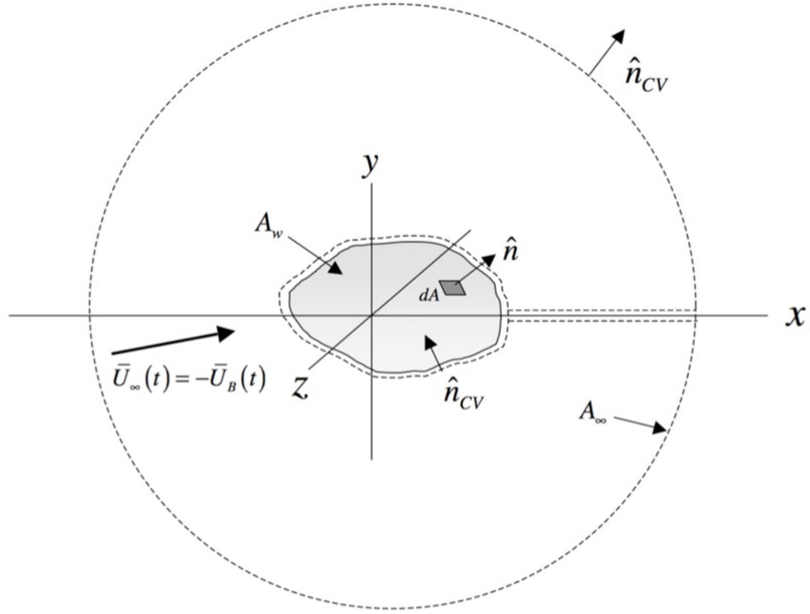


*Figure 10.10 Control volume surrounding a rigid body translating in an inviscid fluid.*

$$\int_V \nabla\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)dV = \int_A \left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV = \int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV + \int_{A_\infty}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV \qquad (10.132)$$

$$\int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}_{CV}dV = \int_V \nabla\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)dV$$
$$\int_{A_w}\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)\hat{n}dV = -\int_V \nabla\left(\frac{\bar{u}\bullet\bar{u}}{2}\right)dV = -\int_V (\bar{u}\bullet\nabla\bar{u})dV \qquad (10.133)$$

$$\int_{A_w}\left(\bar{U}_B(t)\bullet\hat{n}\right)\bar{u}\, dA \qquad (10.134)$$

$$\bar{U} \bullet \hat{n} = \left(\bar{u} - \bar{U}_B\right) \bullet \hat{n} = 0 \Rightarrow \bar{U}_B \bullet \hat{n} = \bar{u} \bullet \hat{n} \tag{10.135}$$

$$\int_{A_w} \left(\bar{U}_B(t) \bullet \hat{n}\right) \bar{u}\, dA = \int_{A_w} \left(\bar{u} \bullet \hat{n}\right) \bar{u}\, dA = \int_{A_w} \left(\bar{u}\, \bar{u}\right) \bullet \hat{n}\, dA \tag{10.136}$$

$$\int_V \nabla \bullet \left(\bar{u}\, \bar{u}\right) dV = \int_A \left(\bar{u}\, \bar{u}\right) \bullet \hat{n}_{CV} dA = \int_{A_w} \left(\bar{u}\, \bar{u}\right) \bullet \hat{n}_{CV} dA + \int_{A_\infty} \left(\bar{u}\, \bar{u}\right) \bullet \hat{n}_{CV} dA \tag{10.137}$$

$$\int_{A_w} \left(\bar{u}\, \bar{u}\right) \bullet \hat{n} dA = -\int_V \nabla \bullet \left(\bar{u}\, \bar{u}\right) dV \tag{10.138}$$

$$\int_{A_w} \left(\bar{U}_B(t) \bullet \hat{n}\right) \bar{u}\, dA = \int_{A_w} \left(\overline{uu}\right) \bullet \hat{n} dA = -\int_V \nabla \bullet \left(\overline{uu}\right) dV = -\int_V \bar{u} \bullet \nabla\left(\bar{u}\right) dV \tag{10.139}$$

$$\int_{A_w} \left(\frac{\bar{u} \bullet \bar{u}}{2}\right) \hat{n}\, dA - \int_{A_w} \left(\bar{U}_B(t) \bullet \hat{n}\right) \bar{u}\, dA = -\int_V \bar{u} \bullet \nabla\left(\bar{u}\right) dV + \int_V \bar{u} \bullet \nabla\left(\bar{u}\right) dV = 0 \tag{10.140}$$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt} \int_{A_w} \Phi \hat{n} dA - \bar{U}_B(t) \times \int_{A_w} \left(\hat{n} \times \bar{u}\right) dA \tag{10.141}$$

$$\bar{U}_B(t) = -\bar{U}_\infty(t) \qquad (10.142)$$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA + \bar{U}_\infty(t)\times\int_{A_w}(\hat{n}\times\bar{u})dA \qquad (10.143)$$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA + \bar{U}_\infty(t)\times\int_{A_w}\left(\hat{n}\times\left(\bar{U}-U_\infty\right)\right)dA =$$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA + \bar{U}_\infty(t)\times\int_{A_w}\left(\hat{n}\times\bar{U}\right)dA - \bar{U}_\infty(t)\times\int_{A_w}(\hat{n})dA\times\bar{U}_\infty(t) \qquad (10.144)$$

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA + \bar{U}_\infty(t)\times\int_{A_w}\left(\hat{n}\times\bar{U}\right)dA$$

Finally the force on a body in potential flow in the body-fixed frame is

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi\hat{n}dA - \bar{U}_\infty(t)\times\int_{A_w}(\nabla\Phi\times\hat{n})dA \qquad (10.145)$$

In steady flow the force is perpendicular to the velocity vector approaching the body

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# 10.9 Virtual mass

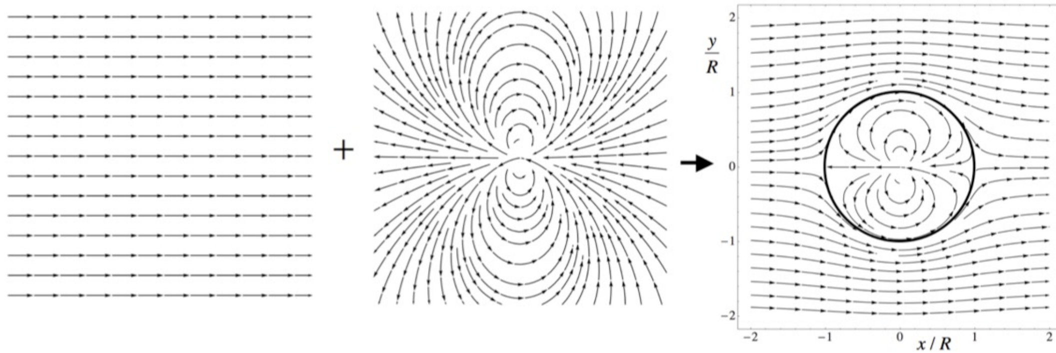


*Figure 10.12 Potential flow past a sphere.*

$$\frac{\bar{F}}{\rho} = \frac{d}{dt}\int_{A_w} \Phi \hat{n}\, dA - \bar{U}_\infty(t) \times \int_{A_w} (\nabla\Phi \times \hat{n})\, dA \qquad (10.153)$$

$$\Phi_{Sphere} = U_\infty f(t)\, x \left(1 + \frac{(R_{Sphere})^3}{2(x^2+y^2+z^2)^{3/2}}\right) \qquad (10.154)$$

$$\frac{F_x}{\rho} = \frac{d}{dt}\int_{A_w} \Phi n_x\, dA \qquad (10.155)$$

$$\frac{F_x}{\rho} = \frac{d}{dt}\int_{A_w} \Phi n_x\, dA =$$

$$U_\infty \frac{df}{dt}\int_0^{2\pi}\int_0^{\pi} R_{Sphere} Cos(\theta)\left(1 + \frac{(R_{Sphere})^3}{2(R_{Sphere})^3}\right) Cos(\theta)(R_{Sphere})^2 Sin(\theta)\, d\theta\, d\phi = \qquad (10.156)$$

$$U_\infty \frac{df}{dt} 2\pi (R_{Sphere})^3$$

Force on the sphere in the body-fixed frame

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# What is the force on the body in the space-fixed frame?

$$\begin{aligned}
&\tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t}) = \Phi(x,y,z,t) + x\dot{X}_B(t) \\
&\tilde{x} = x + \dot{X}_B(t) \\
&\tilde{y} = y \\
&\tilde{z} = z \\
&\tilde{t} = t \\
&\tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t}) = \Phi(\tilde{x} - \dot{X}_B(\tilde{t}),\tilde{y},\tilde{z},\tilde{t}) + (\tilde{x} - \dot{X}_B(\tilde{t}))\dot{X}_B(\tilde{t}) = \\
&\Phi(\tilde{x} + U_\infty(\tilde{t}),\tilde{y},\tilde{z},\tilde{t}) - U_\infty(\tilde{t})(\tilde{x} + U_\infty(\tilde{t}))
\end{aligned} \qquad (10.157)$$

$$\frac{\tilde{\tilde{F}}}{\rho} = \frac{d}{dt}\int_{\tilde{A}_w} \tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\tilde{\tilde{n}}d\tilde{A} \qquad (10.158)$$

$$\frac{\tilde{F}_x}{\rho} = \frac{d}{dt}\int_{A_w}\left(U_\infty f(t)(\tilde{x}+U_\infty(\tilde{t}))\left(\frac{(R_{Sphere})^3}{2\left((\tilde{x}+U_\infty(\tilde{t}))^2+\tilde{y}^2+\tilde{z}^2\right)^{3/2}}\right)\right)\tilde{n}_x d\tilde{A} = \qquad (10.159)$$

$$\frac{U_\infty}{2}\frac{df}{dt}\int_{A_w}(\tilde{x}+U_\infty(\tilde{t}))\tilde{n}_x d\tilde{A}$$

$$\tilde{n}_x = Cos(\tilde{\theta}) \text{ and } \tilde{x} + U_\infty(\tilde{t}) = R_{Sphere}Cos(\tilde{\theta})$$

Force on the sphere in the space-fixed frame

$$\frac{\tilde{F}_x}{\rho} = \frac{U_\infty (R_{Sphere})^3}{2}\frac{df}{dt}\int_0^\pi\int_0^{2\pi} Cos^2(\tilde{\theta})Sin(\tilde{\theta})d\phi d\theta = \frac{2\pi}{3}(R_{Sphere})^3 U_\infty \frac{df}{dt} \qquad (10.160)$$

Virtual mass = 1/2 of the displaced mass

$$\frac{F_x}{\rho} = 2\pi \left( R_{Sphere} \right)^3 U_\infty \frac{df}{dt} \tag{10.161}$$

$$\frac{\tilde{F}_x}{\rho} = \frac{2\pi}{3} \left( R_{Sphere} \right)^3 U_\infty \frac{df}{dt} \tag{10.162}$$

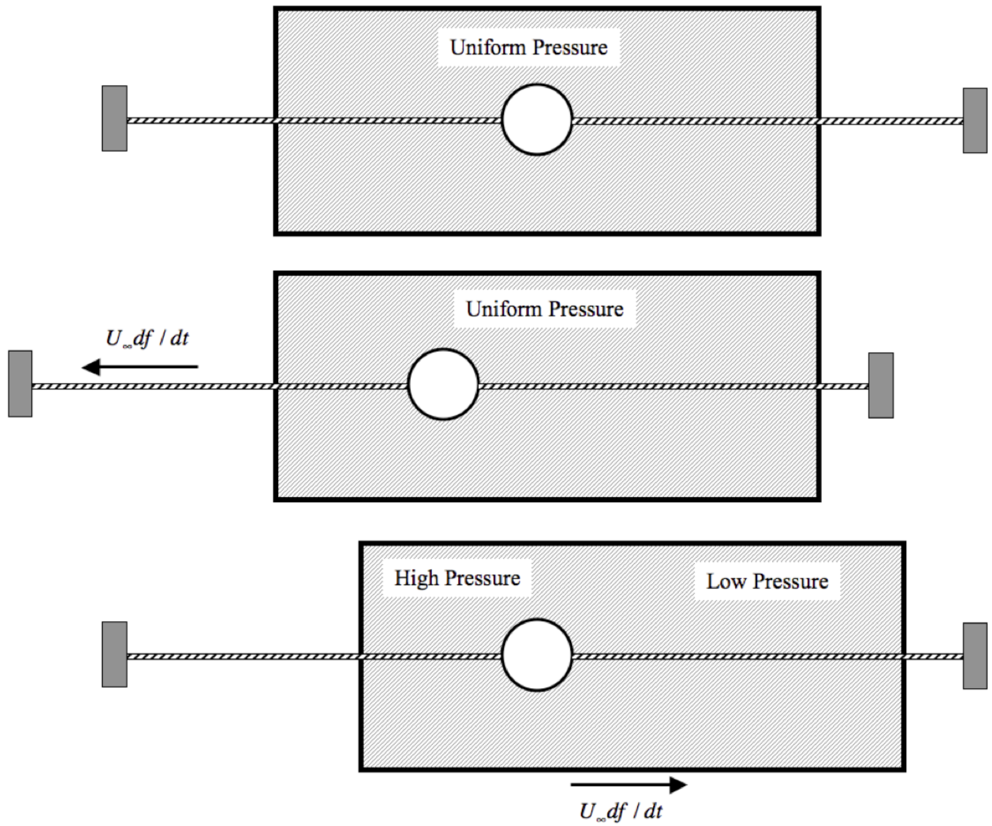


*Figure 10.13 Accelerated sphere versus accelerated fluid*